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The weighted Weiss conjecture and reproducing kernel theses for generalized Hankel operators

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Abstract The weighted Weiss conjecture states that the system theoretic property of weighted admissibility can be characterised by a resolvent growth condition. For positive weights, it is known that the conjecture is true if the system is governed by a normal operator; however, the conjecture fails if the system operator is the unilateral shift on the Hardy space $H^2(\mathbb{D})$ (discrete time) or the right-shift semigroup on $L^2(\mathbb{R}_+)$ (continuous time). To contrast and complement these counterexamples, in this paper positive results are presented characterising weighted admissibility of linear systems governed by shift operators and shift semigroups. These results are shown to be equivalent to the question of whether certain generalized Hankel operators satisfy a reproducing kernel thesis.

Keywords One parameter semigroups, admissibility, Hardy space, weighted Bergman space, Hankel operators, Reproducing kernel thesis.

Mathematics Subject Classification (2000) 30H10, 30H20, 47B32, 47B35, 47D06, 93B28.

1 Introduction

Consider an infinite dimensional control system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t), \quad t \geq 0,$$
$$x(0) = x_0 \in X$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X and the observation operator satisfies $C \in \mathcal{L}(D(A), \mathbb{C})$. For the system to be well-posed, in the sense of [19], a necessary condition is that C is *admissible* for A, that is, there exists $k > 0$ such that

$$\|CT(\cdot)x_0\|_{L^2(\mathbb{R}_+)} \leq k\|x_0\|_X, \qquad x_0 \in D(A).$$

An important consequence of admissibility is that the output y can be well defined even in the case that C is unbounded. In particular, admissibility implies that the map $x_0 \mapsto CT(\cdot)x_0 \in L^2(\mathbb{R}_+)$,

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defined initially on $D(A)$, has a continuous extension to the whole space X, meaning that the output is well defined for any initial condition $x_0 \in X$.

A generalization of admissibility, first considered in [3], is to require that the output is an element of a weighted L^2-space. For $\beta > -1$, C is said to be β-admissible for A if there exists a constant $k > 0$ such that

$$\int_0^\infty t^\beta |CT(t)x_0|^2 dt \leq k^2 \|x_0\|^2, \qquad x_0 \in D(A). \tag{1}$$

To test whether a given system is β-admissible, a frequency-domain characterization is convenient and, to this end, it is not difficult to show that β-admissibility implies the resolvent growth condition

$$\sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{\frac{1+\beta}{2}} \|CR(\lambda, A)^{-(1+\beta)}\|_{X^*} < \infty, \tag{2}$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ denotes the resolvent of the semigroup generator A, and $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ is the right-half plane. The question of whether the converse statement (2) $\Rightarrow$ (1) holds, commonly referred to as a *(weighted) Weiss conjecture*, is much more subtle. Existing results concerning the conjecture are discussed below, but we first describe a discrete time version of the Weiss conjecture, introduced in [5], which will also be studied in this paper.

A discrete-time linear control system on a Hilbert space X has the form

$$x_{n+1} = Tx_n, \quad y_n = Cx_n, \quad x_0 \in X, \qquad n \in \mathbb{N},$$

where $T \in \mathcal{L}(X)$ and $C \in X^*$. In this case, for $\beta > -1$, the observation functional C is said to be (discrete) β-admissible for T if there exists $k > 0$ such that

$$\sum_{n=0}^\infty (1+n)^\beta |CT^n x|^2 \leq k^2 \|x\|_X^2, \qquad x \in X. \tag{3}$$

Analogous to continuous time systems, the resolvent condition

$$\sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^{\frac{1+\beta}{2}} \|C(I - \bar{\omega}T)^{-(1+\beta)}\|_{X^*} < \infty \tag{4}$$

is necessary for (3) and the discrete time form of the weighted Weiss conjecture is to ask when the converse implication is true. The Weiss conjecture is superficially easier to study in discrete time due to the boundedness of the operators involved. However, it should be noted that it is sometimes possible to translate positive and negative results concerning the conjecture via the Cayley transform [5, 20].

The continuous time conjecture (2) $\Rightarrow$ (1) was originally posed [17] in the unweighted case $\beta = 0$. In this situation, the conjecture is true if A generates a C_0-semigroup of contractions [6], which extends the results that the conjecture holds if A is normal [18] and if A is the generator of the right-shift semigroup on $L^2(\mathbb{R}_+)$ [11]. The discrete time version (4) $\Rightarrow$ (3) for $\beta = 0$ and T a contraction was shown in [5].

For non-zero weights, the behaviour of the conjecture is more complicated. In the case that A is normal, the continuous time conjecture (2) $\Rightarrow$ (1) is true [22] for positive weights $\beta \in (0, 1)$, but false [21] in the case that $\beta \in (-1, 0)$. Analogous results also hold for the discrete time conjecture conjecture when T is normal [21, 22]. Furthermore, both continuous and discrete time conjectures are not true for general contraction operators for weights $\beta \in (0, 1)$: in continuous time, the right-shift semigroup on $L^2(\mathbb{R}_+)$ provides the counterexample [20]; while in discrete time (4) $\Rightarrow$ (3) fails if T is the unilateral shift on the Hardy space $H^2(\mathbb{D})$ [21].

It should be noted that the restriction $\beta \in (-1, 1)$ in the above discussion arises from the fact that the growth bound $\sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{\frac{1-\beta}{2}} \|CR(\lambda, A)\| < \infty$ (respectively, the condition $\sup_{\omega \in \mathbb{D}}(1 -$

$|\omega|^2)^{\frac{1-\beta}{2}}\|C(I-\bar{\omega}T)^{-1}\| < \infty$ in discrete time) was considered in the cited literature, i.e. a condition involving only the first power of the resolvent. In this situation, the restriction $\beta < 1$ is natural. However, as shown for example in [20], the truth of the weighted conjecture is not affected by considering instead the resolvent growth bound (2) and in this situation the natural range of weights is $\beta > -1$. Thus, the resolvent condition (2) is considered in the remainder of this paper.

The importance of determining the truth of the conjecture for the right-shift semigroup (or, in discrete time, the unilateral shift) is due to the Sz.Nagy-Foiaş model theory for contractions [14]. This states that a general contraction operator can be decomposed as a sum of operators, one of which is unitarily equivalent to a part of a shift operator. In [6] this decomposition was used in the case $\beta = 0$ (in discrete time, see [5]) to extend the truth of the conjecture for normal semigroups and the right-shift semigroup to general contraction semigroups. Thus, it is disappointing that neither the right-shift semigroup on $L^2(\mathbb{R}_+)$ nor the unilateral shift on $H^2(\mathbb{D})$ satisfy the weighted Weiss conjecture in the case $\beta \in (0,1)$.

The main results of this paper are to obtain positive results characterising β-admissibility for shift operators and semigroups. Results are proven in discrete time for the unilateral shift and in continuous time for the right-shift semigroup. For technical simplicity we first describe results in the discrete time setting. Two approaches are taken. The first is to consider the unilateral shift $(Sf)(z) = zf(z)$ acting on different space to $H^2(\mathbb{D})$. In Section 2, β-admissibility of the shift $S : X \to X$ is considered in the case that X is a weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{D}), \alpha > -1$, which contains analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\|f\|^2_{\mathcal{A}^2_\alpha(\mathbb{D})} = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha dA(z)$ and $dA(z) := \frac{1}{\pi} dxdy$ is area measure on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, for $z = x + iy$. Since the norm $\|f\|_{\mathcal{A}^2_\alpha(\mathbb{D})}$ is equivalent to

$$\left(\sum_{n=0}^{\infty} |f_n|^2 (1+n)^{-(1+\alpha)}\right)^{\frac{1}{2}}, \tag{5}$$

where f_n are the Taylor coefficients of f, naively, the Hardy space $H^2(\mathbb{D})$ may be thought of as the 'corner' of the family of weighted Bergman spaces as $\alpha \to -1^+$. However, the behaviour of the weighted Weiss conjecture changes at this corner: it is shown in Theorem 2.9 that for $\beta > 0$ the resolvent bound characterisation (4) $\Rightarrow$ (3) of β-admissibility holds for the shift $S : \mathcal{A}^2_\alpha(\mathbb{D}) \to \mathcal{A}^2_\alpha(\mathbb{D})$, for any $\alpha > -1$. The second approach, taken in Section 4, is to derive a modified resolvent growth bound characterisation of β-admissibility for the shift $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$. In this case, it is shown in Corollary 4.5 that β-admissibility is characterised by

$$\sup_{\omega \in \mathbb{D}} (1-|\omega|^2)^{\frac{1}{2}} \left\| C(I-\bar{\omega}S)^{-1} \right\|_{\mathcal{A}^2_{\beta-1}(\mathbb{D})^*} < \infty. \tag{6}$$

The difference between this condition and (4), which *does not* characterise β-admissibility of $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$, is that the weight β appears in the space in which the norm of the operator $C(I-\bar{\omega}S)^{-1}$ is tested, rather than as a power of the resolvent and the required growth rate.

That (6) is in some sense the 'correct' resolvent growth condition with which to test weighted admissibility of $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is intrinsically related to the notion of a Reproducing Kernel Thesis (RKT). A Reproducing Kernel Hilbert Space H is a space of analytic functions on a set Ω (in this paper, either $\Omega = \mathbb{D}$ or $\Omega = \mathbb{C}_+ = \{\lambda : \text{Re}\lambda > 0\}$) containing functions $(k_\omega)_{\omega \in \Omega} \subset H$, known as the *reproducing kernels*, which satisfy $f(\omega) = \langle f, k_\omega \rangle_H$ for any $f \in H, \omega \in \Omega$. A linear operator $T : H \to K$, where K is a second Hilbert space, is said to satisfy a Reproducing Kernel Thesis if its boundedness is characterised by

$$\sup_{\omega \in \Omega} \frac{\|Tk_\omega\|_K}{\|k_\omega\|_H} < \infty. \tag{7}$$

The question of which operators satisfy a RKT has received much attention and it is known that many important operators do satisfy a RKT (see, e.g. [10, p. 131] for a brief overview). Of particular relevance to the study of the Weiss conjecture for shifts is the fact that the little Hankel operator $h_{\bar{c}}(f) := \overline{P}(\bar{c}f)$, with symbol $c \in H^2(\mathbb{D})$, mapping from $H^2(\mathbb{D})$ to $\overline{H^2(\mathbb{D})} = \overline{P}H^2(\mathbb{D})$ satisfies a RKT. Here, $\overline{P}$ denotes the projection onto anti-analytic functions. In the case $\beta = 0$, it was shown in [5] that if $T = S$ is the unilateral shift on $H^2(\mathbb{D})$ and $c \in H^2(\mathbb{D})$ satisfies $Cf = \langle f, c\rangle_{H^2}$, then (3) holds if and only if $h_{\bar{c}}$ is bounded on $H^2(\mathbb{D})$. On the other hand, since the reproducing kernels for $H^2(\mathbb{D})$ are $k_w(z) = (1 - \bar{\omega}z)^{-1}$ and $\|k_\omega\|_{H^2(\mathbb{D})} = (1 - |\omega|^2)^{-\frac{1}{2}}$, it is not difficult to show that $h_{\bar{c}} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ satisfies (7) if and only if the resolvent condition (4) holds for $\beta = 0$. Hence, the truth of the discrete Weiss conjecture for the shift S in the unweighted case $\beta = 0$ is equivalent to the fact that each Hankel operator $h_{\bar{c}}$ satisfies a RKT.

In the weighted case $\beta > 0$, it is shown in Propositions 4.2 and 4.3 that β-admissibility of the shift $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is equivalent to boundedness of one/both of the *generalized* Hankel operators $h_{\bar{c}}^{\beta/2,0}$ or $h_{\bar{c}}^{0,\beta/2}$ (definitions of these operators are given in Section 4.1). It turns out that whether $h_{\bar{c}}^{\beta/2,0}$ satisfies (7) is equivalent to the modified resolvent condition (6); while whether $h_{\bar{c}}^{0,\beta/2}$ satisfies (7) is equivalent to the original resolvent condition (4). Consequently, the characterization (3) $\Leftrightarrow$ (6) of β-admissibility follows from the fact that the generalized Hankel operators $h_{\bar{c}}^{\beta/2,0}$ satisfy a RKT (proven in Theorem 4.4); while the failure of the original conjecture (4) $\not\Rightarrow$ (3) can now be explained by the fact that the operators $h_{\bar{c}}^{0,\beta/2}$ do not. The technical reason for this result is that the inclusion $D^{-\beta/2}BMOA \subset \Lambda^+_{\beta/2}$ between two certain classes of operator symbols is strict.

Analogous results to the ones described above are proven for the continuous time case. In Section 3, Theorem 3.1, it is shown that for $\beta > 0$ the weighted Weiss conjecture (2) $\Rightarrow$ (1) holds for the right-shift semigroup acting on any of the weighted spaces $L^2_\alpha(\mathbb{R}_+), \alpha > 0$, where

$$L^2_\alpha(\mathbb{R}_+) := \left\{ f : \mathbb{R}_+ \to \mathbb{C} : f \text{ measurable}, \int_0^\infty t^{-\alpha}|f(t)|^2 dt < \infty \right\}.$$

The 'corner' case of the right-shift semigroup on $L^2(\mathbb{R}_+)$ is discussed in Section 5, where it is shown in Corollary 5.9 that β-admissibility, $\beta > 0$, is characterised by the modified resolvent growth condition

$$\sup_{\lambda\in\mathbb{C}_+} (\mathrm{Re}\lambda)^{\frac{1}{2}} \|CR(\lambda, A)\|_{L^2_{\beta/2}(\mathbb{R}_+)^*} < \infty, \tag{8}$$

where A is the generator of the right-shift semigroup. In the continuous time setting, the characterisation of weighted admissibility is related to whether certain generalised Hankel operators satisfy a RKT on the Hardy space $H^2(\mathbb{C}_+)$.

2 Discrete time β-admissibility of the unilateral shift on weighted Bergman spaces

In this section, discrete-time β-admissibility is studied for the unilateral shift $S : \mathcal{A}^2_\alpha(\mathbb{D}) \to \mathcal{A}^2_\alpha(\mathbb{D})$ given by

$$(Sf)(z) = zf(z), \qquad f \in \mathcal{A}^2_\alpha(\mathbb{D}).$$

In the following, the inner product for $\mathcal{A}^2_\alpha(\mathbb{D})$ is written $\langle\ ,\ \rangle_\alpha$.

2.1 Equivalent conditions for discrete β-admissibility and the resolvent bound (4).

First, β-admissibility of an observation functional $C \in \mathcal{A}^2_\alpha(\mathbb{D})^*$ is characterized in terms of its associated Taylor coefficients.

