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Dirac operators on noncommutative curved spacetimes

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Abstract. We study Dirac operators in the framework of twist-deformed noncommutative geometry. The definition of noncommutative Dirac operators is not unique and we focus on three different ones, each generalizing the commutative Dirac operator in a natural way. We show that the three definitions are mutually inequivalent, and that demanding formal self-adjointness with respect to a suitable inner product singles out a preferred choice. A detailed analysis shows that, if the Drinfeld twist contains sufficiently many Killing vector fields, the three operators coincide, which can simplify explicit calculations considerably. We then turn to the construction of quantized Dirac fields on noncommutative curved spacetimes. We show that there exist unique retarded and advanced Green's operators and construct a canonical anti-commutation relation algebra. In the last part we study noncommutative Minkowski and AdS spacetimes as explicit examples.

Key words: Dirac operators; Dirac fields; Drinfeld twists; Deformation quantization; Noncommutative quantum field theory; Quantum field theory on curved spacetimes

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1 Introduction and summary

Noncommutative geometry has long been of interest from a purely mathematical perspective as a natural generalization of ordinary differential geometry. It is also of crucial interest from a physical perspective, since it generically plays a role when the principles of quantum mechanics are combined with those of general relativity [1, 2]. In both contexts, Dirac operators are of major importance: they are relevant for structural questions in noncommutative geometry [3] and essential for the description of fermionic fields in models for high-energy physics. Interestingly enough, the construction of noncommutative Dirac operators, while straightforward for simple examples of noncommutative spacetimes like the Moyal-Weyl Minkowski spacetime, becomes ambiguous on curved spacetimes and for more general deformations. The origin and potential implications of these ambiguities clearly have to be understood in detail, and it is the aim of this paper to present a survey of possible definitions of noncommutative Dirac operators together with a study of their distinct features. We shall investigate various definitions in the framework of twist-deformed noncommutative geometry, and for the coupling of Dirac fields to the noncommutative background geometry we employ techniques of noncommutative vielbein gravity [4]. This allows us to consider a quite large class of deformations, namely those constructed from Abelian Drinfeld twists, of generic parallelizable Lorentzian manifolds. Of particular interest will be semi-Killing deformations, which play an important role in the construction of exact solutions of the noncommutative Einstein equations in [5, 6, 7].

The outline of this paper is as follows: After reviewing the framework of noncommutative vielbein gravity in Section 2, we present three definitions of noncommutative Dirac operators. That set of examples is not meant to be exhaustive, they are rather chosen for the reason that they arise as natural noncommutative generalizations of the commutative Dirac operator and yield the correct classical limit. For the special class of semi-Killing deformations we prove in Section 3 that all three noncommutative Dirac operators coincide. Furthermore, for actual Killing deformations, which in particular includes the Moyal-Weyl Minkowski spacetime, we show that all noncommutative Dirac operators coincide with the undeformed one. This leaves the question of whether the three definitions are equivalent altogether, which we can answer in the negative after a study of deformed spacetimes which do not satisfy the semi-Killing property. By constructing suitable examples, including the quantum plane and a particular curved Moyal-Weyl spacetime, we show that the three noncommutative Dirac operators are mutually different. We then turn to the question for the preferred choice among the three operators, given that they are all inequivalent. The crucial requirement turns out to be formal self-adjointness with respect to a suitable inner product, which indeed singles out one of the three operators as a preferred choice. As an application we investigate in Section 4 the construction of solutions and Green's operators of the noncommutative Dirac equation. We show that, provided the classical limit of our noncommutative spacetimes yields an oriented and time-oriented globally hyperbolic Lorentzian manifold, there exist unique retarded and advanced Green's operators which also characterize the solution space. These results are used in Section 5 to show that the preferred noncommutative Dirac operator can be used to construct a quantum field theory of Dirac fields on the noncommutative curved spacetimes that we consider. In order to illustrate our constructions we provide in Section 6 explicit examples of noncommutative Dirac operators that are of physical interest.

2 Dirac operators on noncommutative curved spacetimes

In this section we introduce three natural definitions for Dirac operators on noncommutative curved spacetimes. The constructions can in principle be carried out on deformed Lorentzian or Euclidean manifolds of any dimension, but to fix notation we mostly focus on Lorentzian manifolds of dimension 4. In Section 3.2 we will also study 2-dimensional examples and so we collect the analogous definitions also for that case.

2.1 Preliminaries

In the following we review techniques from deformation quantization of smooth manifolds by Drinfeld twists and the framework of noncommutative vielbein gravity.

Twist-deformed noncommutative geometry: Let M be a D-dimensional manifold and $C^\infty(M)$ be the algebra of smooth complex-valued functions on M. The noncommutative geometries that we shall consider are those which arise as deformations of M by an Abelian Drinfeld twist

$$\mathcal{F} := e^{-\frac{i\lambda}{2}\,\Theta^{\alpha\beta}\,X_\alpha\otimes X_\beta}\ , \tag{1}$$

where $\Theta^{\alpha\beta}$ is an antisymmetric, real and constant matrix (not necessarily of rank D) and X_α are mutually commuting real vector fields on M, i.e. $[X_\alpha, X_\beta] = 0$ for all α, β. The deformation parameter λ is assumed to be infinitesimally small, i.e. we work in formal deformation quantization. In this setup a formal power series extension $\mathbb{C}[[\lambda]]$ of the complex numbers, as well as of all vector spaces, algebras, etc., has to be performed, but for notational simplicity we will suppress the square brackets $[[\lambda]]$ denoting these extensions. We can assume, without loss of generality, that $\Theta^{\alpha\beta}$ is of the canonical (Darboux) form

$$\Theta = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{2}$$

The twist (1) is used to define a $\star$-product on $C^\infty(M)$ by

$$f \star g := \mu\big(\mathcal{F}^{-1} f \otimes g\big) = f\, g + \frac{i\lambda}{2}\Theta^{\alpha\beta} X_\alpha(f)\, X_\beta(g) + \dots\ , \tag{3}$$

where μ is the usual point-wise product and the action of the vector fields X_α on the functions f, g is via the (Lie) derivative. Furthermore, we can deform the de Rham calculus $\big(\Omega^\bullet := \bigoplus_{n=0}^{D} \Omega^n, \wedge, \mathrm{d}\big)$ on M into a differential calculus on the $\star$-product algebra $\big(C^\infty(M), \star\big)$ by defining the $\wedge_\star$-product

$$\omega \wedge_\star \tau := \wedge\big(\mathcal{F}^{-1} \omega \otimes \tau\big) = \omega \wedge \tau + \frac{i\lambda}{2}\Theta^{\alpha\beta} \mathcal{L}_{X_\alpha}(\omega) \wedge \mathcal{L}_{X_\beta}(\tau) + \dots\ , \tag{4}$$

where the action of the vector fields X_α on the differential forms ω, τ is via the Lie derivative. The undeformed differential d satisfies the graded Leibniz rule with respect to the $\wedge_\star$-product, i.e. $\mathrm{d}(\omega\wedge_\star\tau) = (\mathrm{d}\omega)\wedge_\star\tau + (-1)^{|\omega|}\,\omega\wedge_\star(\mathrm{d}\tau)$ with $|\omega|$ denoting the degree of ω, and hence $\big(\Omega^\bullet, \wedge_\star, \mathrm{d}\big)$ is a differential calculus over $\big(C^\infty(M), \star\big)$. We extend the involution $*$ on $\big(C^\infty(M), \star\big)$, which is given by point-wise complex conjugation, to a graded involution on $\big(\Omega^\bullet, \wedge_\star, \mathrm{d}\big)$ by applying the rules $(\omega \wedge_\star \tau)^* = (-1)^{|\omega|\,|\tau|}\, \tau^* \wedge_\star \omega^*$ and $(\mathrm{d}\omega)^* = \mathrm{d}(\omega^*)$. The undeformed integral $\int_M : \Omega^D \to \mathbb{C}$ satisfies the graded cyclicity property: for all $\omega, \tau \in \Omega^\bullet$ with compact overlapping support such that $|\omega| + |\tau| = D$,

$$\int_M \omega \wedge_\star \tau = \int_M \omega \wedge \tau = (-1)^{|\omega|\,|\tau|} \int_M \tau \wedge_\star \omega\ . \tag{5}$$

This completes our snapshot review of twist-deformed noncommutative geometry and we refer the reader to [8] for a more detailed discussion.

Noncommutative vielbein gravity: Let now M be a 4-dimensional manifold. Following [4], we describe the noncommutative gravitational field by a noncommutative vierbein field V and a noncommutative spin connection Ω. Both V and Ω are one-forms that are valued in the 4-dimensional Clifford algebra (4×4-matrices). We can expand V and Ω in terms of the gamma-matrix basis $\{1, \gamma_5, \gamma_a, \gamma_a\gamma_5, \gamma_{ab}\}$ as

$$\Omega = \frac{1}{4}\,\omega^{ab}\,\gamma_{ab} + i\,\omega\,1 + \tilde{\omega}\,\gamma_5 \quad , \qquad V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5\ . \tag{6}$$

We use the gamma-matrix conventions of [4], which we include for completeness in Appendix A. Notice that noncommutative vielbein gravity contains more fields than its commutative counterpart, where $\tilde{V}^a = \omega = \tilde{\omega} = 0$. The reason is that $SL(2,\mathbb{C})$ (Lorentz) $\star$-gauge transformations do not close and have to be extended to $GL(2,\mathbb{C})$ $\star$-gauge transformations. As in [4] we require that in the commutative limit $\tilde{V}^a|_{\lambda=0} = \omega|_{\lambda=0} = \tilde{\omega}|_{\lambda=0} = 0$, which means that setting $\lambda = 0$ the noncommutative vierbein and the noncommutative spin connection yield a usual commutative $SL(2,\mathbb{C})$ vierbein and spin connection. Let us denote these classical fields by $V_{(0)} := V^a_{(0)}\,\gamma_a := V|_{\lambda=0}$ and $\Omega_{(0)} := \frac{1}{4}\omega^{ab}_{(0)}\gamma_{ab} := \Omega|_{\lambda=0}$.

The $\star$-gauge transformations of noncommutative vielbein gravity act on V and Ω by

$$\delta_\epsilon V = [\epsilon \stackrel{\star}{,} V] \quad , \qquad \delta_\epsilon \Omega = \mathrm{d}\epsilon + [\epsilon \stackrel{\star}{,} \Omega] \ , \tag{7}$$

where $\epsilon = \frac{1}{4}\,\epsilon^{ab}\,\gamma_{ab} + i\,\varepsilon\,1 + \tilde{\varepsilon}\,\gamma_5$ is a Clifford algebra valued function and $[\epsilon \stackrel{\star}{,} V] := \epsilon \star V - V \star \epsilon$ is the $\star$-commutator. We impose the reality conditions $\epsilon^\dagger = -\gamma_0 \epsilon \gamma_0$, $V^\dagger = \gamma_0 V \gamma_0$, $\Omega^\dagger = -\gamma_0 \Omega \gamma_0$, and we notice that these are consistent with the $\star$-gauge transformations. Furthermore, we assume the noncommutative spin connection to be $\star$-torsion free, i.e. $0 = \mathrm{d}_\Omega V := \mathrm{d}V - \{\Omega \stackrel{\star}{,} V\}$ where $\{\Omega \stackrel{\star}{,} V\} = \Omega \wedge_\star V + V \wedge_\star \Omega$ is the $\star$-anticommutator. The $\star$-torsion constraint is part of the equations of motion of noncommutative vielbein gravity [4].

Let us now consider Dirac fields ψ, i.e. functions valued in the representation vector space of the Clifford algebra. We denote the Dirac adjoint by $\overline{\psi} := \psi^\dagger \gamma_0$. The $\star$-gauge transformations act on ψ and $\overline{\psi}$ by $\delta_\epsilon \psi = \epsilon \star \psi$ and $\delta_\epsilon \overline{\psi} = -\overline{\psi} \star \epsilon$, respectively. Notice that the matrix $\psi \star \overline{\psi}$ transforms in the adjoint representation, $\delta_\epsilon(\psi \star \overline{\psi}) = [\epsilon \stackrel{\star}{,} \psi \star \overline{\psi}]$. For all Dirac fields ψ_1, ψ_2 with compact overlapping support we define the inner product

$$\langle \psi_1, \psi_2 \rangle := i \int_M \mathrm{Tr}\big(\psi_2 \star \overline{\psi_1} \star V \wedge_\star V \wedge_\star V \wedge_\star V\,\gamma_5\big) \ , \tag{8}$$

which is $\star$-gauge invariant due to (5), (7) and the cyclicity of the matrix trace Tr.

Lemma 1. *The inner product (8) is hermitian, it reduces to the canonical commutative one for* $\lambda = 0$ *and it is non-degenerate, i.e.:*

a) $\langle \psi_1, \psi_2 \rangle^* = \langle \psi_2, \psi_1 \rangle$

b) $\langle \psi_1, \psi_2 \rangle = \int_M \overline{\psi_1} \psi_2 \,\mathrm{vol} + \mathcal{O}(\lambda)$, *where* $\mathrm{vol} = V^a_{(0)} \wedge V^b_{(0)} \wedge V^c_{(0)} \wedge V^d_{(0)} \epsilon_{abcd}$

c) If $\langle \psi_1, \psi_2 \rangle = 0$ *for all* ψ_2, *then* $\psi_1 = 0$

Proof. We show a) by the following short calculation

$$\begin{aligned}
\langle \psi_1, \psi_2 \rangle^* &= -i \int_M \mathrm{Tr}\big(\gamma_5^\dagger\, V^\dagger \wedge_\star V^\dagger \wedge_\star V^\dagger \wedge_\star V^\dagger \star (\psi_2 \star \overline{\psi_1})^\dagger\big) \\
&= -i \int_M \mathrm{Tr}\big(\psi_1 \star \overline{\psi_2} \gamma_0 \gamma_5 \gamma_0 \star V \wedge_\star V \wedge_\star V \wedge_\star V\big) \\
&= i \int_M \mathrm{Tr}\big(\psi_1 \star \overline{\psi_2} \star V \wedge_\star V \wedge_\star V \wedge_\star V \gamma_5\big) = \langle \psi_2, \psi_1 \rangle \ .
\end{aligned} \tag{9}$$

In the second equality we have used (graded) cyclicity, the reality condition $V^\dagger = \gamma_0 V \gamma_0$, $(\psi_2 \star \overline{\psi_1})^\dagger = \gamma_0 \psi_1 \star \overline{\psi_2} \gamma_0$, $\gamma_0^2 = 1$ and $\gamma_5^\dagger = \gamma_5$. In the third equality we have used $\gamma_5 \gamma_0 = -\gamma_0 \gamma_5$, $\gamma_0^2 = 1$ and $\gamma_5 V = -V \gamma_5$.