Proposition 2.1. *Let $\alpha > -1, \beta > 0$. Suppose that $C \in \mathcal{A}^2_\alpha(\mathbb{D})^*$ is given by $Cf := \langle f, c\rangle_\alpha$, where $c = \sum_{n=0}^\infty c_n z^n \in \mathcal{A}^2_\alpha(\mathbb{D})$. Then C is discrete β-admissible for S if and only if*

$$\sum_{n=0}^\infty \left| \sum_{m=0}^\infty \frac{(1+n)^{\frac{\beta}{2}}}{(1+n+m)^{1+\alpha}} f_m \bar{c}_{n+m} \right|^2 \leq k\|f\|^2_{\mathcal{A}^2_\alpha(\mathbb{D})}, \qquad f \in \mathcal{A}^2_\alpha(\mathbb{D}).$$

Proof. For $f \in \mathcal{A}^2_\alpha(\mathbb{D})$,

$$\begin{aligned}
\sum_{n=0}^\infty (1+n)^\beta |CS^n f|^2 &= \sum_{n=0}^\infty (1+n)^\beta \left| \left\langle \sum_{m=0}^\infty f_m z^{m+n}, \sum_{m=0}^\infty c_m z^m \right\rangle_\alpha \right|^2 \\
&= \sum_{n=0}^\infty (1+n)^\beta \left| \left\langle \sum_{m=n}^\infty f_{m-n} z^m, \sum_{m=0}^\infty c_m z^m \right\rangle_\alpha \right|^2 \\
&= \sum_{n=0}^\infty (1+n)^\beta \left| \sum_{m=n}^\infty f_{m-n} \bar{c}_m (1+m)^{-(1+\alpha)} \right|^2 \\
&= \sum_{n=0}^\infty \left| \sum_{m=0}^\infty \frac{(1+n)^{\frac{\beta}{2}}}{(1+n+m)^{1+\alpha}} f_m \bar{c}_{n+m} \right|^2. \quad \square
\end{aligned}$$

In order to form a comparable expression for the resolvent condition (4), it is necessary to define the operator $g(S)$ for suitable functions g. To this end, let

$$\mathcal{O}(\overline{\mathbb{D}}) := \left\{ g \in H(\mathbb{D}) : \exists \nu > 1 \text{ such that } \sum_{n=0}^\infty |g_n| \nu^n < \infty \right\},$$

where g has Taylor series $g(z) = \sum_{n=0}^\infty g_n z^n$. Then by [5, Lemma 2.1], $g(S) \in \mathcal{L}(X)$ for any $g \in \mathcal{O}(\overline{D})$. Note that $g(z) = (1 - \bar{\omega} z)^{-(1+\beta)} \in \mathcal{O}(\overline{\mathbb{D}})$ for each $\omega \in \mathbb{D}$.

Proposition 2.2. *Let $\alpha > -1$. Suppose that $C \in \mathcal{A}^2_\alpha(\mathbb{D})^*$ is given by $Cf := \langle f, c\rangle_\alpha$, where $c = \sum_{n=0}^\infty c_n z^n \in \mathcal{A}^2_\alpha(\mathbb{D})$. Then for any $g = \sum_{n=0}^\infty g_n z^n \in \mathcal{O}(\overline{\mathbb{D}})$,*

$$\|Cg(S)\|^2_{\mathcal{A}^2_\alpha(\mathbb{D})^*} = \sum_{n=0}^\infty \left| \sum_{m=0}^\infty \frac{(1+n)^{\frac{1+\alpha}{2}}}{(1+n+m)^{1+\alpha}} g_m \bar{c}_{n+m} \right|^2.$$

Proof. For $f \in \mathcal{A}^2_\alpha(\mathbb{D})$,

$$\begin{aligned}
Cg(S)f &= \langle g(S)f, c\rangle_\alpha \\
&= \left\langle \sum_{m=0}^\infty \left(\sum_{n=0}^m f_n g_{m-n} \right) z^m, \sum_{m=0}^\infty c_m z^m \right\rangle_\alpha \\
&= \sum_{m=0}^\infty (1+m)^{-(1+\alpha)} \left(\sum_{n=0}^m f_n g_{m-n} \right) \bar{c}_m \\
&= \sum_{n=0}^\infty \sum_{m=n}^\infty (1+m)^{-(1+\alpha)} f_n g_{m-n} \bar{c}_m \\
&= \sum_{n=0}^\infty (1+n)^{-(1+\alpha)} \left(\sum_{m=0}^\infty \frac{(1+n)^{1+\alpha}}{(1+m+n)^{1+\alpha}} g_m \bar{c}_{n+m} \right) f_n \\
&= \langle f, h\rangle_\alpha,
\end{aligned}$$

where

$$h(z)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(1+n)^{1+\alpha}}{(1+m+n)^{1+\alpha}}\bar{g}_m c_{n+m}\right)z^n.$$

Therefore,

$$\begin{aligned}\|Cg(S)\|^2_{\mathcal{A}^2_\alpha(\mathbb{D})^*}=\|h\|^2_{\mathcal{A}^2_\alpha(\mathbb{D})}&=\sum_{n=0}^{\infty}(1+n)^{-(1+\alpha)}\left|\sum_{m=0}^{\infty}\frac{(1+n)^{1+\alpha}}{(1+m+n)^{1+\alpha}}\bar{g}_m c_{n+m}\right|^2\\&=\sum_{n=0}^{\infty}\left|\sum_{m=0}^{\infty}\frac{(1+n)^{\frac{1+\alpha}{2}}}{(1+m+n)^{1+\alpha}}g_m\bar{c}_{n+m}\right|^2.\quad\square\end{aligned}$$

2.2 Discrete β-admissibility and the little Hankel operator.

The link between Hankel operators and admissibility has previously been frequently exploited [5, 6, 11, 21] in order to study admissibility. In this section, it is shown that boundedness of little Hankel operators between weighted Bergman spaces characterise weighted admissibility of S on weighted Bergman spaces.

The little Hankel operator $h_f:\mathcal{A}^2_\alpha(\mathbb{D})\to\overline{\mathcal{A}^2_\alpha(\mathbb{D})}$ is defined by

$$h_f g=\overline{P_\alpha}(fg)$$

where $\overline{P_\alpha}$ is the orthogonal projection from $L^2(\mathbb{D},(1-|z|^2)^\alpha dA)$ to the space of anti-analytic functions

$$\overline{\mathcal{A}^2_\alpha(\mathbb{D})}:=\{\bar{f}:f\in\mathcal{A}^2_\alpha(\mathbb{D})\}.$$

Suppose that $f\in\mathcal{A}^2_\alpha(\mathbb{D})$. Then using the fact that $\{\bar{z}^n,n=0,1,2,\ldots\}$ is a basis for $\overline{\mathcal{A}^2_\alpha(\mathbb{D})}$,

$$\begin{aligned}\left\langle h_{\bar{f}}z^m,\bar{z}^n\right\rangle_\alpha&=\left\langle\overline{P_\alpha}(\bar{f}(z)z^m),\bar{z}^n\right\rangle_\alpha\\&=\left\langle\bar{f}(z)z^m,\bar{z}^n\right\rangle_\alpha\\&=\left\langle\bar{f}(z),\bar{z}^{m+n}\right\rangle_\alpha\\&=\bar{f}_{n+m}\left\langle\bar{z}^{m+n},\bar{z}^{m+n}\right\rangle_\alpha\\&=\frac{\bar{f}_{n+m}}{(1+n+m)^{1+\alpha}}.\end{aligned}$$

Therefore, if $g=\sum_{m=0}^{\infty}g_m z^m$,

$$\langle h_{\bar{f}}g,\bar{z}^n\rangle_\alpha=\sum_{m=0}^{\infty}\frac{g_m\bar{f}_{n+m}}{(1+n+m)^{1+\alpha}}.$$

and since $\langle\bar{z}^n,\bar{z}^n\rangle_\alpha=(1+n)^{-(1+\alpha)}$, it follows that

$$\left(h_{\bar{f}}g\right)_n=(1+n)^{1+\alpha}\sum_{m=0}^{\infty}\frac{g_m\bar{f}_{n+m}}{(1+n+m)^{1+\alpha}},$$

where $\left(h_{\bar{f}}g\right)_n$ is the n^{th} Fourier coefficient with respect to the basis $\{\bar{z}^n,n=0,1,2,\ldots\}$. Hence,

$$\|h_{\bar{f}}g\|^2_{\overline{\mathcal{A}^2_\alpha(\mathbb{D})}}=\sum_{n=0}^{\infty}\left|\sum_{m=0}^{\infty}\frac{(1+n)^{\frac{1+\alpha}{2}}}{(1+n+m)^{1+\alpha}}g_m\bar{f}_{n+m}\right|^2.\qquad(9)$$

Therefore, there is a link between weighted admissibility on weighted Bergman spaces, and boundedness of the little Hankel operator on weighted Bergman spaces. In the following,

$$k_\omega^\alpha(z) := \frac{(1-|\omega|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{\omega}z)^{2+\alpha}}, \qquad \omega \in \mathbb{D}, z \in \mathbb{D}$$

are the normalized reproducing kernels for $\mathcal{A}_\alpha^2(\mathbb{D})$.

Proposition 2.3. *Let $\alpha > -1, \beta > 0$. Suppose that $C \in \mathcal{A}_\alpha^2(\mathbb{D})^*$ is given by $Cf := \langle f, c\rangle_\alpha$, where $c = \sum_{n=0}^\infty c_n z^n \in \mathcal{A}_\alpha^2(\mathbb{D})$. Then*

(i) C is discrete β-admissible for S if and only if $h_{\bar{c}} : \mathcal{A}_{\beta-1}^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ is bounded.
(ii) For any $\omega \in \mathbb{D}$,

$$(1-|\omega|^2)^{\frac{1+\beta}{2}} \|C(I-\bar{\omega}S)^{-(1+\beta)}\|_{\mathcal{A}_\alpha^2(\mathbb{D})^*} = \|h_{\bar{c}} k_\omega^{\beta-1}\|_{\overline{\mathcal{A}_\alpha^2(\mathbb{D})}}.$$

Proof. (*i*) By Proposition 2.1, and the equivalent expression (5) for norm $\|\cdot\|_{\mathcal{A}_\alpha^2(\mathbb{D})}$, C is discrete β-admissible for S if and only if the matrix $A = (a_{nm})$ with coefficients

$$a_{nm} = \frac{(1+m)^{\frac{1+\alpha}{2}}(1+n)^{\frac{1+(\beta-1)}{2}}}{(1+n+m)^{1+\alpha}} \bar{c}_{n+m},$$

is bounded from ℓ^2 to ℓ^2. On the other hand, $h_{\bar{c}} : \mathcal{A}_{\beta-1}^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ is bounded if and only if

$$\|h_{\bar{c}} g\|_{\overline{\mathcal{A}_\alpha^2(\mathbb{D})}} \le k\|g\|_{\mathcal{A}_{\beta-1}^2(\mathbb{D})}, \qquad g \in \mathcal{A}_{\beta-1}^2(\mathbb{D}),$$

which by (9) and (5) occurs if and only if $A : \ell^2 \to \ell^2$ is bounded.

(*ii*) Follows from Proposition 2.2, (9) and the fact that $k_\omega^{\beta-1} \in \mathcal{O}(\overline{\mathbb{D}})$ for each $\omega \in \mathbb{D}$. □

Proposition 2.3 implies that the question of whether the discrete weighted Weiss conjecture (3) ⇔ (4) holds for $S : \mathcal{A}_\alpha^2(\mathbb{D}) \to \mathcal{A}_\alpha^2(\mathbb{D})$ in the case $\beta > 0$ is equivalent to the following question: does the Hankel operator $h_{\bar{c}} : \mathcal{A}_{\beta-1}^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ satisfy a RKT? Specifically, does (7) with $T = h_{\bar{c}}, H = \mathcal{A}_{\beta-1}^2(\mathbb{D})$ and $K = \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ imply boundedness of $h_{\bar{c}}$?

It is shown in [23, Theorem 8.39] that the question has a positive answer in the case $h_{\bar{c}} : \mathcal{A}_\alpha^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$. In fact the method of proof can be adapted to show that $h_{\bar{c}} : \mathcal{A}_\gamma^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ also satisfies RKT for any $\alpha, \gamma > -1$. For $\alpha, \gamma > -1$, and $f \in L^2(\mathbb{D}, dA_\alpha)$ define the integral operator

$$(Vf)(z) := \left(\frac{3+\alpha+\gamma}{1+\alpha}\right)(1-|z|^2)^{2+\gamma} \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{\omega})^{4+\alpha+\gamma}} dA_\alpha(\omega), \qquad z \in \mathbb{D}$$

and the projection $P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to \mathcal{A}_\alpha^2(\mathbb{D})$ by

$$(P_\alpha f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{\omega})^{2+\alpha}} dA_\alpha(w).$$

The following properties of V can now be deduced.

Lemma 2.4. *Suppose that $\alpha, \gamma > -1$. Then*

(i) The operator V is bounded on $L^2(\mathbb{D}, dA_\alpha)$;
(ii) $P_\alpha V = P_\alpha$.

Proof. (*i*) By [Zhu, Theorem 3.11], $V = T_{2+\gamma,\,\alpha,\,4+\alpha+\gamma}$, which is bounded on $L^2(\mathbb{D}, dA_\alpha)$.

(*ii*) For $f \in L^2(\mathbb{D}, dA_\alpha)$,

$$\begin{aligned}
(P_\alpha V f)(z) &= \int_{\mathbb{D}} \frac{(Vf)(w)}{(1-z\bar{\omega})^{2+\alpha}} dA_\alpha(\omega) \\
&= \left(\frac{3+\alpha+\gamma}{1+\alpha}\right) \int_{\mathbb{D}} \frac{(1-|\omega|^2)^{2+\gamma}}{(1-z\bar{\omega})^{2+\alpha}} \int_{\mathbb{D}} \frac{f(u)}{(1-w\bar{u})^{4+\alpha+\gamma}} dA_\alpha(u) dA_\alpha(w) \\
&= \left(\frac{3+\alpha+\gamma}{1+\alpha}\right) \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{(1-|\omega|^2)^{2+\gamma}}{(1-z\bar{\omega})^{2+\alpha}(1-\omega\bar{u})^{4+\alpha+\gamma}} dA_\alpha(\omega) \right) f(u) dA_\alpha(u) \\
&= \int_{\mathbb{D}} \left(\overline{\int_{\mathbb{D}} \frac{dA_{2+\alpha+\gamma}(\omega)}{(1-\bar{z}\omega)^{2+\alpha}(1-\bar{\omega}u)^{4+\alpha+\gamma}}} \right) f(u) dA_\alpha(u) \\
&= \int_{\mathbb{D}} \overline{\frac{1}{(1-\bar{u}z)^{2+\alpha}}} f(u) dA_\alpha(u) \\
&= \int_{\mathbb{D}} \frac{f(u)}{(1-z\bar{u})^{2+\alpha}} dA_\alpha(u) \\
&= (P_\alpha f)(z). \quad \square
\end{aligned}$$

Lemma 2.5. *Let $\alpha, \gamma > -1$. If $(1-|z|^2)^{\frac{\alpha-\gamma}{2}} f(z) \in L^\infty(\mathbb{D})$, then*

$$h_f \in \mathcal{L}(\mathcal{A}^2_\gamma(\mathbb{D}), \overline{\mathcal{A}^2_\alpha(\mathbb{D})}).$$

Proof. For $g \in \mathcal{A}^2_\gamma(\mathbb{D})$,

$$\begin{aligned}
\|h_f g\|_{\overline{\mathcal{A}^2_\alpha(\mathbb{D})}} &= \|\overline{P_\alpha}(fg)\|_{\overline{\mathcal{A}^2_\alpha(\mathbb{D})}} \\
&\le \|fg\|_{L^2(\mathbb{D}, dA_\alpha)} \\
&= \left((1+\alpha) \int_{\mathbb{D}} |f(z)|^2 |g(z)|^2 (1-|z|^2)^\alpha dA(z) \right)^{\frac{1}{2}} \\
\text{(by assumption)} &\le k \left(\int_{\mathbb{D}} |g(z)|^2 (1-|z|^2)^\gamma dA(z) \right)^{\frac{1}{2}} = k\|g\|_{\mathcal{A}^2_\gamma(\mathbb{D})}. \quad \square
\end{aligned}$$

The action of the little Hankel operator on reproducing kernels is now related to the integral operator V.

Lemma 2.6. *Let $\alpha, \gamma > -1$. Then for $f \in L^2(\mathbb{D}, dA_\alpha)$,*

$$\left\langle \bar{k}^\alpha_z, h_{\bar{f}} k^\gamma_z \right\rangle_\alpha = \frac{(1+\alpha)}{(3+\alpha+\gamma)} (1-|z|^2)^{\frac{\alpha-\gamma}{2}} (Vf)(z), \qquad z \in \mathbb{D}.$$

Proof. Using the fact that $\overline{h_{\bar{f}} k_z^\gamma}$ is analytic,

$$\begin{aligned}\left\langle \bar{k}_z^\alpha, h_{\bar{f}} k_z^\gamma \right\rangle_\alpha &= \left\langle \overline{h_{\bar{f}} k_z^\gamma}, k_z^\alpha \right\rangle_\alpha \\ &= (1-|z|^2)^{1+\frac{\alpha}{2}} \overline{h_{\bar{f}} k_z^\gamma}(z) \\ &= (1-|z|^2)^{1+\frac{\alpha}{2}} \overline{\int_{\mathbb{D}} \frac{\overline{f(\omega)} k_z^\gamma(\omega)}{(1-\bar{z}\omega)^{2+\alpha}} dA_\alpha(\omega)} \\ &= (1-|z|^2)^{2+\frac{\alpha+\gamma}{2}} \int_{\mathbb{D}} \frac{f(\omega)}{(1-z\bar{\omega})^{4+\alpha+\gamma}} dA_\alpha(\omega) \\ &= \frac{(1+\alpha)}{(3+\alpha+\gamma)} (1-|z|^2)^{\frac{\alpha-\gamma}{2}} (Vf)(z). \quad \square\end{aligned}$$

As a consequence, the little Hankel operators $h_{\bar{f}} : \mathcal{A}_\gamma^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ satisfy the reproducing kernel thesis.

Theorem 2.7. *Let $\alpha, \gamma > -1$. Then $\sup\{\|h_{\bar{f}} k_z^\gamma\|_{\overline{\mathcal{A}_\alpha^2(\mathbb{D})}} : z \in \mathbb{D}\} < \infty$ if and only if the little Hankel operator $h_{\bar{f}} : \mathcal{A}_\gamma^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ is bounded.*

Proof. If $f \in L^2(\mathbb{D}, dA_\alpha)$ then by Lemma 2.4,

$$h_{\bar{f}} = h_{\overline{P_\alpha f}} = h_{\overline{P_\alpha V f}} = h_{\overline{Vf}}. \tag{10}$$

By assumption and Lemma 2.6,

$$\frac{(1+\alpha)}{(3+\alpha+\gamma)} (1-|z|^2)^{\frac{\alpha-\gamma}{2}} |(Vf)(z)| = \left| \left\langle \bar{k}_z^\alpha, h_{\bar{f}} k_z^\gamma \right\rangle_\alpha \right| \le \|h_{\bar{f}} k_z^\gamma\|_{L^2(\mathbb{D}, dA_\alpha)} < k, \qquad z \in \mathbb{D}.$$

By (10) and Lemma 2.5, $h_{\bar{f}}$ is bounded. $\square$

Remark 2.8. *It should be noted that boundedness of the little Hankel operator $h_{\bar{f}} : \mathcal{A}_\gamma^2(\mathbb{D}) \to \overline{\mathcal{A}_\alpha^2(\mathbb{D})}$ has been characterised, in terms of symbols, in [7]. Theorem 2.7 therefore provides an additional characterisation of boundedness.*

Using Proposition 2.3 and Theorem 2.7, we can prove that the weighted Weiss conjecture is true for the shift on weighted Bergman spaces.

Theorem 2.9. *Let $\alpha > -1, \beta > 0$. Suppose that $C \in \mathcal{A}_\alpha^2(\mathbb{D})^*$ is given by $Cf := \langle f, c \rangle_\alpha$, where $c = \sum_{n=0}^\infty c_n z^n \in \mathcal{A}_\alpha^2(\mathbb{D})$. Then C is discrete β-admissible for S if and only if*

$$\sup_{\omega \in \mathbb{D}} (1-|\omega|^2)^{\frac{1+\beta}{2}} \left\| C(I - \bar{\omega} S)^{-(1+\beta)} \right\|_{\mathcal{A}_\alpha^2(\mathbb{D})^*} < \infty.$$

3 β-admissibility of the right-shift semigroup on $L_\alpha^2(\mathbb{R}_+)$

In this section Theorem 2.9 is translated to continuous time for the right-shift C_0-semigroup on $L_\alpha^2(\mathbb{R}_+)$ given by

$$(S(t)f)(\tau) := \begin{cases} f(\tau - t), & \tau \ge t; \\ 0, & \tau < t; \end{cases} \qquad t \ge 0, f \in L_\alpha^2(\mathbb{R}_+).$$

The Laplace transform is an isometric isomorphism $\mathcal{L} : L_\alpha^2(\mathbb{R}_+) \to \mathcal{A}_{\alpha-1}^2(\mathbb{C}_+)$. Here, for each $\gamma > -1$,

$$\mathcal{A}_\gamma^2(\mathbb{C}_+) := \left\{ F : \mathbb{C}_+ \to \mathbb{C} : \|F\|_{\mathcal{A}_\gamma^2(\mathbb{C}_+)}^2 := \int_{-\infty}^\infty \int_0^\infty x^\gamma |F(x+iy)|^2 dx dy < \infty \right\}$$

is the weighted Bergman space on the right-half-plane $\mathbb{C}_+$. Under the isomorphism provided by the Laplace transform, $(S(t))_{t\geq 0}$ is equivalent to the semigroup

$$(T(t)f)(z) = e^{-zt}f(z), \qquad f \in \mathcal{A}^2_{\alpha-1}(\mathbb{C}_+), z \in \mathbb{C}_+ \tag{11}$$

on the Bergman space $\mathcal{A}^2_{\alpha-1}(\mathbb{C}_+)$. Using this link, Theorem 2.9 can now be translated from discrete to continuous time.

Theorem 3.1. *Let $\alpha, \beta > 0$. Let A be the generator of the right-shift semigroup $(S(t))_{t\geq 0}$ on $L^2_\alpha(\mathbb{R}_+)$. Then an observation operator $C \in \mathcal{L}(D(A), \mathbb{C})$ is β-admissible for A if and only if*

$$\sup_{\lambda\in\mathbb{C}_+} (\mathrm{Re}\lambda)^{\frac{1+\beta}{2}} \left\|CR(\lambda, A)^{1+\beta}\right\|_{L^2_\alpha(\mathbb{R}_+)^*} < \infty.$$

Proof. Let

$$(J_\alpha f)(z) := \frac{c_\alpha}{(1+z)^{1+\alpha}} f\left(\frac{1-z}{1+z}\right), \qquad f \in \mathcal{A}_\alpha(\mathbb{C}_+), z \in \mathbb{D}$$

be the isometric isomorphism $J_\alpha : \mathcal{A}^2_\alpha(\mathbb{C}_+) \to \mathcal{A}^2_\alpha(\mathbb{D})$. Then

$$\Phi := J_{\alpha-1} \circ \mathcal{L} : L^2_\alpha(\mathbb{R}_+) \to \mathcal{A}^2_{\alpha-1}(\mathbb{D})$$

is also an isometric isomorphism, under which the shift semigroup $(S(t))_{t\geq 0}$ on $L^2_\alpha(\mathbb{R}_+)$ is equivalent to the semigroup

$$(Q(t)f)(z) := e^{-\left(\frac{1-z}{1+z}\right)t} f(z), \qquad f \in \mathcal{A}^2_{\alpha-1}(\mathbb{D}), z \in \mathbb{D}.$$

Notice that the unilateral shift S is the co-generator of $(Q(t))_{t\geq 0}$ on $\mathcal{A}^2_{\alpha-1}(\mathbb{D})$.

Given $C \in \mathcal{L}(D(A), \mathbb{C})$, define an observation operator by $\tilde{C} := C\Phi^{-1}$. If $\tilde{A}$ is the generator of $(Q(t))_{t\geq 0}$, then by assumption and the fact that $R(\lambda, \tilde{A})^{1+\beta} = \Phi R(\lambda, A)^{1+\beta}\Phi^{-1}$,

$$\sup_{\lambda\in\mathbb{C}_+} (\mathrm{Re}\lambda)^{\frac{1+\beta}{2}} \left\|\tilde{C}R(\lambda, \tilde{A})^{1+\beta}\right\|_{\mathcal{A}^2_{\alpha-1}(\mathbb{D})^*} < \infty.$$

By the above equation and an argument from [20], it follows that if $D := \tilde{C}(I - \tilde{A})^{-(1+\beta)}$ then

$$\sup_{\omega\in\mathbb{D}} (1-|\omega|^2)^{\frac{1+\beta}{2}} \left\|D(I - \bar{\omega}S)^{-(1+\beta)}\right\|_{\mathcal{A}^2_{\alpha-1}(\mathbb{D})^*} < \infty$$

and Theorem 2.9 implies that D is discrete β-admissible for S. It is shown in [20] that D is discrete β-admissible for S if and only if $\tilde{C}$ is β-admissible for $(Q(t))_{t\geq 0}$. Since $(Q(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are equivalent semigroups, it follows that C is β-admissible for $(S(t))_{t\geq 0}$. □

4 Discrete β-admissibility of the unilateral shift on the Hardy space

In this section discrete β-admissibility, $\beta > 0$, is characterised for the unilateral shift $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ given by $(Sf)(z) = zf(z), z \in \mathbb{D}, f \in H^2(\mathbb{D})$. The Hardy space $H^2(\mathbb{D})$ is the set of complex-valued analytic functions $f(z) = \sum_{n=0}^\infty f_n z^n$ such that

$$\|f\|^2_{H^2(\mathbb{D})} := \sup_{0<r<1} \int_{\theta=0}^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^\infty |f_n|^2 < \infty.$$

The space $H^2(\mathbb{D})$ is a reproducing kernel space with the (non-normalized) reproducing kernel with respect to $w \in \mathbb{D}$ given by

$$k_w(z) = \frac{1}{1-\bar{w}z}, \qquad z \in \mathbb{D}.$$

Note that $\|k_w\|_{H^2(\mathbb{D})} = (1-|w|^2)^{-\frac{1}{2}}$.

For a function $f \in \mathrm{Hol}(\mathbb{D}) + \overline{\mathrm{Hol}(\mathbb{D})}$ we associate the sequence of Taylor coefficients $\{f_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ for which

$$f(z) = \sum_{n=0}^{\infty} f_n z^n + \sum_{n=1}^{\infty} f_{-n} \bar{z}^n, \qquad z \in \mathbb{D}.$$

If $\{f_n\}_{n\in\mathbb{Z}}$ has finite support we say that f is polynomial. The pairing between two functions $f, g \in \mathrm{Hol}(\mathbb{D}) + \overline{\mathrm{Hol}(\mathbb{D})}$ is defined by

$$\langle f, g\rangle = \sum_n f_n \overline{g_n},$$

whenever the series converges. We are particularly interested in the cases when either both sequences are square summable, or one of the functions is polynomial. We will often work with $H^2(\mathbb{D})$ and $\overline{H^2(\mathbb{D})}$ as closed subspaces of $L^2(\mathbb{T})$, where the Taylor coefficients are interpreted as Fourier coefficients. Note that the pairing between $f, g \in L^2(\mathbb{T})$ coincides with the usual inner product.

We now introduce spaces of analytic functions which will be required to study discrete β-admissibility of the unilateral shift on $H^2(\mathbb{D})$.

For an integer $n \geq 1$, define the trigonometric polynomial W_n by the Fourier coefficients

$$\hat{W}_n(k) = \begin{cases} \frac{k-2^{n-1}}{2^{n-1}} & \text{if } k \in [2^{n-1}, 2^n), \\ \frac{2^{n+1}-k}{2^n} & \text{if } k \in [2^n, 2^{n+1}), \\ 0 & \text{otherwise.} \end{cases}$$

For $n \leq -1$, let $W_n = \overline{W_{-n}}$, and finally $W_0(\theta) = e^{-i\theta} + 1 + e^{i\theta}$. For $s \in \mathbb{R}$, the *Hölder-Zygmund space* Λ_s consists of distributions f on $\mathbb{T}$ such that

$$\|f\|_{\Lambda_s} = \sup_{n\in\mathbb{Z}} 2^{|n|s} \|W_n * f\|_\infty < \infty.$$

These spaces are introduced in [13, Appendix 2]. The parameter s indicates in this way how quickly the Fourier coefficients of f decay, and therefore the defining property of the Hölder-Zygmund spaces is a smoothness condition. We will often consider the subspace Λ_s^+ of holomorphic distributions in Λ_s. The space Λ_0^+ is called the Bloch space.