To show b) let us set in (8) $\lambda = 0$ and use that $V|_{\lambda=0} = V_{(0)} = V^a_{(0)}\,\gamma_a$ (i.e. that $\tilde{V}^a$ vanishes at order λ^0). Using further that the antisymmetrized product of 4 gamma-matrices is $\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} = -i\gamma_5\,\epsilon_{abcd}$ and that $\gamma_5^2 = 1$ we obtain the desired result.

c) is a consequence of b) and the fact that the classical inner product $\int_M \overline{\psi_1}\psi_2 \,\mathrm{vol}$ is non-degenerate. ■

2.2 Construction of noncommutative Dirac operators

We now turn to the construction of Dirac operators on our noncommutative curved spacetimes. To this end we define the $\star$-covariant differential acting on Dirac fields by $\mathrm{d}_\Omega \psi := \mathrm{d}\psi - \Omega \star \psi$. On adjoint Dirac fields we analogously define $\mathrm{d}_\Omega \overline{\psi} := \overline{\mathrm{d}_\Omega \psi} = \mathrm{d}\overline{\psi} + \overline{\psi} \star \Omega$ and note that $\mathrm{d}_\Omega(\psi_1 \star \overline{\psi_2}) := \mathrm{d}(\psi_1 \star \overline{\psi_2}) - [\Omega \stackrel{\star}{,} \psi_1 \star \overline{\psi_2}] = (\mathrm{d}_\Omega \psi_1) \star \overline{\psi_2} + \psi_1 \star (\mathrm{d}_\Omega \overline{\psi_2})$.

The Aschieri-Castellani Dirac operator: The first noncommutative Dirac operator is motivated by the noncommutative Dirac field action proposed in [4]

$$S^{\mathrm{AC}} = -4 \int_M \mathrm{Tr}\big((\mathrm{d}_\Omega \psi) \star \overline{\psi} \wedge_\star V \wedge_\star V \wedge_\star V\, \gamma_5\big) \ . \tag{10}$$

Since the inner product (8) is non-degenerate, we can define a Dirac operator $\not{D}^{\mathrm{AC}}$ by requiring that, for all ψ_1 of compact support,

$$\langle \psi_1, \not{D}^{\mathrm{AC}} \psi_2 \rangle = -4 \int_M \mathrm{Tr}\big((\mathrm{d}_\Omega \psi_2) \star \overline{\psi_1} \wedge_\star V \wedge_\star V \wedge_\star V\, \gamma_5\big) \ . \tag{11}$$

This yields precisely the equation of motion operator corresponding to the action (10). Since this Dirac operator naturally arises from a real action which reduces to the standard commutative one for $\lambda = 0$, it has the following nice properties:

Lemma 2. *The following properties hold true:*

a) $\langle \psi_1, \not{D}^{\mathrm{AC}} \psi_2 \rangle = \langle \not{D}^{\mathrm{AC}} \psi_1, \psi_2 \rangle$, *i.e.* $\not{D}^{\mathrm{AC}}$ *is formally self-adjoint with respect to* $\langle\ ,\ \rangle$.

b) $\not{D}^{\mathrm{AC}}|_{\lambda=0} = \not{D}_{(0)}$, *where* $\not{D}_{(0)}$ *is the classical Dirac operator corresponding to* $\big(V_{(0)}, \Omega_{(0)}\big)$.

Proof. We show a) by the following calculation

$$\begin{aligned}
\langle \not{D}^{\mathrm{AC}} \psi_1, \psi_2 \rangle &= \langle \psi_2, \not{D}^{\mathrm{AC}} \psi_1 \rangle^* = -4 \int_M \mathrm{Tr}\big(\gamma_5^\dagger\, V^\dagger \wedge_\star V^\dagger \wedge_\star V^\dagger \wedge_\star ((\mathrm{d}_\Omega \psi_1) \star \overline{\psi_2})^\dagger\big) \\
&= 4 \int_M \mathrm{Tr}\big(\psi_2 \star \overline{\mathrm{d}_\Omega \psi_1} \gamma_0 \gamma_5 \gamma_0 \wedge_\star V \wedge_\star V \wedge_\star V\big) \\
&= 4 \int_M \mathrm{Tr}\big(\psi_2 \star (\mathrm{d}_\Omega \overline{\psi_1}) \wedge_\star V \wedge_\star V \wedge_\star V\, \gamma_5\big) \\
&= -4 \int_M \mathrm{Tr}\Big(\big((\mathrm{d}_\Omega \psi_2) \star \overline{\psi_1} - \mathrm{d}_\Omega(\psi_2 \star \overline{\psi_1})\big) \wedge_\star V \wedge_\star V \wedge_\star V\, \gamma_5\Big) \\
&= \langle \psi_1, \not{D}^{\mathrm{AC}} \psi_2 \rangle + 4 \int_M \mathrm{d}\, \mathrm{Tr}\big(\psi_2 \star \overline{\psi_1} \star V \wedge_\star V \wedge_\star V\, \gamma_5\big) = \langle \psi_1, \not{D}^{\mathrm{AC}} \psi_2 \rangle \ .
\end{aligned} \tag{12}$$

In the third equality we have used graded cyclicity, $\gamma_0^2 = 1$ and $\gamma_5^\dagger = \gamma_5$. In the fourth equality we have used that $\overline{\mathrm{d}_\Omega \psi} = \mathrm{d}_\Omega \overline{\psi}$, $\gamma_5 \gamma_0 = -\gamma_0 \gamma_5$, $\gamma_0^2 = 1$ and $\gamma_5 V = -V \gamma_5$. In equality five we have used that $\mathrm{d}_\Omega(\psi_2 \star \overline{\psi_1}) = (\mathrm{d}_\Omega \psi_2) \star \overline{\psi_1} + \psi_2 \star (\mathrm{d}_\Omega \overline{\psi_1})$, in equality six the $\star$-torsion constraint $\mathrm{d}_\Omega V = 0$ and in equality seven Stokes' theorem.

To prove b) notice that $V \wedge_\star V \wedge_\star V\, \gamma_5|_{\lambda=0} = i\, V^a_{(0)} \wedge V^b_{(0)} \wedge V^c_{(0)}\, \epsilon_{abcd} \gamma^d$. Let us denote by $V^{-1}_{(0)} =: E_{(0)\,a} \gamma^a$ the inverse of the vierbein $V_{(0)}$, that is a vector field on M with values in the Clifford algebra. Expanding $\mathrm{d}_\Omega|_{\lambda=0} \psi_2$ in the vierbein basis, $\mathrm{d}_\Omega|_{\lambda=0} \psi_2 = V^a_{(0)} \big(E_{(0)\,a}(\psi_2) - \Omega_{(0)\,a}\, \psi_2\big) =: V^a_{(0)}\, \nabla_{(0)\,a} \psi_2$, we obtain for (11) at $\lambda = 0$

$$\begin{aligned}
\langle \psi_1, \not{D}^{\mathrm{AC}} \psi_2 \rangle|_{\lambda=0} &= -4i \int_M \mathrm{Tr}\big((\nabla_{(0)\,e} \psi_2) \overline{\psi_1}\, \gamma^d\big) V^e_{(0)} \wedge V^a_{(0)} \wedge V^b_{(0)} \wedge V^c_{(0)} \epsilon_{abcd} \\
&= \int_M \overline{\psi_1}\, i\, \gamma^d \nabla_{(0)\,d} \psi_2\, \mathrm{vol} = \int_M \overline{\psi_1} \not{D}_{(0)} \psi_2\, \mathrm{vol} \ .
\end{aligned} \tag{13}$$

■

We emphasize already at this point that the definition of the Dirac operator via (11), while certainly useful to establish its nice features, is also rather implicit, which would eventually complicate explicit applications. We shall therefore introduce two more options to construct noncommutative Dirac operators in the next two paragraphs.

The contraction Dirac operator: For the next noncommutative Dirac operator we shall follow closely the usual construction of a Dirac operator on commutative spacetimes, which goes as follows: Let $V_{(0)} = V^a_{(0)}\gamma_a$ be a classical vierbein, $\Omega_{(0)}$ a classical spin connection and let us denote by $V^{-1}_{(0)} = E_{(0)\,a}\gamma^a$ the inverse vierbein. The classical Dirac operator is $\not{D}_{(0)}\psi = i\,\gamma^a\nabla_{(0)\,a}\psi = i\,\gamma^a\big(E_{(0)\,a}(\psi) - \Omega_{(0)\,a}\psi\big)$, where we have expressed $\Omega_{(0)}$ in the vierbein basis $\Omega_{(0)} = V^a\,\Omega_{(0)\,a}$. Notice that this operator can be written in an index-free form $\not{D}_{(0)}\psi = i\,\iota_{V^{-1}_{(0)}}\big(\mathrm{d}\psi - \Omega_{(0)}\psi\big)$, where ι is the contraction operator (interior product) extended to matrix-valued vector fields and vector-valued one-forms in the obvious way.