Given a function $f \in L^1(\mathbb{T})$ we define the quantity

$$\|f\|_{BMO} = \sup_{I\subset\mathbb{T}} \frac{1}{|I|} \int_I \left| f(t) - \frac{1}{|I|}\int_I f(s)ds \right| dt.$$

We then define the space

$$BMOA(\mathbb{D}) = \{f \in H^2(\mathbb{D}) : \|f\|_{BMO} < \infty\}.$$

The space $BMOA(\mathbb{D})$ can be characterized using wavelets. Given a function $\psi : \mathbb{R} \to \mathbb{C}$ we let $\psi_j(x) = 2^{j/2}\psi(2^j x)$ for $j \in \mathbb{Z}$ and $\psi_J(x) = \psi_j(x - x_J)$ where x_J is the left endpoint of the dyadic interval $J = [2\pi k 2^{-j}, 2\pi(k+1)2^{-j})$. We will need a function $\psi \in \mathcal{S}(\mathbb{R})$ such that

$$\operatorname{supp} \hat{\psi} \subseteq \left\{\xi : \frac{1}{3} \leq |\xi| \leq \frac{4}{3}\right\}, \tag{12}$$

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0, \qquad k \in \mathbb{Z}, \tag{13}$$

and moreover is such that $\{\psi_J\}_J$, where J ranges over all dyadic intervals, is an orthonormal basis for $L^2(\mathbb{R})$. Such a ψ exists and is exemplified by the Littlewood-Paley wavelet constructed in [9, pp. 21–25, p. 75].

We identify $[0, 2\pi)$, $\mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}$ with each other, using the mapping $x \mapsto e^{ix}$. For any dyadic interval $J \subseteq [0, 2\pi)$, we define the 2π-periodification of ψ_J by

$$\varphi_J(x) = \sum_{k\in\mathbb{Z}} \psi_J(x - 2\pi k), \qquad x \in \mathbb{R}.$$

The family $\{\varphi_J\}_{J\subseteq\mathbb{T}\,\text{dyadic}}$ together with the constant function 1 is an orthonormal wavelet on $\mathbb{T}$. Using this language we have the following proposition [9, page 162].

Proposition 4.1. *A function $f \in H^2(\mathbb{D})$ is in BMOA if and only if there exists $M > 0$ such that*

$$\sum_{J\subseteq I} |\langle f, \varphi_J\rangle|^2 \leq M^2|I|,$$

for any dyadic interval $I \subseteq [0, 2\pi)$. Moreover $\inf M \approx \|f\|_{BMO}$.

Let $\alpha \in \mathbb{R}$. In order to discuss weighted admissibility we introduce the following operator, defined for double sided sequences of numbers:

$$D^\alpha : (a_n)_{n\in\mathbb{Z}} \mapsto ((1 + |n|)^\alpha a_n)_{n\in\mathbb{Z}}.$$

By letting D^α act on the sequence of Taylor coefficients of a function, D^α may be regarded as an operator acting on $\mathrm{Hol}(\mathbb{D}) + \overline{\mathrm{Hol}(\mathbb{D})}$. Note that for $\alpha > 0$, $D^\alpha : H^2(\mathbb{D}) \to \mathcal{A}^2_{2\alpha-1}(\mathbb{D})$ isomorphically. For $\alpha, s \in \mathbb{R}$ we also have that $D^\alpha \Lambda_s = \Lambda_{s-\alpha}$ [13, Equation (A2.15)]. It is well known the $BMOA \subset \Lambda_0^+$ with strict inclusion and so it follows that $D^{-s}BMOA \subset \Lambda_s^+$ with strict inclusion for all $s \in \mathbb{R}$.

4.1 Admissibility and the little Hankel operator on $H^2(\mathbb{D})$.

Given a function $f \in H^2(\mathbb{D})$, define the little Hankel operator $h_{\bar{f}} : H^\infty(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ by

$$h_{\bar{f}} g = \overline{P}(\bar{f} g),$$

where $\overline{P}$ is the orthogonal projection from $L^2(\mathbb{T})$ to $\overline{H^2(\mathbb{D})}$. We will investigate when this operator has a continuous extension $h_{\bar{f}} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$.

In the same manner as before, we see that

$$\begin{aligned}
\langle h_{\bar{f}} z^m, \bar{z}^n\rangle &= \langle \overline{P}(\bar{f}(z) z^m), \bar{z}^n\rangle \\
&= \langle \bar{f}(z) z^m, \bar{z}^n\rangle \\
&= \langle \bar{f}(z), \bar{z}^{m+n}\rangle \\
&= \bar{f}_{n+m}\langle \bar{z}^{m+n}, \bar{z}^{m+n}\rangle \\
&= \bar{f}_{n+m},
\end{aligned}$$

so that the matrix for the operator $h_{\bar{f}}$, in the monomial bases $\{z^n : n = 0, 1, 2, \ldots\}$ and $\{\bar{z}^n : n = 0, 1, 2, \ldots\}$ in $H^2(\mathbb{D})$ and $\overline{H^2(\mathbb{D})}$ respectively, becomes

$$\{\bar{f}_{n+m}\}_{n,m\geq 0} = \begin{pmatrix} \bar{f}_0 & \bar{f}_1 & \bar{f}_2 & \cdots \\ \bar{f}_1 & \bar{f}_2 & \bar{f}_3 & \cdots \\ \bar{f}_2 & \bar{f}_3 & \bar{f}_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Define the generalized Hankel operator

$$h_{\bar{f}}^{\alpha,\beta} : g \mapsto D^\alpha h_{\bar{f}} D^\beta g,$$

where $f \in H^2(\mathbb{D})$ and $\alpha, \beta \geq 0$. The operator is defined, at least for $g \in \mathcal{O}(\mathbb{D})$, in the sense that $h_{\bar{f}}^{\alpha,\beta} g \in \overline{A^2_{2\alpha-1}(\mathbb{D})}$. The operator can be represented by the generalized Hankel matrix

$$\{(1+n)^\alpha (1+m)^\beta \bar{f}_{n+m}\}_{n,m\geq 0} = \begin{pmatrix} \bar{f}_0 & 2^\beta \bar{f}_1 & 3^\beta \bar{f}_2 & \dots \\ 2^\alpha \bar{f}_1 & 2^\alpha 2^\beta \bar{f}_2 & 2^\alpha 3^\beta \bar{f}_3 & \dots \\ 3^\alpha \bar{f}_2 & 3^\alpha 2^\beta \bar{f}_3 & 3^\alpha 3^\beta \bar{f}_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The following proposition links the generalized Hankel operator to weighted admissibility.

Proposition 4.2. *Let $C \in H^2(\mathbb{D})^*$, and let $c \in H^2(\mathbb{D})$ be given by $Cf = \langle f, c\rangle$. If $\beta \geq 0$, then C is discrete 2β-admissible for S if and only $h_{\bar{c}}^{\beta,0} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ is bounded.*

Proof. Observe that

$$(1+n)^\beta C S^n f = (1+n)^\beta \sum_{m=0}^{\infty} f_m \bar{c}_{n+m} = (1+n)^\beta (h_{\bar{c}} f)_n = (h_{\bar{c}}^{\beta,0} f)_n.$$

Parseval's identity now completes the proof:

$$\sum_{n=0}^{\infty} (1+n)^{2\beta} |CS^n f|^2 = \sum_{n=0}^{\infty} |(h_{\bar{c}}^{\beta,0} f)_n|^2 = \|h_{\bar{c}}^{\beta,0} f\|^2_{\overline{H^2(\mathbb{D})}}. \quad \square$$

The boundedness of the operators $h_{\bar{f}}^{\alpha,\beta}$ has been characterized in [8] and [12]. The results have been collected in [13, Chapter 6.8].

Proposition 4.3. *Let $f \in H^2(\mathbb{D})$. With the notation above we have that:*

(i) Let $\beta \geq 0$. Then each of the operators $h_{\bar{f}}^{0,\beta} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ and $h_{\bar{f}}^{\beta,0} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ is bounded if and only if $D^\beta f \in BMOA$, with $\|h_{\bar{f}}^{0,\beta}\|_{H^2(\mathbb{D})\to\overline{H^2(\mathbb{D})}}$ and $\|h_{\bar{f}}^{\beta,0}\|_{H^2(\mathbb{D})\to\overline{H^2(\mathbb{D})}}$ comparable to $\|D^\beta f\|_{BMO}$.

(ii) Let $\alpha, \beta > 0$. Then the operator $h_{\bar{f}}^{\alpha,\beta} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ is bounded if and only if $f \in \Lambda_{\alpha+\beta}$, with $\|h_{\bar{f}}^{\alpha,\beta}\|_{H^2(\mathbb{D})\to\overline{H^2(\mathbb{D})}}$ comparable to $\|f\|_{\Lambda_{\alpha+\beta}}$.

4.2 Characterizing discrete β-admissibility of the shift on $H^2(\mathbb{D})$.

Similar to the case for the shift on weighted Bergman spaces considered in Section 2, the truth of the weighted Weiss conjecture for the shift on $H^2(\mathbb{D})$ is related to whether the operators $h_{\bar{c}}^{\beta,0} = D^\beta h_{\bar{c}}$ and $h_{\bar{c}}^{0,\beta} = h_{\bar{c}} D^\beta$ satisfy a RKT. We show that this is true for the former class of operators but false for the latter.

Theorem 4.4. *Let $c \in H^2(\mathbb{D})$ and $\beta \geq 0$. The following are equivalent:*

(i) The operator $h_{\bar{c}}^{\beta,0} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ is bounded.

(ii) The operator $h_{\bar{c}}^{\beta,0} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ *is bounded on reproducing kernels, i.e.*

$$M = \sup_{w\in\mathbb{D}} (1-|w|^2)^{1/2} \|h_{\bar{c}}^{\beta,0} k_w\|_{\overline{H^2(\mathbb{D})}} < \infty.$$

Moreover $\|h_{\bar{c}}^{\beta,0}\|_{H^2(\mathbb{D})\to\overline{H^2(\mathbb{D})}} \approx M$.

Proof. It is obvious that $(i) \Rightarrow (ii)$. By Proposition 4.3 we conclude that $M \lesssim \|D^\beta c\|_{BMO}$ and also that in order to prove $(ii) \Rightarrow (i)$ it is sufficient to show that $\|D^\beta c\|_{BMO} \lesssim M$. By Proposition 4.1 it is enough to show that

$$\sum_{J\subseteq I} |\langle D^\beta c, \varphi_J\rangle|^2 \lesssim M^2|I|,$$

for any dyadic interval $I \subseteq \mathbb{T}$.

Consider a fixed I and define $s_J = \langle D^\beta c, \varphi_J\rangle$ for $J \subseteq I$. It is immediate that $s_J = \langle c, D^\beta \varphi_J\rangle$. Since c is analytic,

$$\begin{aligned} s_J &= \langle c, D^\beta P\varphi_J\rangle \\ &= \langle c\bar{k}_w, hD^\beta P\varphi_J\rangle \\ &= \langle P(c\bar{k}_w), hD^\beta P\varphi_J\rangle \\ &= \langle g, D^{-\beta}(hD^\beta P\varphi_J)\rangle, \end{aligned}$$

where $w = \sqrt{1-\frac{|I|}{2\pi}}e^{ix_I}$, $\bar{g} = D^\beta h_{\bar{c}} k_w$ and $h : \mathbb{D} \to \mathbb{C}$ is the analytic function given by $h(z) = (1-\bar{w}z)$. Note that $1-|w|^2 = \frac{|I|}{2\pi}$ and that by hypothesis $\|g\|^2 \lesssim \frac{M^2}{|I|}$.

It is an elementary exercise to show that

$$(D^{-\beta}(hD^\beta P\varphi_J))\hat{}(n) = \Big(\hat{\varphi}_J(n) - \bar{w}\Big(\frac{n}{1+n}\Big)^\beta \hat{\varphi}_J(n-1)\Big), \qquad n \in \mathbb{N},$$

from which it quickly follows that

$$D^{-\beta}(hD^\beta P\varphi_J) = h\varphi_J + \bar{w}FP\varphi_J,$$

where

$$(FP\varphi_J)\hat{}(n) = \Big(1 - \Big(\frac{n}{1+n}\Big)^\beta\Big)\hat{\varphi}_J(n-1), \qquad n \in \mathbb{N}.$$

An important observation is that

$$\left|1 - \Big(\frac{x}{1+x}\Big)^\beta\right| \lesssim \frac{1}{1+x}, \qquad x \geq 1. \tag{14}$$

So far, it has been shown that

$$s_J = \langle g\bar{h}, P\varphi_J\rangle + \bar{w}\langle g, FP\varphi_J\rangle$$

and this expression is now decomposed further. Let χ be the characteristic function of the interval $[x_I - 2|I|, x_I + 2|I|)$. Then,

$$\begin{aligned} s_J &= \langle g\bar{h}, P\varphi_J\rangle + \bar{w}\langle g, FP\varphi_J\rangle \\ &= \langle \chi g\bar{h}, P\varphi_J\rangle + \bar{w}\langle g, FP\varphi_J\rangle + \langle g, (1-\chi)hP\varphi_J\rangle, \end{aligned}$$

so that

$$\sum_{J\subseteq I} |\langle D^\beta f, \varphi_J\rangle|^2 \leq 9 \sum_{j=1}^{3} \sum_{J\subset I} |s_J^{(j)}|^2,$$

where

$$s_J^{(1)} = \langle \chi g\bar{h}, P\varphi_J\rangle,$$
$$s_J^{(2)} = \langle g, FP\varphi_J\rangle,$$
$$s_J^{(3)} = \langle g, (1-\chi)hP\varphi_J\rangle.$$

We now handle these three parts separately.