Using the deformed contraction operator $\iota^\star$ between vector fields and one-forms as defined in [9], we generalize the above construction to the noncommutative setting. Explicitly, for a vector field v and a one-form ω we define

$$\iota^\star_v(\omega) := \iota\big(\mathcal{F}^{-1} v\otimes\omega\big) = \iota_v(\omega) + \frac{i\lambda}{2}\Theta^{\alpha\beta}\,\iota_{\mathcal{L}_{X_\alpha}(v)}\big(\mathcal{L}_{X_\beta}(\omega)\big) + \dots\ , \tag{14}$$

where the action of the vector fields X_α on v and ω is via the Lie derivative. Since by hypothesis the V^a components of the noncommutative vierbein are invertible, we can define the $\star$-inverse E_a (which are vector fields) by the $\star$-contraction condition $\iota^\star_{E_a}\big(V^b\big) = \delta^b_a$. We collect all E_a in the Clifford algebra valued vector field $V^{-1_\star} := E_a\,\gamma^a$. Following the same strategy as in the commutative case we define the contraction Dirac operator by

$$\not{D}^{\mathrm{contr}}\psi := i\,\iota^\star_{V^{-1_\star}}\big(\mathrm{d}_\Omega\psi\big) = i\,\gamma^a\iota^\star_{E_a}\big(\mathrm{d}_\Omega\psi\big)\ . \tag{15}$$

Since the involved operations reduce to the classical ones for $\lambda = 0$, this Dirac operator also has the correct classical limit $\not{D}^{\mathrm{contr}}|_{\lambda=0} = \not{D}_{(0)}$.

The deformed Dirac operator: The last noncommutative Dirac operator is motivated by the framework of Connes for noncommutative spin geometry [3]. In this formalism the Dirac operator enters as a fundamental degree of freedom of the theory. Hence, one reasonable option to obtain a noncommutative Dirac operator is to deform the classical Dirac operator $\not{D}_{(0)}$ via the techniques developed in [10]. More precisely, denoting the inverse twist by $\mathcal{F}^{-1} = \bar{f}^\alpha\otimes\bar{f}_\alpha$ we define the deformed Dirac operator by applying the deformation map constructed in [10]

$$\not{D}^{\mathcal{F}}\psi := \big(\bar{f}^\alpha \blacktriangleright \not{D}_{(0)}\big)\bar{f}_\alpha(\psi) = \not{D}_{(0)}\psi + \frac{i\lambda}{2}\Theta^{\alpha\beta}\big(X_\alpha\blacktriangleright\not{D}_{(0)}\big)X_\beta(\psi) + \dots\ , \tag{16}$$

where $X_\alpha \blacktriangleright \not{D}_{(0)} := X_\alpha\circ\not{D}_{(0)} - \not{D}_{(0)}\circ X_\alpha$ is the adjoint action. Also this Dirac operator has the correct classical limit for $\lambda = 0$.

2.3 Two-dimensional Dirac operators

We now turn to the two-dimensional case which will be relevant for the examples in Subsection 3.2. Our conventions for the 2-dimensional Clifford algebra are as follows: $\eta_{ab} = \mathrm{diag}(1,-1)_{ab}$ is the 2-dimensional Minkowski metric and the Clifford relation $\{\gamma_a,\gamma_b\} = 2\eta_{ab}$ is satisfied by the 2×2-matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\quad,\qquad \gamma_1 = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\ . \tag{17}$$

We also define $\gamma_3 := \gamma_0\gamma_1$ and note that $\gamma_{ab} := \frac{1}{2}[\gamma_a,\gamma_b] = \epsilon_{ab}\,\gamma_3$, where ϵ_{ab} is the 2-dimensional ϵ-tensor, $\epsilon_{01} = 1$. We further have $\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0$, $\gamma_3^\dagger = \gamma_3$, $\gamma_3^2 = 1$ and $\gamma_3\gamma_a = -\gamma_a\gamma_3$.

The noncommutative twobein and spin connection have the following expansion in terms of the gamma-matrix basis $\{1, \gamma_3, \gamma_a\}$

$$V = V^a \gamma_a \quad , \qquad \Omega = \omega\, \gamma_3 + \tilde{\omega}\, 1 \ . \tag{18}$$

We define for $\epsilon = \varepsilon\, \gamma_3 + \tilde{\varepsilon}\, 1$ the $\star$-gauge transformations $\delta_\epsilon V := [\epsilon \overset{\star}{,} V]$ and $\delta_\epsilon \Omega := \mathrm{d}\epsilon + [\epsilon \overset{\star}{,} \Omega]$. As in the case of 4-dimensional noncommutative vielbein gravity we had to introduce the extra fields $\tilde{\omega}$ and $\tilde{\varepsilon}$ such that the $\star$-gauge transformations close. In contrast to the case of $D = 4$ we do not need additional terms in the twobein field and thus the interpretation of V as a soldering form remains valid in $D = 2$ noncommutative vielbein gravity. We again impose the reality conditions $\epsilon^\dagger = -\gamma_0 \epsilon \gamma_0$, $V^\dagger = \gamma_0 V \gamma_0$ and $\Omega^\dagger = -\gamma_0 \Omega \gamma_0$.

Let us consider Dirac fields ψ. The Dirac adjoint is $\overline{\psi} := \psi^\dagger \gamma_0$ and $\star$-gauge transformations act on ψ and $\overline{\psi}$ via $\delta_\epsilon \psi := \epsilon \star \psi$ and $\delta_\epsilon \overline{\psi} = -\overline{\psi} \star \epsilon$. We define in analogy to (8) a $\star$-gauge invariant and hermitian inner product

$$\langle \psi_1, \psi_2 \rangle := \int_M \mathrm{Tr}\big(\psi_2 \star \overline{\psi_1} \star V \wedge_\star V \gamma_3\big) \ . \tag{19}$$

For $\lambda = 0$ we obtain the usual inner product $\langle \psi_1, \psi_2 \rangle|_{\lambda=0} = \int_M \overline{\psi_1} \psi_2\, \mathrm{vol}$, since $V \wedge_\star V \gamma_3|_{\lambda=0} = V_{(0)} \wedge V_{(0)} \gamma_3 = V^a_{(0)} \wedge V^b_{(0)} \epsilon_{ab} \gamma_3^2 = \mathrm{vol}$.

Using the $\star$-covariant differential $\mathrm{d}_\Omega \psi := \mathrm{d}\psi - \Omega \star \psi$ we define the $D = 2$ analog of the Aschieri-Castellani Dirac operator (11) by requiring that, for all ψ_1 of compact support,

$$\langle \psi_1, \not{D}^{\mathrm{AC}} \psi_2 \rangle = 2i \int_M \mathrm{Tr}\big((\mathrm{d}_\Omega \psi_2) \star \overline{\psi_1} \wedge_\star V \gamma_3\big) \ . \tag{20}$$

Using that $\gamma_a \gamma_3 = -\epsilon_{ab} \gamma^b$ we find that $\not{D}^{\mathrm{AC}}$ has the correct classical limit $\not{D}^{\mathrm{AC}}|_{\lambda=0} = \not{D}_{(0)}$. Furthermore, $\not{D}^{\mathrm{AC}}$ is formally self-adjoint with respect to the inner product (19) if the $\star$-torsion constraint $0 = \mathrm{d}_\Omega V = \mathrm{d}V - \{\Omega \overset{\star}{,} V\}$ holds. The contraction and deformed Dirac operator of Subsection 2.2 are easily adapted to the case of $D = 2$.

3 Comparison of the noncommutative Dirac operators

In this section we study the three noncommutative Dirac operators defined in Subsection 2.2 in more detail. In the first part we focus on the special class of semi-Killing deformations, for which the three operators turn out to be equivalent. In the second part we study explicit examples of more general deformations to work out the differences among the operators. To keep the technical part as simple as possible we focus on two-dimensional spacetimes in that part.

3.1 Semi-Killing twists: Dirac operators unisono

Of particular interest for the explicit examples to be presented in Section 6 are those noncommutative curved spacetimes that solve the noncommutative Einstein equations. As noted in [5, 6, 7], any undeformed metric field solving the classical Einstein equations also solves the noncommutative Einstein equations [9] if the twist is semi-Killing. Explicitly, an Abelian twist (1) is semi-Killing if $\Theta^{\alpha\beta} X_\alpha \otimes X_\beta \in \Xi \otimes \mathfrak{K} + \mathfrak{K} \otimes \Xi$, where Ξ is the Lie algebra of vector fields on M and $\mathfrak{K} := \{X \in \Xi : \mathcal{L}_X(V) = 0\}$ is the Killing Lie algebra. Using the canonical form of $\Theta^{\alpha\beta}$ (2), this condition is equivalent to requiring that either X_{2n} or X_{2n-1} is a Killing vector field, for all $n = 1, 2, \dots$. We note that all our examples of noncommutative Klein-Gordon operators studied in [11] are of this type. The semi-Killing requirement allowed us in many cases to calculate explicitly the noncommutative Klein-Gordon operators to all orders in λ. As we shall show in Section 6 the same holds true for the noncommutative Dirac operators.

Let us assume that the twist $\mathcal{F}$ is semi-Killing and furthermore that $V = V_{(0)}$, i.e. that V only contains λ^0-terms. The classical Levi-Civita connection $\Omega_{(0)}$, specified uniquely by the commutative torsion constraint $\mathrm{d}V_{(0)} - \{\Omega_{(0)}, V_{(0)}\} = 0$, has the invariance property $\mathcal{L}_X(\Omega_{(0)}) = 0$ for all $X \in \mathfrak{K}$. This implies that $\Omega := \Omega_{(0)}$ fulfills the $\star$-torsion constraint $\mathrm{d}V - \{\Omega \overset{\star}{,} V\} = 0$. Those (V, Ω)-pairs solve the noncommutative Einstein equations (in vielbein form) [4] whenever they solve the commutative Einstein equations.

We denote by E_a the basis for the vector fields on M which is specified by the undeformed contraction condition $\iota_{E_a}(V^b) = \delta_a^b$. Notice that this condition implies that $\mathcal{L}_X(E_a) = 0$ for all $X \in \mathfrak{K}$. Hence, also the deformed contraction condition $\iota^\star_{E_a}(V^b) = \delta^b_a$ holds true for semi-Killing twists. Due to the latter property, the $\star$-inverse vierbein reads $V^{-1_\star} = E_a \gamma^a$. We expand the $\star$-covariant differential into this basis, i.e. $\mathrm{d}_\Omega \psi = V^a \star \nabla^\star_a \psi$ or equivalently $\nabla^\star_a \psi = \iota^\star_{E_a}\big(\mathrm{d}_\Omega\psi\big)$. Defining the components Ω_a of the spin connection by $\Omega =: V^a\,\Omega_a$, the conditions $\mathcal{L}_X(\Omega) = 0$ and $\mathcal{L}_X(V) = 0$ imply that $\mathcal{L}_X(\Omega_a) = 0$, for all $X \in \mathfrak{K}$, and thus $\Omega = V^a\,\Omega_a = V^a \star \Omega_a$. Furthermore, defining the differential operator $E^\star_a$ by the basis expansion $\mathrm{d}\psi =: V^a \star E^\star_a(\psi)$ (which is equivalent to $E^\star_a(\psi) = \iota^\star_{E_a}(\mathrm{d}\psi)$) we obtain

$$\nabla^\star_a \psi = \iota^\star_{E_a}\big(\mathrm{d}_\Omega \psi\big) = E^\star_a(\psi) - \Omega_a \star \psi \ . \tag{21}$$

The contraction Dirac operator (15) expressed in this basis reads

$$\not{D}^{\mathrm{contr}} \psi = i\,\gamma^a \nabla^\star_a \psi = i\,\gamma^a \big(E^\star_a(\psi) - \Omega_a \star \psi\big) \ . \tag{22}$$

To evaluate the deformed Dirac operator $\not{D}^{\mathcal{F}}$ we first note that the classical Dirac operator expressed in the vierbein basis reads $\not{D}_{(0)}\psi = i\,\gamma^a\,\iota_{E_a}\big(\mathrm{d}\psi - \Omega\psi\big) = i\,\gamma^a\,\big(E_a(\psi) - \Omega_a\,\psi\big)$. Evaluating the deformation map in (16) we obtain that the deformed Dirac operator coincides for semi-Killing twists with the contraction Dirac operator,

$$\not{D}^{\mathcal{F}}\psi = \big(\bar{f}^\alpha \blacktriangleright \not{D}_{(0)}\big)\bar{f}_\alpha(\psi) = i\,\gamma^a \nabla^\star_a \psi = \not{D}^{\mathrm{contr}}\psi \ . \tag{23}$$

In this calculation we have used that $\mathcal{L}_X(\Omega_a) = 0$ whenever $X \in \mathfrak{K}$. Finally, we compute explicitly the Aschieri-Castellani Dirac operator (11) for semi-Killing twists. Due to the semi-Killing property the following holds true

$$i\,V \wedge_\star V \wedge_\star V \wedge_\star V \gamma_5 = i\,V \wedge V \wedge V \wedge V \gamma_5 = 1\,\mathrm{vol} \ , \tag{24a}$$

$$V \wedge_\star V \wedge_\star V \gamma_5 = V \wedge V \wedge V \gamma_5 = i\,V^a \wedge V^b \wedge V^c\,\epsilon_{abcd}\gamma^d \ . \tag{24b}$$

Hence, the inner product (8) reads

$$\langle \psi_1, \psi_2 \rangle = \int_M \overline{\psi_1} \star \mathrm{vol} \star \psi_2 \ , \tag{25}$$

where $\mathrm{vol} = V^a \wedge V^b \wedge V^c \wedge V^d\,\epsilon_{abcd}$ is the classical volume form, and (11) simplifies to

$$\langle \psi_1, \not{D}^{\mathrm{AC}}\psi_2 \rangle = 4i \int_M \overline{\psi_1} \star (V^a \wedge V^b \wedge V^c)\epsilon_{abcd}\,\gamma^d \wedge_\star \mathrm{d}_\Omega\psi_2 = \int_M \overline{\psi_1} \star \mathrm{vol} \star i\,\gamma^d \nabla^\star_d \psi_2 \ . \tag{26}$$

This shows that for semi-Killing twist all three definitions of noncommutative Dirac operators coincide. We would like to stress that for generic semi-Killing twists the noncommutative Dirac operators do not coincide with the classical one $\not{D}_{(0)}$, cf. Section 6 for explicit examples. In summary, we have obtained the following

Proposition 3. *For semi-Killing deformations the three noncommutative Dirac operators defined in Subsection 2.2 are all equivalent. For practical purposes one can therefore choose the technically most convenient one.*

If all vector fields X_α in the twist $\mathcal{F}$ are Killing, then the differential operator $E^\star_a$ defined by $\mathrm{d}\psi =: V^a \star E^\star_a(\psi)$ coincides with the vector field E_a (the inverse vierbein). Furthermore, $\Omega_a \star \psi = \Omega_a\, \psi$ and thus we obtain for the contraction Dirac operator (22) $\not{D}^{\mathrm{contr}}\psi = i\,\gamma^a\big(E_a(\psi) - \Omega_a\,\psi\big) = \not{D}_{(0)}\psi$. Since actual Killing twists are contained in the class of semi-Killing twists, all three noncommutative Dirac operators coincide and we thus have $\not{D}^{\mathrm{contr}} = \not{D}^{\mathcal{F}} = \not{D}^{\mathrm{AC}} = \not{D}_{(0)}$ for Killing twists. The same result has been obtained for noncommutative Klein-Gordon operators in [11, 12].