Since $\{\varphi_J\}_{J\subseteq I}$ forms an orthonormal set in $L^2(\mathbb{T})$ it follows immediately from Bessel's inequality that

$$\begin{aligned}\sum_{J\subseteq I}|s_J^{(1)}|^2 &\leq \left\|P\chi\bar{h}g\right\|^2\\ &\leq \left\|\chi\bar{h}g\right\|^2\\ &= \int_{|x-x_I|<2|I|}|g(e^{ix})|^2|h(e^{-ix})|^2dx\\ &\leq \sup_{|x-x_I|<2|I|}|h(e^{-ix})|^2\frac{M^2}{|I|}.\end{aligned}$$

Using simple geometric arguments it is easy to show that $\sup_{|x-x_I|<2|I|}|h(e^{-ix})|^2 \lesssim |I|^2$ which in turn gives

$$\sum_{J\subseteq I}|s_J^{(1)}|^2 \lesssim M^2|I|.$$

The second set of terms is estimated using Hölder's inequality:

$$|s_J^{(2)}|^2 \leq \|g\|^2\|FP\varphi_J\|^2 \leq \frac{M^2}{|I|}\|FP\varphi_J\|^2.$$

It is easy to show that

$$\hat{\varphi}_J(n) = 2^{-j/2}e^{inx_J}\hat{\psi}\left(\frac{n}{2^j}\right).$$

Since $\hat{\psi}(\xi)=0$ for $|\xi|<1/3$ we have that all $\hat{\varphi}_J(n)$ vanish for $n<2^j/3$. Using this together with (14) gives

$$\begin{aligned}\|FP\varphi_J\|^2 &= \sum_{n=0}^{\infty}\left|1-\left(\frac{1+n}{2+n}\right)^{\beta}\right|^2|\hat{\varphi}_J(n)|^2\\ &\leq \left|1-\left(\frac{1+2^j/3}{2+2^j/3}\right)^{\beta}\right|^2\sum_{n=0}^{\infty}|\hat{\varphi}_J(n)|^2\\ &\lesssim \left(\frac{1}{2+2^j/3}\right)^2\\ &\lesssim |J|^2.\end{aligned}$$

Hence,

$$\sum_{J\subseteq I}|s_J^{(2)}|^2 \lesssim \frac{M^2}{|I|}\sum_{J\subseteq I}|J|^2 = \frac{M^2}{|I|}\sum_{n=0}^{\infty}2^n\left(\frac{|I|}{2^n}\right)^2 = 2M^2|I|.$$

Estimating the final set of terms is similar to estimating the second, although somewhat more sophisticated. Using Hölder's inequality,

$$|s_J^{(3)}|^2 \leq \frac{M^2}{|I|}\|(1-\chi)hP\varphi_J\|^2.$$

We now need to show that estimate $\|(1-\chi)h\varphi_J\|^2 \lesssim |J|^2$. To do this, notice that $h = h_1 + rh_2$ where $h_1(x) = 1 - r + r(e^{i(x-x_J)} - e^{i(x-x_I)})$, $h_2(x) = 1 - e^{i(x-x_J)}$ and $r = |w|$.

Assuming that $J \subseteq I$ we have that

$$|h_1(x)| \lesssim (1-r) + |x_J - x_I| \lesssim |I|.$$

Now, note that $P\varphi_J$ is a periodification of the function $\psi_+ = \mathcal{F}^{-1}(\chi_{\mathbb{R}_+}\hat{\psi})$, where $\chi_{\mathbb{R}_+}$ is the indicator function of $\mathbb{R}_+$. This gives

$$\begin{aligned}
|(h_2 P\varphi_J)(x)| &= 2^{j/2}|(1-e^{i(x-x_J)})| \left|\sum_{k\in\mathbb{Z}} \psi_+(2^j(x-x_J-2\pi k))\right| \\
&\lesssim 2^{j/2}|(x-x_J)| \left|\sum_{k\in\mathbb{Z}} \psi_+(2^j(x-x_J-2\pi k))\right| \\
&\lesssim 2^{j/2} \left|\sum_{k\in\mathbb{Z}} (x-x_J-2\pi k+2\pi k)\psi_+(2^j(x-x_J-2\pi k))\right| \\
&\leq 2^{-j/2} \left|\sum_{k\in\mathbb{Z}} 2^j(x-x_J-2\pi k)\psi_+(2^j(x-x_J-2\pi k))\right| \\
&\quad + 2^{j/2} \left|\sum_{k\in\mathbb{Z}} 2\pi k\psi_+(2^j(x-x_J-2\pi k))\right|.
\end{aligned}$$

Consequently,

$$|(hP\varphi_J)(x)| \lesssim \phi_1(x) + \phi_2(x) + \phi_3(x),$$

where

$$\begin{aligned}
\phi_1(x) &= |I||P\varphi_J(x)|, \\
\phi_2(x) &= 2^{-j/2} \left|\sum_{k\in\mathbb{Z}} 2^j(x-x_J-2\pi k)\psi_+(2^j(x-x_J-2\pi k))\right|, \\
\phi_3(x) &= 2^{j/2} \left|\sum_{k\in\mathbb{Z}} 2\pi k\psi_+(2^j(x-x_J-2\pi k))\right|.
\end{aligned}$$

Now, since ψ_+ is a Schwartz function,

$$\begin{aligned}
\|(1-\chi)\phi_1\|^2 &= \int_{2|I|<|x-x_I|<\pi} |\phi_1(x)|^2 dx \\
&\leq |I|^2 \int_{|I|<|x-x_J|<\pi} 2^j \left|\sum_{k\in\mathbb{Z}} \psi_+(2^j(x-x_J-2\pi k))\right|^2 dx \\
&\lesssim |I|^2 \int_{|I|<|x-x_J|<\pi} 2^j \sum_{k\in\mathbb{Z}} \left|\psi_+(2^j(x-x_J-2\pi k))\right| dx \\
&\leq |I|^2 \int_{|I|<|x-x_J|} 2^j \left|\psi_+(2^j(x-x_J))\right| dx \\
\left(\text{letting } 2^j(x-x_J)=u\right) &= |I|^2 \int_{2^j|I|<|u|} |\psi_+(u)| dx \\
&\lesssim |I|^2 \int_{2^j|I|}^{\infty} \frac{1}{u^3} dx \\
&\approx \frac{1}{2^{2j}} \approx |J|^2.
\end{aligned}$$

In the computation above we have used that $\left|\sum_{k\in\mathbb{Z}} \psi_+(2^j(x - x_J - 2\pi k))\right|$ is uniformly bounded for $x, x_J \in \mathbb{R}, j \in \mathbb{N}$.

Similarly $\|(1-\chi)\phi_2\|^2, \|(1-\chi)\phi_3\|^2 \lesssim |J|^2$. This finally gives

$$\begin{aligned}
\sum_{J\subseteq I} |s_J^{(3)}|^2 &\le \frac{M^2}{|I|} \sum_{J\subseteq I} \|(1-\chi)h\varphi_J\|^2 \\
&\lesssim \frac{M^2}{|I|} \sum_{J\subseteq I} \sum_{k=1}^{3} \|(1-\chi)\phi_k\|^2 \\
&\lesssim \frac{M^2}{|I|} \sum_{J\subseteq I} |J|^2 \\
&= \frac{M^2}{|I|} \sum_{n=0}^{\infty} 2^n \left(\frac{|I|}{2^n}\right)^2 = 2M^2|I|,
\end{aligned}$$

which completes the proof. □

Corollary 4.5. *Let $C \in H^2(\mathbb{D})^*$ and $\beta > 0$. Then C is discrete β-admissible for the shift operator $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ if and only if*

$$M = \sup_{w\in\mathbb{D}} (1-|w|^2)^{1/2} \left\|C(I - \bar{w}S)^{-1}\right\|_{\mathcal{A}^2_{\beta-1}(\mathbb{D})^*} < \infty.$$

Moreover M is comparable to the constant of admissibility.

Proof. Let $Cf = \langle f, c\rangle$ for $f \in H^2(\mathbb{D})$. Then

$$\begin{aligned}
\langle D^{\beta/2} h_{\bar{c}} k_w, \bar{g}\rangle &= \langle k_w D^{\beta/2} g, c\rangle \\
&= (1-|w|^2)^{1/2} C(I - \bar{w}S)^{-1} D^{\beta/2} g,
\end{aligned}$$

for analytic polynomials g. Recalling that $D^{\beta/2} : H^2(\mathbb{D}) \to \mathcal{A}^2_{\beta-1}(\mathbb{D})$ is an isomorphism,

$$\sup_{w\in\mathbb{D}} \left\|D^{\beta/2} h_{\bar{c}} k_w\right\|_{\overline{H^2(\mathbb{D})}} = \sup_{w\in\mathbb{D}} (1-|w|^2)^{1/2} \left\|C(I - \bar{w}S)^{-1}\right\|_{\mathcal{A}^2_{\beta-1}(\mathbb{D})^*}.$$

The result now follows from Proposition 4.2 and Theorem 4.4. □

4.3 Regarding the failure of (4) ⇒ (3).

For $\alpha \in \mathbb{R}$ and $w \in \mathbb{D}$, define the function

$$g_w^\alpha(z) = \frac{1}{(1-\bar{w}z)^{1+\alpha}}, \qquad z \in \mathbb{D}.$$

For positive α, g_w^α is a (non-normalized) reproducing kernel in the Bergman space $\mathcal{A}^2_{\alpha-1}(\mathbb{D})$.

Lemma 4.6. *Let $\beta > 0$ and assume that $c \in \Lambda_\beta$. Then for any $\alpha > \beta - 1/2$ there is a constant M_α such that*

$$\sup_{w\in\mathbb{D}} (1-|w|^2)^{1/2+\alpha-\beta} \|h_{\bar{c}} g_w^\alpha\|_{\overline{H^2(\mathbb{D})}} \le M_\alpha < \infty.$$

Proof. Let $w \in \mathbb{D}$. We will approximate $\|h_{\bar{c}} g_w^\alpha\|_{\overline{H^2(\mathbb{D})}}$ by $\|h_{\bar{c}} g_w^\gamma\|_{\overline{H^2(\mathbb{D})}}$ for $\gamma \in (\alpha - \beta, \alpha)$.

Since $c \in \Lambda_\beta$, Proposition 4.3 implies that the operator $h_{\bar{c}}^{\alpha-\gamma,\beta-\alpha+\gamma} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ is bounded. Using this and boundedness of $D^{-(\beta-\alpha+\gamma)} : \mathcal{A}^2_{2(\beta-\alpha+\gamma)-1}(\mathbb{D}) \to H^2(\mathbb{D})$ gives

$$\begin{aligned}
\|h_{\bar{c}} g_w^\gamma\|_{\overline{H^2(\mathbb{D})}} &\le \|D^{\alpha-\gamma} h_{\bar{c}} g_w^\gamma\|_{\overline{H^2(\mathbb{D})}} \\
&= \|D^{\alpha-\gamma} h_{\bar{c}} D^{\beta-\alpha+\gamma} D^{-(\beta-\alpha+\gamma)} g_w^\gamma\|_{\overline{H^2(\mathbb{D})}} \\
&\lesssim \|D^{-(\beta-\alpha+\gamma)} g_w^\gamma\|_{H^2(\mathbb{D})} \\
&= \|g_w^\gamma\|_{\mathcal{A}^2_{2(\beta-\alpha+\gamma)-1}(\mathbb{D})}.
\end{aligned}$$

It is well known from the theory of Bergman spaces that

$$\sup_{w\in\mathbb{D}} (1-|w|^2)^{1/2+\alpha-\beta} \|g_w^\gamma\|_{\mathcal{A}^2_{2(\beta-\alpha+\gamma)-1}(\mathbb{D})} = M_\alpha < \infty, \tag{15}$$

for some constant M_α. Hence, for a fixed $\omega \in \mathbb{D}$,

$$\|h_{\bar{c}} g_w^\gamma\|_{\overline{H^2(\mathbb{D})}} \le \frac{M_\alpha}{(1-|w|^2)^{1/2+\alpha-\beta}}.$$

In particular $\{\|h_{\bar{c}} g_w^\gamma\|_{\overline{H^2(\mathbb{D})}}\}_{\gamma\in(\alpha-\beta,\alpha)}$ is a bounded family of functions, so by reflexivity it has a weakly convergent subsequence as $\gamma \to \alpha$. Consequently, if it can be shown that

$$weak - \lim_{\gamma\to\alpha} h_{\bar{c}} g_w^\gamma = h_{\bar{c}} g_w^\alpha, \tag{16}$$

then the proof is complete, since then $\|h_{\bar{c}} g_w^\alpha\|_{\overline{H^2(\mathbb{D})}} \le \sup_{\gamma\in(\alpha-\beta,\alpha)} \|h_{\bar{c}} g_w^\gamma\|_{\overline{H^2(\mathbb{D})}}$.

The functions $h_{\bar{c}} g_w^\gamma$ and $h_{\bar{c}} g_w^\alpha$ are well defined elements of $H^2(\mathbb{D})$ so, for $z \in \mathbb{D}$,

$$\begin{aligned}
(h_{\bar{c}} g_w^\gamma)(z) &= \langle \bar{c} g_w^\gamma, k_z \rangle \\
&= \int_{\mathbb{T}} \overline{c(\zeta)} \frac{1}{(1-\bar{w}\zeta)^{1+\gamma}} \frac{1}{1-z\bar{\zeta}} dm(\zeta) \\
&\to \int_{\mathbb{T}} \overline{c(\zeta)} \frac{1}{(1-\bar{w}\zeta)^{1+\alpha}} \frac{1}{1-z\bar{\zeta}} dm(\zeta) = h_{\bar{c}} g_\omega^\alpha(z), \qquad \gamma \to \alpha,
\end{aligned}$$

where the limit is justified by dominated convergence. This implies that (16) holds. □

A consequence of Lemma 4.6 is the following theorem, which is a partial generalization the main result of [21] to $\beta \ge 1$.

Theorem 4.7. *Let $\beta > 0$. Then there exists $C_\beta \in H^2(\mathbb{D})^*$ that is not discrete time β-admissible for S, but still satisfies*

$$\sup_{w\in\mathbb{D}} (1-|w|^2)^{\frac{1-\beta}{2}+\alpha} \|C_\beta (I - \bar{w}S)^{-(1+\alpha)}\|_{H^2(\mathbb{D})^*} < \infty$$

for any $\alpha > \frac{\beta-1}{2}$.

Proof. The space $D^{-\beta/2} BMOA$ is strictly included in $\Lambda^+_{\beta/2}$. Choose $c_\beta \in \Lambda^+_{\beta/2} \setminus D^{-\beta/2} BMOA$ and define C_β by

$$C_\beta f = \langle f, c_\beta \rangle, \qquad f \in H^2(\mathbb{D}).$$

By Theorem 4.3, this operator is not β-admissible for S. However, it satisfies the resolvent condition by Lemma 4.6. □

We remark that Lemma 4.6 still holds if g_w^α is replaced by $D^\alpha k_w$. The proof is preserved, word for word, except that the standard Bergman space estimate (15) is replaced by the estimate

$$\left\|D^{\alpha-\beta}k_w\right\|_{H^2(\mathbb{D})} \lesssim \frac{1}{(1-|w|^2)^{1/2+\alpha-\beta}}.$$

This in turn follows from the estimate

$$\frac{\prod_{i=1}^n (i+\gamma)}{n!} = \frac{n^\gamma}{\Gamma(1+\gamma)}\left\{1+O\Big(\frac{1}{n}\Big)\right\}, \qquad n \in \mathbb{N}, \gamma \notin \{-1,-2,\ldots\},$$

see [24, Chap. 3]. This modified version of Lemma 4.6 immediately implies the following counterexample.