3.2 Non-semi-Killing twists: bringing up the differences

We have seen in the previous subsection that the three noncommutative Dirac operators are equivalent for semi-Killing deformations. This naturally raises the question whether the operators are equivalent altogether, which we answer now. Let us first provide an example which shows that the Aschieri-Castellani Dirac operator differs from the others. To this end, we consider the 2-dimensional Minkowski spacetime $M = \mathbb{R}^2$ described by the twobein $V = V^a\,\gamma_a = dt\,\gamma_0 + dx\,\gamma_1$ and deform it by the twist (1) with $X_1 = t\partial_t$ and $X_2 = x\partial_x$. Notice that this deformation leads to the commutation relations of the quantum plane $t \star x = e^{i\lambda}\, x \star t$. The $\star$-torsion constraint is solved by $\Omega = 0$. The $\star$-inverse E_a of V^a is defined by $\iota^\star_{E_a}(V^b) = \delta^b_a$ and it is given by $E_0 = \partial_t$, $E_1 = \partial_x$. We further find for the $\star$-covariant derivative $\mathrm{d}_\Omega\psi = \mathrm{d}\psi = V^a \star \nabla^\star_a\psi = dt \star e^{-\frac{i\lambda}{2} x\partial_x}\partial_t\psi + dx \star e^{\frac{i\lambda}{2} t\partial_t}\partial_x\psi$. This leads to the following contraction Dirac operator on the quantum plane

$$\not{D}^{\mathrm{contr}}\psi = i\gamma^a\nabla^\star_a\psi = i\Big(\gamma^0\, e^{-\frac{i\lambda}{2} x\partial_x}\partial_t\psi + \gamma^1\, e^{\frac{i\lambda}{2} t\partial_t}\partial_x\psi\Big)\ . \tag{27}$$

The deformed Dirac operator (16) also can be explicitly evaluated by using that $t\partial_t \blacktriangleright \partial_t = -\partial_t$ and $t\partial_t \blacktriangleright \partial_x = 0$ (and similarly for $x\partial_x$). We find that it agrees with the contraction Dirac operator in the case of the quantum plane. Finally, we evaluate the Aschieri-Castellani Dirac operator. Using that $dt \wedge_\star dx = e^{\frac{i\lambda}{2}}\, dt \wedge dx = e^{\frac{i\lambda}{2}}\,\mathrm{vol}/2$, with vol denoting the classical volume form, we obtain for the inner product (19)

$$\langle\psi_1,\psi_2\rangle = \cos(\lambda/2)\int_M \overline{\psi_1} \star \mathrm{vol} \star \psi_2\ . \tag{28}$$

Furthermore, evaluating (20) we obtain

$$\langle\psi_1, \not{D}^{\mathrm{AC}}\psi_2\rangle = i\int_M \overline{\psi_1} \star \mathrm{vol} \star \Big(e^{-\frac{i\lambda}{2}}\,\gamma^0\nabla^\star_0\psi_2 + e^{\frac{i\lambda}{2}}\,\gamma^1\nabla^\star_1\psi_2\Big)\ , \tag{29}$$

which yields the Aschieri-Castellani Dirac operator on the quantum plane

$$\not{D}^{\mathrm{AC}}\psi = \frac{i}{\cos(\lambda/2)}\Big(e^{-\frac{i\lambda}{2}}\,\gamma^0\, e^{-\frac{i\lambda}{2} x\partial_x}\partial_t\psi + e^{\frac{i\lambda}{2}}\,\gamma^1\, e^{\frac{i\lambda}{2} t\partial_t}\partial_x\psi\Big)\ . \tag{30}$$

Comparing (27) and (30) we observe that the noncommutative Dirac operators $\not{D}^{\mathrm{contr}} = \not{D}^{\mathcal{F}}$ and $\not{D}^{\mathrm{AC}}$ do not coincide on the quantum plane. Notice that the difference is not just in the overall factor, but the two terms have also acquired different phases. Furthermore, calculating the formal adjoint of the operator (27) with respect to the inner product (28) we obtain

$$(\not{D}^{\mathrm{contr}})^*\,\psi = i\,\Big(e^{-i\lambda}\,\gamma^0\, e^{-\frac{i\lambda}{2} x\partial_x}\partial_t\psi + e^{i\lambda}\,\gamma^1\, e^{\frac{i\lambda}{2} t\partial_t}\partial_x\psi\Big)\ . \tag{31}$$

The formal adjoint operator does not agree with (27), hence $\not{D}^{\mathrm{contr}}$ and also $\not{D}^{\mathcal{F}}$ are not formally self-adjoint on the quantum plane. Since formal self-adjointness is essential for the construction of a quantum field theory, cf. Section 5, this example singles out the Aschieri-Castellani

Dirac operator, which is always formally self-adjoint by Lemma 2, from the three candidates of noncommutative Dirac operators.

For completeness, we consider another example of a noncommutative spacetime to prove that the contraction and deformed Dirac operators do not always coincide. Let us consider $M = \mathbb{R}^2$ with Cartesian coordinates (t, x) and the deformation given by the Moyal-Weyl twist, i.e. (1) with $X_1 = \partial_t$ and $X_2 = \partial_x$. We equip this deformed manifold with the noncommutative twobein $V = dt\,\gamma_0 + \Phi\, dx\,\gamma_1$, where $\Phi \in C^\infty(M)$ is a strictly positive function. We notice that the spin connection $\Omega = dx\,\frac{\partial_t \Phi}{2}\,\gamma_3$ satisfies the ordinary and $\star$-torsion constraint. The $\star$-inverse twobein reads $E_0 = \partial_t$ and $E_1 = \Phi^{-1_\star}\,\partial_x$, where $\Phi^{-1_\star}$ is the $\star$-inverse function of Φ. For the contraction Dirac operator (15) we obtain

$$\not{D}^{\mathrm{contr}}\psi = i\Big(\gamma^0\,\partial_t\psi + \gamma^1\,\Phi^{-1_\star}\star\partial_x\psi + \frac{\gamma_0}{2}\,\Phi^{-1_\star}\star\partial_t\Phi\star\psi\Big)~. \tag{32}$$

Evaluating the deformed Dirac operator (16) we find for this model

$$\not{D}^{\mathcal{F}}\psi = i\Big(\gamma^0\,\partial_t\psi + \gamma^1\,\Phi^{-1}\star\partial_x\psi + \frac{\gamma_0}{2}\,(\Phi^{-1}\,\partial_t\Phi)\star\psi\Big)~. \tag{33}$$

Notice that in this expression there appears the usual inverse Φ^{-1} of Φ as well as the usual product $\Phi^{-1}\,\partial_t\Phi$. In general, the expressions (32) and (33) do not coincide, which implies that the contraction and deformed Dirac operators are distinct. Calculating the Aschieri-Castellani Dirac operator (20) for this model we obtain that it coincides with (32).

In summary, we have obtained the following

Proposition 4. *In general, the three noncommutative Dirac operators defined in Subsection 2.2 are mutually different.*

As shown in Section 3.1, the differences disappear when we restrict ourselves to semi-Killing deformations. Since the Aschieri-Castellani Dirac operator is in general formally self-adjoint, cf. Lemma 2, while the others are not, the discussion in this section provides good arguments singling out this particular noncommutative Dirac operator.

4 Solution theory of the noncommutative Dirac equation

In [12] we have discussed the solution theory of formal deformations of wave operators on deformed globally hyperbolic spacetimes. In particular, we have constructed retarded/advanced Green's operators as well as the solution space of the deformed wave equation. In this section we generalize these results to formal deformations of Dirac operators. This is essential to construct a quantum field theory corresponding to the noncommutative Dirac operators introduced above.

Since it does not complicate our proofs, we shall work in the following more abstract setting which contains the noncommutative Dirac operators above: Let $\pi : V \to M$ be a complex vector bundle over a classical time-oriented and connected globally hyperbolic Lorentzian manifold M. Let us denote the space of sections of this bundle by $\Gamma^\infty(V)$. We shall suppress as before the brackets $[[\lambda]]$ for the formal power series extension of this vector space. The class of operators $P_\star := \sum_{n=0}^\infty \lambda^n\, P_{(n)} : \Gamma^\infty(V) \to \Gamma^\infty(V)$ that we shall consider is characterized as follows: 1.) $P := P_{(0)} : \Gamma^\infty(V) \to \Gamma^\infty(V)$ is an operator of Dirac-type, i.e. $-P^2$ is a wave operator. 2.) $P_{(n)} : \Gamma^\infty(V) \to \Gamma^\infty_0(V)$, $n \geq 1$, is a finite-order differential operator mapping to the space of sections of compact support $\Gamma^\infty_0(V)$. The support condition is sufficient, however not in all cases necessary, to construct the Green's operators for $P_\star$. Notice that the massive Aschieri-Castellani Dirac operator $\not{D}^{\mathrm{AC}} + m$, with $m \in \mathbb{R}$, as well as all other noncommutative Dirac operators studied above are examples for such $P_\star$, if we restrict to deformations of compact support.

It is well known that for classical operators of Dirac-type $P : \Gamma^\infty(V) \to \Gamma^\infty(V)$ there exist a unique retarded and advanced Green's operator $G^\pm$, see e.g. [13] for a proof employing a modern language. We remind the reader that a retarded/advanced Green's operator is a linear map $G^\pm : \Gamma^\infty_0(V) \to \Gamma^\infty(V)$ satisfying the inhomogeneous equation of motion $G^\pm \circ P = P \circ G^\pm = \mathrm{id}$ and the support condition $\mathrm{supp}(G^\pm\psi) \subseteq J^\pm(\mathrm{supp}(\psi))$, for all $\psi \in \Gamma^\infty_0(V)$, where $J^\pm(\mathrm{supp}(\psi))$ is the forward/backward lightcone of the set $\mathrm{supp}(\psi)$.

Theorem 5. *Let us define the map* $G^\pm_\star := \sum_{n=0}^\infty \lambda^n\, G^\pm_{(n)} : \Gamma^\infty_0(V) \to \Gamma^\infty(V)$ *by* $G^\pm_{(0)} := G^\pm$ *and, for* $n > 0$,

$$G^\pm_{(n)} := \sum_{k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} (-1)^k \delta_{j_1+\cdots+j_k,n}\; G^\pm \circ P_{(j_1)} \circ G^\pm \circ P_{(j_2)} \circ \cdots \circ G^\pm \circ P_{(j_k)} \circ G^\pm \ , \tag{34}$$

where $\delta_{n,m}$ *is the Kronecker delta. Then* $G^\pm_\star$ *is the unique retarded/advanced Green's operator for* $P_\star$*, i.e.* $G^\pm_\star \circ P_\star = P_\star \circ G^\pm_\star = \mathrm{id}$ *and* $\mathrm{supp}(G^\pm_{(n)}\psi) \subseteq J^\pm(\mathrm{supp}(\psi))$*, for all* $n \in \mathbb{N}^0$ *and* $\psi \in \Gamma^\infty_0(V)$.

Proof. The proof is straightforward by following the steps in the proof of [12, Theorem 1]. ■

We now provide an explicit characterization of the solution space $\mathrm{Sol} := \big\{\psi \in \Gamma^\infty_{\mathrm{sc}}(V) : P_\star\psi = 0\big\}$, where the subscript $_{\mathrm{sc}}$ denotes sections of spacelike compact support. All solutions are obtained by the causal propagator $G_\star := G^+_\star - G^-_\star : \Gamma^\infty_0(V) \to \Gamma^\infty_{\mathrm{sc}}(V)$ due to the following

Theorem 6. *The following sequence of linear maps is a complex which is exact everywhere:*

$$\{0\} \longrightarrow \Gamma^\infty_0(V) \xrightarrow{P_\star} \Gamma^\infty_0(V) \xrightarrow{G_\star} \Gamma^\infty_{\mathrm{sc}}(V) \xrightarrow{P_\star} \Gamma^\infty_{\mathrm{sc}}(V) \tag{35}$$

Proof. This proof is obtained by following the steps in the proof of [12, Theorem 2]. However, a small modification is required, since in [12, Theorem 2] we had given an inner product on $\Gamma^\infty(V)$ and we have assumed that the equation of motion operator $P_\star$ is formally self-adjoint. These properties were only used in [12, Equation (34)]. Since we did not assume an inner product on $\Gamma^\infty(V)$, we have to replace the inner product in [12, Equation (34)] by the pairing between $\Gamma^\infty(V)$ and $\Gamma^\infty(V^*)$, where V^* is the dual bundle of V. The Green's and equation of motion operators in the left entry of the inner product in [12, Equation (34)] also have to be replaced by their formal adjoints acting on sections of V^*. With this small modification the proof of [12, Theorem 2] generalizes easily to our present setting. ■

5 CAR quantization

With the tools developed in the previous section we can construct the canonical anti-commutation relation (CAR) algebra corresponding to deformed Dirac-type operators $P_\star : \Gamma^\infty(V) \to \Gamma^\infty(V)$. This is the observable algebra of the quantized noncommutative Dirac field. For this construction we also require a hermitian inner product $\langle\ ,\ \rangle$ on $\Gamma^\infty(V)$ and that $P_\star$ is formally self-adjoint with respect to $\langle\ ,\ \rangle$. Due to Lemma 2, the massive Aschieri-Castellani Dirac operator $\not{D}^{\mathrm{AC}} + m$, with $m \in \mathbb{R}$, is of this kind, where the inner product is given by (8).

Let us define another inner product on $\Gamma^\infty_0(V)$ by using the causal propagator $G_\star$ corresponding to $P_\star$: for all $\psi_1, \psi_2 \in \Gamma^\infty_0(V)$,

$$\langle\langle \psi_1, \psi_2 \rangle\rangle := i\, \langle \psi_1, G_\star \psi_2 \rangle \ . \tag{36}$$

Since $\langle\psi_1,\psi_2\rangle^* = \langle\psi_2,\psi_1\rangle$ and $P_\star$ is formally self-adjoint (which implies that $G_\star$ is formally skew-adjoint) we obtain

$$\langle\langle\psi_1,\psi_2\rangle\rangle^* = -i\,\langle G_\star\psi_2,\psi_1\rangle = i\,\langle\psi_2,G_\star\psi_1\rangle = \langle\langle\psi_2,\psi_1\rangle\rangle~. \tag{37}$$

Notice that, due to Theorem 6 and the fact that $G_\star$ is formally skew-adjoint, $\langle\langle~,~\rangle\rangle$ induces a well-defined hermitian inner product on the quotient $H_\star := \Gamma^\infty_0(V)/P_\star\big[\Gamma^\infty_0(V)\big]$.