Theorem 4.8. *Let $\beta > 0$. Then there exists $c \in H^2(\mathbb{D})$ such that the generalized Hankel operator $h_{\bar{c}}^{0,\beta} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ does not satisfy the reproducing kernel thesis.*

5 Admissibility of the right-shift semigroup on $L^2(\mathbb{R}_+)$

We begin with a number of technical definitions which are required in order to characterise β-admissibility, $\beta > 0$, of the right-shift semigroup on $L^2(\mathbb{R}_+)$.

5.1 Distribution spaces and the φ-transform.

Let $\check{\phi}$ denote the inverse Fourier transform $\mathcal{F}^{-1}\phi$ and define the space

$$\mathcal{Z} = \{\phi \in \mathcal{S} : \check{\phi}^{(k)} = 0, k = 0,1,2,\ldots\}$$

with the topology inherited from the Schwartz space $\mathcal{S}$. Its topological dual $\mathcal{Z}'$ is isomorphic to $\mathcal{S}'/\mathcal{P}$ (the space of tempered distributions modulo polynomials). Much of the notation used in this section is taken from [1]. For a slightly more detailed introduction to the space $\mathcal{Z}'$ we refer to [15].

Let $\varphi \in \mathcal{S}$ be a function such that

$$\operatorname{supp} \hat{\varphi} \subseteq \{\xi \in \mathbb{R} : 1/2 \le |\xi| \le 2\}, \tag{17}$$

$$|\hat{\varphi}(\xi)| \ge c > 0 \quad \text{for } 3/5 \le |\xi| \le 5/3, \tag{18}$$

and also

$$\sum_{n=-\infty}^{\infty} \hat{\varphi}(2^{-n}\xi) = 1, \qquad \xi \in \mathbb{R}\setminus\{0\}. \tag{19}$$

For each $n \in \mathbb{Z}$ define

$$\varphi_n(x) = 2^n \varphi(2^n x), \qquad x \in \mathbb{R}.$$

From [2, Lemma 6.9] we cite the following lemma.

Lemma 5.1. *Assume that φ satisfies* (17) *and* (18). *Then there exists $\psi \in \mathcal{S}$ that also satisfies* (17) *and* (18) *and that*

$$\sum_{n\in\mathbb{Z}} \overline{\hat{\varphi}(2^n\xi)}\hat{\psi}(2^n\xi) = 1, \qquad \xi \neq 0.$$

We remark that in [1] the existence of ψ as in the above lemma was stated as a requirement in the choice of φ.

Let $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. The Besov space $\dot{B}_p^{\alpha,q}$ is defined as the set of $f \in \mathcal{Z}'$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}} := \|\{2^{n\alpha}\|\varphi_n * f\|_{L^p}\}_{n\in\mathbb{Z}}\|_{l^q(\mathbb{Z})} < \infty.$$

If $p < \infty$, define the Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ as the set of $f \in \mathcal{Z}'$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} := \|\Big(\sum_{n\in\mathbb{Z}} (2^{n\alpha}|\varphi_n * f|)^q\Big)^{1/q}\|_{L^p} < \infty.$$

The exceptional space $\dot{F}_\infty^{\alpha,q}$ is defined as the set of $f \in \mathcal{Z}'$ such that

$$\|f\|_{\dot{F}_\infty^{\alpha,q}} := \sup \Big(\frac{1}{|P|}\int_P \sum_{n=-\log_2|P|}^{\infty} (2^{n\alpha}|\varphi_n * f|)^q\Big)^{1/q} < \infty,$$

where the supremum is taken over all dyadic intervals $P = [k2^{-j}, (k+1)2^{-j})$. The last definition is the one given in [1] where it is also proved to be equivalent to the definition given in [15]. The definitions can be proved to be independent of the choice of φ, [15, p. 240].

Given a function $g : \mathbb{R} \to \mathbb{C}$ we define the multiplication operator $M_{g(x)} : \mathcal{Z}' \to \mathcal{Z}'$ by

$$(M_{g(x)}f)(x) = g(x)f(x), \qquad f \in \mathcal{Z}',$$

provided that this is well defined. For $\alpha \in \mathbb{R}$, define the inverse Riesz potential $D^\alpha : \mathcal{Z}' \to \mathcal{Z}'$ by

$$D^\alpha f = \mathcal{F}(M_{|\xi|^\alpha}\check{f}).$$

From [15, p. 242, Theorem 1; p. 244, Theorem] we collect the following results.

Proposition 5.2. *Let $\alpha, \beta \in \mathbb{R}$ and $0 < p, q \le \infty$. Then $D^\beta : \dot{B}_p^{\alpha,q} \to \dot{B}_p^{\alpha-\beta,q}$ is a surjective isomorphism. If $p < \infty$, then $D^\beta : \dot{F}_p^{\alpha,q} \to \dot{F}_p^{\alpha-\beta,q}$ is a surjective isomorphism. Moreover we have the special cases $\dot{F}_2^{0,2} = L^2$ and $\dot{F}_\infty^{0,2} = BMO(\mathbb{R})$.*

We remark that by the duality identity $(F_1^{\alpha,q})^* = F_\infty^{-\alpha,q'}$ (see [1, Equation (5.2)]) valid for $q \in [1,\infty)$ the conclusion of Proposition 5.2 holds also for the spaces $\dot{F}_\infty^{\alpha,q}$ whenever $q \in (1,\infty]$. This shows in particular that $D^{-\alpha}BMO(\mathbb{R}) = \dot{F}_\infty^{\alpha,2}$.

Given $\phi \in \mathcal{Z}$, define $P_+\phi = \mathcal{F}(\chi_{\mathbb{R}_+}\check{\phi})$ where $\chi_{\mathbb{R}_+}$ denotes the indicator function of the positive real numbers. Also define $P_+ : \mathcal{Z}' \to \mathcal{Z}'$ by

$$\langle \phi, P_+ f\rangle = \langle P_+\phi, f\rangle, \qquad \phi \in \mathcal{Z}, f \in \mathcal{Z}',$$

and $P_- = I - P_+$. Test functions and distributions belonging to $\mathcal{Z}_+ = P_+\mathcal{Z}$ and $\mathcal{Z}'_+ = P_+\mathcal{Z}'$ respectively will be referred to as analytic. The main reason for this is that

$$P_+L^2 = H^2(\mathbb{C}_+),$$

the Hardy space of the right half plane.

Let $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. We will work with sequences $s = \{s_Q\}_Q$ indexed by the set of dyadic intervals on $\mathbb{R}$. If $p < \infty$, define the space $\dot{f}_p^{\alpha,q}$ of sequences such that

$$\|s\|_{\dot{f}_p^{\alpha,q}} := \left\|\Big(\sum_Q (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\Big)^{1/q}\right\|_{L^p} < \infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized indicator function of Q. In the special case $p = q = 2$ the integral in the above norm is easily computed and we have that

$$\|s\|_{\dot{f}_2^{\alpha,2}} = \Big(\sum_Q |Q|^{-2\alpha}|s_Q|^2\Big)^{1/2}.$$

The space $\dot{f}_\infty^{\alpha,q}$ is defined by the norm

$$\|s\|_{\dot{f}_\infty^{\alpha,q}} := \sup_{P \text{ dyadic}} \Big(\frac{1}{|P|}\int_P \sum_{Q\subseteq P} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\Big)^{1/q}.$$

In particular we have that

$$\|s\|_{\dot{f}_\infty^{\alpha,2}} = \sup_{P \text{ dyadic}} \Big(\frac{1}{|P|}\sum_{Q\subseteq P} |Q|^{-2\alpha}|s_Q|^2\Big)^{1/2}.$$

For each dyadic interval Q, define the dilation translation

$$\varphi_Q(x) = |Q|^{-1/2}\varphi\left(\frac{x - x_Q}{|Q|}\right), \qquad x \in \mathbb{R}, \tag{20}$$

where x_Q is the left endpoint of Q. The sequence $\{\psi_Q\}_Q$ is defined similarly. We define the φ-transform of a distribution $f \in \mathcal{Z}'$ by

$$S_\varphi f = \{\langle f, \varphi_Q\rangle\}_Q.$$

The inverse φ-transform of a sequence $s = \{s_Q\}_Q$ is defined by

$$T_\psi s = \sum_Q s_Q \psi_Q,$$

where ψ is as in Lemma 5.1. The importance of the φ-transform is the following ([1, Theorem 2.2]).

Proposition 5.3. *Let $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. The operators $S_\varphi : \dot{F}_p^{\alpha,q} \to \dot{f}_p^{\alpha,q}$ and $T_\psi : \dot{f}_p^{\alpha,q} \to \dot{F}_p^{\alpha,q}$ are bounded. Moreover $T_\psi \circ S_\phi$ is the identity on $\dot{F}_p^{\alpha,q}$. In particular $\|f\|_{\dot{F}_p^{\alpha,q}} \approx \|S_\phi f\|_{\dot{f}_p^{\alpha,q}}$.*

5.2 Continuous time admissibility and Hankel operators.

Let $c \in \mathcal{Z}'_-$. Define the Hankel type operator $H_c : \mathcal{Z}_+ \to \mathcal{Z}'_-$ by

$$H_c f = P_-(cf), \qquad f \in \mathcal{Z}_+.$$

First we observe that the operator H_c can be defined in a natural way on a larger class of functions that just $\mathcal{Z}_+$. Let $f \in \mathcal{Z}'_+$. Formally,

$$\begin{aligned}\langle \phi, H_c f\rangle &= \langle \phi, cf\rangle \\ &= \langle \check{\phi}, \check{c} * \check{f}\rangle \\ &= \langle\langle \check{\phi}(\xi+\eta), \check{f}(\eta)\rangle, \check{c}(\xi)\rangle, \qquad \phi \in \mathcal{Z}_-.\end{aligned}$$

The last expression is well defined provided that $\langle \check{\phi}(\xi+\eta), \check{f}(\eta)\rangle$ is a Schwartz function that vanishes on $\mathbb{R}_+$. This is the case if, for example, $f = K_\lambda$ where

$$K_\lambda(x) = \frac{1}{\bar{\lambda} + ix}, \qquad x \in \mathbb{R}$$

for some $\lambda \in \mathbb{C}_+$, since then

$$\check{K}_\lambda(\xi) = \begin{cases} e^{-\xi\bar{\lambda}} & \text{if } \xi \geq 0, \\ 0 & \text{if } \xi < 0. \end{cases}$$

If K_λ is thought of as a function of ix rather than of x, then K_λ is a reproducing kernel of $H^2(\mathbb{C}_+)$ with respect to λ. We therefore say that the operator H_c is defined on the set of reproducing kernels. For future convenience we also define $K_\lambda^\gamma = D^\gamma K_\lambda$ whenever $\gamma \in \mathbb{R}$. If $\gamma > 0$ then K_λ^γ (multiplied with a constant) is a reproducing kernel for the space $\mathcal{A}^2_{\gamma-1}(\mathbb{C}_+)$. As long as $\gamma > -1$ we have that the map $H_c K_\lambda^\gamma : \mathcal{Z}_- \to \mathbb{C}$ is well defined.

Let $(S(t))_{t\geq 0}$ be the right shift semigroup on $L^2(\mathbb{R}_+)$. Its infinitesimal generator A is given by

$$A = -\frac{d}{dt}, \qquad D(A) = W_0^{1,2}(\mathbb{R}_+) = \left\{ f \in L^2(\mathbb{R}_+) : f' \in L^2(\mathbb{R}_+), f(0) = 0 \right\},$$

see [16, Example 2.4.5] for details. If $C \in D(A)^*$ then $C(I-A)^{-1} \in L^2(\mathbb{R}_+)^*$ and so there is a unique $c_0 \in L^2(\mathbb{R}_+)$ such that

$$C(I-A)^{-1} f = \langle f, c_0 \rangle, \qquad f \in L^2(\mathbb{R}_+). \tag{21}$$

Conversely this equation generates an A-bounded linear functional for any $c_0 \in L^2(\mathbb{R}_+)$.

In order to compute fractional power resolvents, the following lemma ([4, Proposition 3.3.5]) is useful.

Lemma 5.4. *Let B be the generator of the contractive C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then for any $\lambda \in \mathbb{C}_+$ and $\beta > 0$*

$$(\lambda I - B)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} T(t) e^{-t\lambda} dt.$$

A simple consequence is the following.

Lemma 5.5. *Let $f \in \mathcal{Z}_+$, $\lambda \in \mathbb{C}_+$, $\beta > 0$ and A be the infinitesimal generator of the right-shift semigroup on $L^2(\mathbb{R}_+)$. Then*

$$\mathcal{F}((\bar{\lambda} I - A)^{-(1+\beta)} \check{f}) = f K_\lambda^\beta.$$

The following proposition relates reproducing kernels and generalised Hankel operators to the weighted Weiss conjecture.

Proposition 5.6. *Let A be the infinitesimal generator of the right shift semigroup, $C \in D(A)^*$ and $\beta \geq 0$. If $c_0 \in L^2(\mathbb{R}_+)$ is related to C through (21) and $c \in \mathcal{Z}_-'$ is given by*

$$c(\xi) = (1 + i\xi)(\mathcal{F}\tilde{c}_0)(\xi), \qquad \xi \in \mathbb{R},$$

where $\tilde{c}_0(s) = \overline{c_0(-s)}$. Then:

(i) Whenever $f \in \mathcal{Z}_+$,

$$\int_0^\infty t^{2\beta} \left| C S(t) \check{f} \right|^2 dt = \left\| D^\beta H_c f \right\|^2_{L^2(\mathbb{R})}.$$

In particular, since $\mathcal{F}^{-1}\mathcal{Z}_+$ is dense in $D(A)$, C is 2β-admissible for $(S(t))_{t\geq 0}$ if and only if $D^\beta H_c : \mathcal{Z}_+ \to L^2(\mathbb{R})$ extends to a bounded linear operator from $\mathcal{F}L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R})$.

(ii)

$$\left\| C(\bar{\lambda} I - A)^{-1} M_{|x|^\beta} \right\|_{L^2(\mathbb{R}_+, dx)^*} = \left\| D^\beta H_c K_\lambda \right\|_{L^2(\mathbb{R})}.$$

In particular the resolvent estimate

$$\sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2} \left\| C(\bar{\lambda} I - A)^{-1} M_{|x|^\beta} \right\|_{L^2(\mathbb{R}_+, dx)^*} < \infty$$

holds if and only if $D^\beta H_c : \mathcal{Z}_+ \to L^2(\mathbb{R})$ is bounded on reproducing kernels.

(iii)

$$\left\|C(\bar{\lambda}I - A)^{-(1+\beta)}\right\|_{L^2(\mathbb{R}_+,dx)^*} = \left\|H_c K_\lambda^\beta\right\|_{L^2(\mathbb{R})}.$$

In particular the resolvent estimate

$$\sup_{\lambda\in\mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2}\left\|C(\bar{\lambda}I - A)^{-(1+\beta)}\right\|_{L^2(\mathbb{R}_+,dx)^*} < \infty$$

holds if and only if $H_c D^\beta : \mathcal{Z}_+ \to L^2(\mathbb{R})$ is bounded on reproducing kernels.

Proof. (i) Take $f \in \mathcal{Z}_+$ and consider the function

$$g_f : t \mapsto \begin{cases} CS(-t)\check{f} & \text{if } t \le 0, \\ 0 & \text{if } t > 0. \end{cases}$$

Then, for $t \le 0$,

$$\begin{aligned} g_f(t) &= C(I - A)^{-1}(I - A)S(-t)\check{f} \\ &= \langle (I - A)S(-t)\check{f}, c_0\rangle \\ &= \int_{s=-t} (\check{f} + (\check{f})')(s + t)\overline{c_0(s)}ds \\ &= (\check{f} + (\check{f})') * \tilde{c}_0(t). \end{aligned}$$

Hence,

$$\begin{aligned} \int_{t=0}^{\infty} t^{2\beta}|CS(t)\check{f}|^2 dt &= \int_{t=-\infty}^{0} |t|^{2\beta}\left|CS(-t)\check{f}\right|^2 dt \\ &= \left\|M_{|t|^\beta}\chi_{\mathbb{R}_-}((\check{f} + (\check{f})') * \tilde{c}_0)\right\|_{L^2}^2 \\ &= \left\|D^\beta P_-((1 + i\xi)f\hat{\tilde{c}}_0)\right\|_{L^2} \\ &= \left\|D^\beta H_c f\right\|_{L^2}. \end{aligned}$$

(ii) Using Lemma 5.5 together with the elementary identity $\bar{f} = \mathcal{F}\tilde{\check{f}}$ we obtain,

$$\begin{aligned} C(\bar{\lambda}I - A)^{-1}M_{|x|^\beta}\check{f} &= C(I - A)^{-1}(I - A)(\bar{\lambda}I - A)^{-1}M_{|x|^\beta}\check{f} \\ &= \left\langle (I - A)(\bar{\lambda}I - A)^{-1}M_{|x|^\beta}\check{f}, c_0\right\rangle \\ &= \left\langle (1 + i\xi)K_\lambda D^\beta f, \hat{c}_0\right\rangle \\ &= \left\langle (1 + i\xi)K_\lambda \hat{\tilde{c}}_0, D^\beta \bar{f}\right\rangle \\ &= \left\langle D^\beta H_c K_\lambda, \hat{\tilde{\bar{f}}}\right\rangle. \end{aligned}$$

The result follows by taking the supremum over all $f \in \mathcal{Z}_+$ of unit length.

(iii) This is similar to the proof of (ii). □

In the paper [8] the authors characterize boundedness of the operators $D^\alpha H_c D^\beta$ for $\alpha, \beta \ge 0$. In particular we need [8, Theorem 5.1] and [8, Theorem 5.3] (together with a few technical comments from the examples).

Proposition 5.7. *For $c \in \mathcal{Z}'_-$ we have that:*

(*i*) *Let* $\beta \geq 0$. *Then both* $D^\beta H_c : \mathcal{Z}_+ \to L^2(\mathbb{R})$ *and* $H_c D^\beta : \mathcal{Z}_+ \to L^2(\mathbb{R})$ *extend to bounded linear operators from* $\mathcal{F}L^2(\mathbb{R}_+)$ *to* $L^2(\mathbb{R})$ *if and only if* $D^\beta c \in BMO(\mathbb{R})$, *with* $\|D^\beta H_c\|_{\mathcal{F}L^2(\mathbb{R}_+)\to L^2(\mathbb{R})}$ *and* $\|H_c D^\beta\|_{\mathcal{F}L^2(\mathbb{R}_+)\to L^2(\mathbb{R})}$ *comparable to* $\|D^\beta c\|_{BMO(\mathbb{R})}$.

(*ii*) *Let* $\alpha, \beta > 0$. *Then* $D^\alpha H_c D^\beta : \mathcal{Z}_+ \to L^2(\mathbb{R})$ *extends to a bounded linear operator from* $\mathcal{F}L^2(\mathbb{R}_+)$ *to* $L^2(\mathbb{R})$ *if and only if* $c \in \dot{B}^{\alpha+\beta}_\infty$, *with* $\|D^\alpha H_c D^\beta\|_{\mathcal{F}L^2(\mathbb{R}_+)\to L^2(\mathbb{R})}$ *comparable to* $\|c\|_{\dot{B}^{\alpha+\beta}_\infty}$.

5.3 The Reproducing Kernel Thesis for $D^\alpha H_c$.

The main result of this section is the following.

Theorem 5.8. *Let* $c \in \mathcal{Z}'_-$ *and* $\beta \geq 0$. *The following are equivalent:*

(*i*) *The operator* $D^\beta H_c : \mathcal{Z}_+ \to L^2(\mathbb{R})$ *extends to a bounded operator on* $H^2(\mathbb{C}_+)$;
(*ii*) *The operator* $D^\beta H_c : \mathcal{Z}_+ \to L^2(\mathbb{R})$ *is bounded on reproducing kernels, i.e.*

$$M = \sup_{\lambda\in\mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2} \left\| D^\beta H_c K_\lambda \right\|_{L^2(\mathbb{R})} < \infty.$$

Moreover $\|D^\beta H_c\|_{H^2(\mathbb{C}_+)\to L^2(\mathbb{R})} \approx M$.

Proof. $(i) \Rightarrow (ii)$ is trivial. By Proposition 5.7 the converse statement follows if $\|D^\beta c\|_{BMO(\mathbb{R})} \lesssim M$. By Proposition 5.2 we need to show that $c \in \dot{F}^{\beta;2}_\infty$ which by Proposition 5.3 is equivalent to that $S_\varphi c \in \dot{f}^{\beta;2}_\infty$. Define the sequence $s_Q = \langle c, \varphi_Q\rangle$. It is sufficient to prove that

$$\sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \sum_{Q\subseteq P} |Q|^{-2\beta} |s_Q|^2 \right)^{1/2} \lesssim M.$$

To this end, consider a fixed dyadic interval P. Since $c \in \mathcal{Z}'_-$,

$$s_Q = \langle cK_\lambda, \bar{h}P_-\varphi_Q\rangle = \langle H_c K_\lambda, \bar{h}P_-\varphi_Q\rangle = \langle g, D^{-\beta}(\bar{h}P_-\varphi_Q)\rangle,$$

where $\lambda = |P| + ix_P$, $g = D^\beta H_c K_\lambda$ and $h(x) = \bar{\lambda} + ix$. In the above calculation we have used that $\bar{h}P_-\varphi_Q \in \mathcal{Z}_-$. Note that

$$\|g\|^2_{L^2(\mathbb{R})} \leq \frac{M^2}{|P|}.$$

Let $\chi \in \mathcal{S}$ be a smooth cutoff such that $\chi(x) = 1$ when $|x - x_P| \leq 2|P|$, $0 < \chi(x) < 1$ when $2|P| < |x - x_P| < 3|P|$ and $\chi(x) = 0$ when $|x - x_P| \geq 3|P|$. We have

$$\begin{aligned} s_Q = \left\langle g, D^{-\beta}(\bar{h}P_-\varphi_Q)\right\rangle &= \left\langle \chi g + (1-\chi)g, D^{-\beta}(\bar{h}P_-\varphi_Q)\right\rangle \\ &= \left\langle hD^{-\beta}(\chi g), P_-\varphi_Q\right\rangle + \left\langle g, (1-\chi)D^{-\beta}(hP_-\varphi_Q)\right\rangle. \end{aligned}$$

A calculation shows that

$$\begin{aligned} hD^{-\beta}(\chi g) &= \mathcal{F}\left(\left[\bar{\lambda} + \frac{d}{d\xi}\right] M_{|\xi|^{-\beta}}(\check{\chi} * \check{g}) \right) \\ &= \mathcal{F}\left(\bar{\lambda} M_{|\xi|^{-\beta}}(\check{\chi} * \check{g})\right) + \mathcal{F}\left(M_{|\xi|^{-\beta}} \frac{d}{d\xi}(\check{\chi} * \check{g})\right) - \mathcal{F}\left(\beta M_{\mathrm{sgn}(\xi)|\xi|^{-(1+\beta)}}(\check{\chi} * \check{g})\right) \\ &= D^{-\beta}(h\chi g) - \beta D^{-(1+\beta)} \mathcal{F}\left(M_{\mathrm{sgn}(\xi)}(\check{\chi} * \check{g})\right). \end{aligned}$$

This implies that $s_Q = s_Q^{(1)} + s_Q^{(2)} + s_Q^{(3)}$, where

$$\begin{aligned}
s_Q^{(1)} &= \left\langle D^{-\beta}\left(h\chi g\right), P_-\varphi_Q\right\rangle,\\
s_Q^{(2)} &= -\beta\left\langle D^{-(1+\beta)}\mathcal{F}\left(M_{\mathrm{sgn}(\xi)}(\check{\chi} * \check{g})\right), P_-\varphi_Q\right\rangle,\\
s_Q^{(3)} &= \left\langle g, (1-\chi)D^{-\beta}\left(\bar{h}P_-\varphi_Q\right)\right\rangle.
\end{aligned}$$

The proof is completed by showing that

$$\sum_{Q\subseteq P}|Q|^{-2\beta}|s_Q^{(j)}|^2 \lesssim M^2|P|, \qquad j = 1,2,3.$$

First, using Propositions 5.2 and 5.3,

$$\begin{aligned}
\sum_{Q\subseteq P}|Q|^{-2\beta}\left|s_Q^{(1)}\right|^2 &= \sum_{Q\subseteq P}|Q|^{-2\beta}\left|\left\langle D^{-\beta}P_-\left(h\chi g\right), \varphi_Q\right\rangle\right|^2\\
&\leq \sum_{Q}|Q|^{-2\beta}\left|\left\langle D^{-\beta}P_-\left(h\chi g\right), \varphi_Q\right\rangle\right|^2\\
&= \left\|S_\varphi\left(D^{-\beta}P_-\left(h\chi g\right)\right)\right\|_{\dot{f}_2^{\beta,2}}\\
&\lesssim \left\|D^{-\beta}P_-\left(h\chi g\right)\right\|_{\dot{F}_2^{\beta,2}}\\
&\approx \left\|P_-\left(h\chi g\right)\right\|_{\dot{F}_2^{0,2}}\\
&\approx \left\|P_-\left(h\chi g\right)\right\|_{L^2}\\
&\leq \int_{|x-x_P|<3|P|}|\bar{\lambda}+ix|^2|g(x)|^2dx \lesssim |P|M^2.
\end{aligned}$$

The second part is estimated using Hölders inequality on each term.

$$\begin{aligned}
\left|s_Q^{(2)}\right|^2 &= \beta^2\left|\left\langle D^{-(1+\beta)}\mathcal{F}\left(M_{\mathrm{sgn}(\xi)}\left(\check{\chi} * \check{g}\right)\right), P_-\varphi_Q\right\rangle\right|^2\\
&= \beta^2\left|\left\langle M_{\mathrm{sgn}(\xi)|\xi|^{-(1+\beta)}}\left(\check{\chi} * \check{g}\right), \chi_{\mathbb{R}_-}\check{\varphi}_Q\right\rangle\right|^2\\
&\leq \beta^2\left(\int_{-\frac{2}{|Q|}}^{-\frac{1}{2|Q|}}|\xi|^{-(1+\beta)}\left|\left(\check{\chi} * \check{g}\right)(\xi)\right|\left|\check{\varphi}_Q(\xi)\right|d\xi\right)^2\\
&\lesssim |Q|^{2+2\beta}\left\|\check{\chi} * \check{g}\right\|^2_{L^2(\mathbb{R})}\\
&\leq |Q|^{2+2\beta}\|g\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{|Q|^{2+2\beta}M^2}{|P|}.
\end{aligned}$$

In the calculation we have used that $\{\varphi_Q\}_Q$ is an L^2-normalized sequence. Summing up,

$$\sum_{Q\subseteq P}|Q|^{-2\beta}\left|s_Q^{(2)}\right|^2 \leq \sum_{Q\subseteq P}\frac{|Q|^2M^2}{|P|} = \sum_{n=0}^{\infty}2^n\left(\frac{|P|}{2^n}\right)^2\frac{M^2}{|P|} = 2M^2|P|.$$

Finally, apply Hölders inequality to each term in the third and final sum to give

$$\left|s_Q^{(3)}\right|^2 \leq \frac{M^2}{|P|}\left\|\left(1-\chi\right)D^{-\beta}\left(\bar{h}P_-\varphi_Q\right)\right\|^2_{L^2}. \tag{22}$$

Before summing up we need to approximate $\left\|\left(1-\chi\right)D^{-\beta}\left(\bar{h}P_-\varphi_Q\right)\right\|^2_{L^2}$. It is elementary to show that

$$\check{\varphi}_Q(\xi) = |Q|^{1/2}e^{ix_Q\xi}\check{\varphi}(|Q|\xi), \qquad \xi\in\mathbb{R}.$$

This gives that for $x \in \mathbb{R}$,

$$\begin{aligned} D^{-\beta}\left(\bar{h}P_{-}\varphi_Q\right)(x) &= \mathcal{F}\left(M_{|\xi|^{-\beta}}\left[\lambda - \frac{d}{d\xi}\right]\left(\chi_{\mathbb{R}_-}\check{\varphi}_Q\right)\right)(x) \\ &= \int_{-\infty}^{0} |\xi|^{-\beta}|Q|^{1/2}e^{ix_Q\xi}\Big[\left(\lambda - ix_Q\right)\check{\varphi}\left(|Q|\xi\right) - |Q|\left(\check{\varphi}\right)'\left(|Q|\xi\right)\Big]e^{-i\xi x}d\xi \\ (\text{letting } u = |Q|\xi) &= |Q|^{\beta-1/2}\int_{-\infty}^{0}|u|^{-\beta}\Big[\left(\lambda - ix_Q\right)\check{\varphi}(u) + |Q|\left(\check{\varphi}\right)'(u)\Big]e^{-iu\left(\frac{x-x_Q}{|Q|}\right)}du \\ &= |Q|^{\beta-1/2}\Big[\left(\lambda - ix_Q\right)D^{-\beta}P_{-}\varphi + |Q|D^{-\beta}\left(M_{ix}P_{-}\varphi\right)\Big]\left(\frac{x-x_Q}{|Q|}\right). \end{aligned}$$

Assuming $Q \subseteq P$ we then obtain

$$\left|D^{-\beta}\left(\bar{h}P_{-}\varphi_Q\right)(x)\right|^2 \lesssim |P|^2|Q|^{2\beta-1}\left|\phi\left(\frac{x-x_Q}{|Q|}\right)\right|, \qquad x \in \mathbb{R}, \tag{23}$$

where

$$\phi = \left|D^{-\beta}P_{-}\varphi\right|^2 + \left|D^{-\beta}(M_{ix}P_{-}\varphi)\right|^2.$$

Using (23),

$$\begin{aligned} \left\|(1-\chi)\,D^{-\beta}\left(\bar{h}P_{-}\varphi_Q\right)\right\|_{L^2}^2 &\le \int_{|x-x_P|>2|P|}\left|D^{-\beta}(\bar{h}P_{-}\varphi_Q)(x)\right|^2dx \\ &\le \int_{|x-x_Q|>|P|}|P|^2|Q|^{2\beta-1}\left|\phi\Big(\frac{x-x_Q}{|Q|}\Big)\right|dx \\ \left(\text{letting } u = \frac{x-x_Q}{|Q|}\right) &= \int_{|u|>\frac{|P|}{|Q|}}|P|^2|Q|^{2\beta}|\phi(u)|du. \end{aligned}$$

Since ϕ decays like a Schwartz function,

$$|\phi(x)| \lesssim \frac{1}{|x|^3}, \qquad x \in \mathbb{R},$$

and hence,

$$\begin{aligned} \left\|(1-\chi)\,D^{-\beta}\left(\bar{h}\varphi_Q\right)\right\|_{L^2}^2 &\lesssim \int_{|x|>\frac{|P|}{|Q|}}|P|^2|Q|^{2\beta}\frac{1}{|x|^3}dx \\ &\lesssim |Q|^{2+2\beta}. \end{aligned}$$

Combining the above inequality with (22) gives

$$|s_Q^{(3)}|^2 \lesssim \frac{M^2}{|P|}|Q|^{2+2\beta}.$$

Summing up,

$$\sum_{Q\subseteq P}|Q|^{-2\beta}|s_Q^{(3)}|^2 \lesssim \sum_{Q\subseteq P}\frac{M^2}{|P|}|Q|^2 = \sum_{n=0}^{\infty}\frac{M^2}{|P|}2^n\left(\frac{|P|}{2^n}\right)^2 = 2M^2|P|. \qquad \square$$

Corollary 5.9. *Let A denote the infinitesimal generator of the right shift semigroup $(S(t))_{t\ge0}$ and $\beta \ge 0$. Then $C \in D(A)^*$ is 2β-admissible for $(S(t))_{t\ge0}$ if and only if*

$$M = \sup_{\lambda\in\mathbb{C}_+}(\mathrm{Re}\lambda)^{1/2}\left\|C(\bar{\lambda}I - A)^{-1}M_{|x|^\beta}\right\|_{L^2(\mathbb{R}_+,dx)^*} < \infty.$$

Moreover the constant M is comparable to the constant of admissibility.

Proof. This follows immediately from Theorem 5.8 and Proposition 5.6. $\square$

5.4 Regarding the failure of (2) $\Rightarrow$ (1).

We have the following analogue of Lemma 4.6.

Lemma 5.10. *Let $\beta > 0$ and $c \in \mathcal{Z}'_-$. Then the following statements are true:*

(i) *If for some $\alpha > \beta - 1/2$ there exists a constant M_α such that*

$$M_\alpha = \sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2+\alpha-\beta} \left\| H_c K_\lambda^\alpha \right\|_{H^2(\mathbb{C}_+)} < \infty$$

then $c \in \dot{B}^\beta_\infty$.

(ii) *If $\alpha > \max\{\beta - 1/2, 0\}$, $c \in \dot{B}^\beta_\infty$ and moreover*

$$\frac{c(x)}{1+ix} \in L^2(\mathbb{R}), \tag{24}$$

then there exists a constant M_α such that

$$M_\alpha = \sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2+\alpha-\beta} \left\| H_c K_\lambda^\alpha \right\|_{H^2(\mathbb{C}_+)} < \infty.$$

Proof. In order to prove the first statement let $\{\varphi_n\}_{n\in\mathbb{Z}}$ be as in section 5.1. We need to show that

$$|\langle g, \varphi_n * c\rangle| \lesssim 2^{-n\beta} \|g\|_{L^1}, \qquad n \in \mathbb{Z}, g \in \mathcal{Z}.$$

Since $c \in \mathcal{Z}'_-$ we may replace φ_n with $P_-\varphi_n$.

Consider fixed $n \in \mathbb{Z}$, $g \in \mathcal{Z}$. By definition of the convolution,

$$\langle g, \varphi_n * c\rangle = \langle \phi, c\rangle,$$

where $\phi : x \mapsto \langle g(x+y), \varphi_n(y)\rangle$.

Introduce the functions $g_k = g\chi_{[k2^{-n},(k+1)2^{-n})}$ and $\phi_k : x \mapsto \langle g_k(x+y), \varphi_n(y)\rangle$, where $\chi_{[k2^{-n},(k+1)2^{-n})}$ denotes the characteristic function of $[k2^{-n}, (k+1)2^{-n})$. A quick calculation shows that $\hat{\phi}_k = \hat{g}_k \overline{\hat{\varphi}_n}$ so that $\phi_k \in \mathcal{Z}_-$. It is also easy to show that $\phi = \sum_{k\in\mathbb{Z}} \phi_k$ with convergence in the Schwartz topology.

Choose the sequence $\lambda_k = 2^{-n} + ik2^{-n}$, $k \in \mathbb{Z}$. Then

$$\begin{aligned}
\langle g, \varphi_n * c\rangle &= \sum_{k\in\mathbb{Z}} \langle \phi_k, c\rangle \\
&= \sum_{k\in\mathbb{Z}} \left\langle \phi_k(x)(\lambda_k - ix)^{1+\alpha}, c(x) D^\alpha K_{\lambda_k}(x)\right\rangle \\
&= \sum_{k\in\mathbb{Z}} \left\langle \phi_k(x)(\lambda_k - ix)^{1+\alpha}, (H_c D^\alpha K_{\lambda_k})(x)\right\rangle.
\end{aligned}$$

This gives

$$\begin{aligned}
|\langle g, \varphi_n * c\rangle| &\le \sum_{k\in\mathbb{Z}} \left|\left\langle \phi_k(x)(\lambda_k - ix)^{1+\alpha}, (H_c D^\alpha K_{\lambda_k})(x)\right\rangle\right| \\
&\le \sum_{k\in\mathbb{Z}} \left\|\phi_k(x)(\lambda_k - ix)^{1+\alpha}\right\|_{L^2} \left\|(H_c D^\alpha K_{\lambda_k})(x)\right\|_{L^2} \\
&\le \frac{M_\alpha}{2^{-n(1/2+\alpha-\beta)}} \sum_{k\in\mathbb{Z}} \left\|\phi_k(x)(\lambda_k - ix)^{1+\alpha}\right\|_{L^2}.
\end{aligned}$$

We come down to approximating $\|\phi_k(x)(\lambda_k - ix)^{1+\alpha}\|_{L^2}$. Applying Minkowski's inequality in the first step gives

$$\begin{aligned}\left\|\phi_k(x)(\lambda_k - ix)^{1+\alpha}\right\|_{L^2} &\le \int |g_k(y)|\left\|\varphi_n(y-x)(\lambda_k - ix)^{1+\alpha}\right\|_{L^2(dx)}dy \\ &= \|g_k\|_{L^1} \sup_{y\in[k2^{-n},(k+1)2^{-n})} \left\|\varphi_n(x)(\lambda_k - i(y-x))^{1+\alpha}\right\|_{L^2(dx)}.\end{aligned}$$

By a change of variables, it follows that for each $y \in [k2^{-n}, (k+1)2^{-n})$,

$$\begin{aligned}\left\|\varphi_n(x)(\bar\lambda_k - i(y-x))^{1+\alpha}\right\|^2_{L^2(dx)} &\lesssim 2^n\Big(|2^{-n} + i(y-k2^{-n})|^{2+2\alpha}\int|\varphi_0(x)|^2dx \\ &\quad + 2^{-2n(1+\alpha)}\int|x|^{2+2\alpha}|\varphi_0(x)|^2dx\Big) \\ &\lesssim 2^{-n(1+2\alpha)}.\end{aligned}$$

Consequently,

$$\begin{aligned}|\langle g, \varphi_n * c\rangle| &\le \frac{M_\alpha}{2^{-n(1/2+\alpha-\beta)}}\sum_{k\in\mathbb{Z}}\left\|\phi_k(x)(\lambda_k + ix)^{1+\alpha}\right\|_{L^2} \\ &\lesssim \frac{M_\alpha}{2^{n\beta}}\sum_{k\in\mathbb{Z}}\|g_k\|_{L^1} = M_\alpha 2^{-n\beta}\|g\|_{L^1},\end{aligned}$$

and hence, $c \in \dot{B}^\beta_\infty$.

To prove the second statement, fix $\lambda \in \mathbb{C}_+$, let $\gamma \in (\max\{\alpha-\beta, 0\}, \alpha)$ and consider the functions $H_cK^\gamma_\lambda$. Note that

$$\begin{aligned}\left\|H_cK^\gamma_\lambda\right\| &\le \left\|\chi_{(-1,0)}M_{|\xi|^{\alpha-\gamma}}(\check{c} * \check{K}^\gamma_\lambda)\right\| + \left\|\chi_{(-1,0)}M_{1-|\xi|^{\alpha-\gamma}}(\check{c} * \check{K}^\gamma_\lambda)\right\| + \left\|\chi_{(-\infty,-1)}M_{|\xi|^{\alpha-\gamma}}(\check{c} * \check{K}^\gamma_\lambda)\right\| \\ &\lesssim \left\|\chi_{(-1,0)}M_{|\xi|^{\alpha-\gamma}}(\check{c} * \check{K}^\gamma_\lambda)\right\| + \left\|D^{\alpha-\gamma}H_cD^{\beta-\alpha+\gamma}K^{\alpha-\beta}_\lambda\right\|.\end{aligned}$$

By the assumption (24), $H_cK^\gamma_\lambda \to H_cK^\alpha_\lambda$ with convergence in $L^2(\mathbb{R})$ as $\gamma \to \alpha$. Hence, it is easy to see that $\|\chi_{(-1,0)}M_{1-|\xi|^{\alpha-\gamma}}(\check{c} * \check{K}^\gamma_\lambda)\| \to 0$ as $\gamma \to \alpha$. Analogous to the proof of Lemma 4.6, it follows that

$$\left\|D^{\alpha-\gamma}H_cD^{\beta-\alpha+\gamma}K^{\alpha-\beta}_\lambda\right\| \lesssim \left\|K^{\alpha-\beta}_\lambda\right\| \approx (\mathrm{Re}\lambda)^{-1/2+\beta-\alpha},$$

which completes the proof. □

Lemma 5.10 will provide a counterexample to the weighted Weiss conjecture once it has been proven that a certain set of operator symbols is nonempty.

Lemma 5.11. *Let $\beta > 0$. Then there exists $c \in \mathcal{Z}'_-$ with the following properties:*

(i)

$$\frac{c(x)}{1+ix} \in L^2(\mathbb{R}).$$

(ii)

$$D^\beta c \notin BMO(\mathbb{R}).$$

(iii)

$$D^\beta c \in \dot{B}^0_\infty.$$

Proof. We construct c explicitly. Let $f_0 \in \mathcal{S}$ be a smooth function with $\operatorname{supp}(\check{f}_0) \subset [-1, -1/2]$ and let $f = M_{1+ix} f_0$. Define $c \in \mathcal{Z}'_-$ by

$$\check{c}(\xi) = \sum_{n=0}^{\infty} \frac{\check{f}(\xi + 2^n)}{|\xi|^\beta}, \qquad \xi \in \mathbb{R}.$$

It is clear that $c \in L^2(\mathbb{R})$ and consequently (i) holds.

Furthermore $D^\beta c = M_{1+ix} g$ where

$$\check{g}(\xi) = \sum_{n=0}^{\infty} \check{f}_0(\xi + 2^n), \qquad \xi \in \mathbb{R}.$$

Since clearly $g \notin L^2(\mathbb{R})$ we have that $D^\beta c \notin BMO(\mathbb{R})$.

Finally, since the support of $\check{c}$ is sparse,

$$\left|\varphi_n * (D^\beta c)(x)\right| = \left| \int \varphi_n(x-y) f(y) \left(e^{i2^n y} + e^{i2^{n+1} y}\right) dy \right| \le 2\|f\|_{L^\infty(\mathbb{R})} \|\varphi_n\|_{L^1(\mathbb{R})},$$

which by the normalization of $\{\varphi_n\}_{n\in\mathbb{Z}}$ is uniformly bounded for $x \in \mathbb{R}, n \in \mathbb{Z}$. □

Theorem 5.12. *Let $\beta > 0$ and let A be the infinitesimal generator of the right shift semigroup on $L^2(\mathbb{R}_+)$. There exists an operator $C_\beta \in D(A)^*$ which is not 2β-admissible but still satisfies*

$$\sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{1/2+\alpha-\beta} \left\| C(\bar{\lambda} I - A)^{-(1+\alpha)} \right\|_{L^2(\mathbb{R}_+)^*} < \infty$$

for any $\alpha > \max\{\beta - 1/2, 0\}$.

Proof. Let $c_\beta \in \mathcal{Z}'_-$ have the properties stated in Lemma 5.11. By the first property there is a corresponding observation operator $C_\beta \in D(A)^*$. By Lemma 5.10 this operator satisfies the resolvent condition while by Propositions 5.6 and 5.7 it is not 2β-admissible. □

Analogous to the discrete time case, we obtain the following result.

Theorem 5.13. *Let $\beta > 0$. There exists $c \in \mathcal{Z}'_-$ such that the operator $H_c D^\beta : H^2(\mathbb{C}_+) \to L^2(\mathbb{R})$ does not satisfy the reproducing kernel thesis.*

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