If $V = DM$ is a Dirac spinor bundle and $P_\star$ is such that $P := P_{(0)} = \not{D}_{(0)} + m$, the inner product $\langle\langle~,~\rangle\rangle$ on $H_\star$ is positive-definite. To show this statement let us consider the classical limit $\lambda = 0$. We find by using Green's formula [14, p. 160, Prop. 9.1] for the inner product in this limit, that for all $\psi_1,\psi_2\in H_\star|_{\lambda=0} = \Gamma^\infty_0(DM)/P\big[\Gamma^\infty_0(DM)\big]$,

$$\langle\langle\psi_1,\psi_2\rangle\rangle|_{\lambda=0} = i\,\langle PG^\pm\psi_1,G\psi_2\rangle|_{\lambda=0} = i\int_\Sigma \overline{i\,\gamma_a n^a G\psi_1}\,G\psi_2\,\mathrm{vol}_\Sigma = \int_\Sigma (G\psi_1)^\dagger\,G\psi_2\,\mathrm{vol}_\Sigma~.$$

Here Σ is any Cauchy surface and $n = E_a\,n^a$ its future-pointing normal vector field. In the last equality we have used that we can choose $n = E_0$. Since the inner product $\langle\langle~,~\rangle\rangle$ is positive-definite at order λ^0 it is positive-definite to all orders in the deformation parameter.

The inner-product space $\big(H_\star,\langle\langle~,~\rangle\rangle\big)$ can be quantized in terms of a CAR algebra, see e.g. [15] for a modern review of these techniques: To any element $\psi\in H_\star$ we associate an abstract operator $a(\psi)$ and consider the free unital $\ast$-algebra A^{free} generated by all $a(\psi)$, $\psi\in H_\star$. We define the CAR algebra $A^{\mathrm{CAR}} := A^{\mathrm{free}}/\mathcal{I}$ as the quotient of A^{free} by the both-sided $\ast$-ideal $\mathcal{I}$ generated by the elements, for all $\psi_1,\psi_2\in H_\star$ and $\alpha_1,\alpha_2\in\mathbb{C}$,

$$a\big(\alpha_1\,\psi_1+\alpha_2\,\psi_2\big) - \alpha_1\,a(\psi_1) - \alpha_2\,a(\psi_2)~, \tag{38a}$$
$$\big\{a(\psi_1),a(\psi_2)\big\}~, \tag{38b}$$
$$\big\{a(\psi_1)^*,a(\psi_2)\big\} - \langle\langle\psi_1,\psi_2\rangle\rangle\,1~, \tag{38c}$$

where $\{~,~\}$ is the anti-commutator. The interpretation of this quotient is as follows: (38a) allows us to regard $a(\psi)$ as smeared linear field operators. (38b) and (38c) encode the CAR. The on-shell condition is already implemented in $H_\star$. In the physics literature the Dirac field operator is typically denoted by $\Psi(x)$ and its adjoint by $\overline{\Psi}(x)$. This notation is related to ours by $a(\psi) = \langle\Psi,\psi\rangle$ and $a(\psi)^* = \langle\psi,\Psi\rangle$, where by the inner products we (formally) denote the smearing of the field operators by test sections. In this notation (38b) states that $\Psi(x)$ anti-commutes with $\Psi(y)$ and (38c) that the anti-commutator between $\overline{\Psi}(x)$ and $\Psi(y)$ is non-trivial.

6 Explicit examples

In this section we will explicitly study the noncommutative Dirac operators discussed in Section 2 on two noncommutative (curved) spacetimes. For their attractive features, e.g. as solutions to noncommutative Einstein equations, we will focus on semi-Killing deformations. As shown in Subsection 3.1 all three noncommutative Dirac operators coincide in this case and we will collectively denote them by $\not{D}_\star$. These studies are complementing our explicit examples of deformed Klein-Gordon operators [11].

6.1 κ-Minkowski spacetime

As a first example we consider $M = \mathbb{R}^4$ with global coordinates denoted by $x^\mu = (t,x^j)$ and the Minkowski vierbein $V = \gamma_a\delta^a_\mu\,dx^\mu$, along with the spin connection $\Omega = 0$. For the twist (1) we use $X_1 = \partial_t$ and $X_2 = x^j\partial_j$, which yields a semi-Killing twist. The commutation relations of the coordinate functions are those of κ-Minkowski spacetime, i.e. $[t \stackrel{\star}{,} x^j] = i\lambda\,x^j$ and $[x^i \stackrel{\star}{,} x^j] = 0$.

Various fields with their equation of motion operators have been studied on this particular noncommutative spacetime, see [11] for the scalar field and [16] for the $U(1)$ gauge field. We supplement these studies by the Dirac field with equation of motion operator given by any of the noncommutative Dirac operators introduced in Subsection 2.2, which all coincide for this model since $\mathcal{F}$ is semi-Killing.

Using that $\mathcal{L}_{X_1}(V^a) = 0$, $\mathcal{L}_{X_2}(V^0) = 0$ and $\mathcal{L}_{X_2}(V^j) = V^j$ we obtain for the $\star$-covariant derivative, which is defined by $\mathrm{d}_\Omega\psi = V^a \star \nabla^\star_a\psi$, the following expression

$$\nabla^\star_0 = \partial_t \ , \quad \nabla^\star_j = e^{\frac{i\lambda}{2}\partial_t}\partial_j \ . \tag{39}$$

Since the three noncommutative Dirac operators coincide for this model we choose to calculate the simplest one, which is the contraction Dirac operator (15), and find

$$\not{D}_\star\psi = i\,\gamma^a\nabla^\star_a\psi = i\left(\gamma^0\partial_t\psi + \gamma^j e^{\frac{i\lambda}{2}\partial_t}\partial_j\psi\right) \ . \tag{40}$$

For the solutions of the noncommutative Dirac equation $\not{D}_\star\psi = 0$ we can then derive a dispersion relation by squaring the equation of motion operator $\not{D}_\star$. More explicitly, this yields

$$\Box_\star := -\not{D}_\star^2 = \eta^{ab}\,\nabla^\star_a\nabla^\star_b = \partial_t^2 - \triangle\, e^{i\lambda\partial_t} \ , \tag{41}$$

where $\triangle := \partial_1^2 + \partial_2^2 + \partial_3^2$ is the spatial Laplacian. To study the dispersion relation we make a plane wave ansatz $\psi = \chi\, e^{i(Et+k_jx^j)}$, where E is the energy, k_j the momentum and $\chi \in \mathbb{C}^4$ a polarization spinor. Since $\not{D}_\star\psi = 0$ implies $\Box_\star\psi = 0$ we obtain the deformed energy-momentum relation

$$E^2 - e^{-\lambda E}\,\mathbf{k}^2 = 0 \quad \Leftrightarrow \quad E^2\, e^{\lambda E} = \mathbf{k}^2 \ . \tag{42a}$$

From the equation of motion $\not{D}_\star\psi = 0$ we further obtain a condition on the polarization spinor

$$\big(\gamma^0\, E + \gamma^j k_j\, e^{-\frac{\lambda}{2}E}\big)\chi = 0 \ . \tag{42b}$$

Without loss of generality we choose the spatial momentum along the third direction, i.e. $\mathbf{k} = (0,0,k)$, such that (42b) becomes $\big(\gamma^0\, E + \gamma^3 k\, e^{-\frac{\lambda}{2}E}\big)\chi = 0$. Using the on-shell condition (42a) this becomes independent of λ and reduces to the analogous equation in the commutative case. We thus find that the physical spin polarizations, which are characterized as the solutions of (42b), do not receive noncommutative corrections. Hence, this type of noncommutative geometry does not introduce an anomalous spin precession.

6.2 Noncommutative anti de Sitter space

We now turn to a curved spacetime example, for which the natural first candidates are the maximally symmetric (anti) de Sitter ((A)dS) spacetimes. We choose AdS which is of relevance e.g. for model building in particle physics and AdS/CFT, but note that similar calculations for the cosmologically relevant dS are fully analogous. A particle-physics model employing a deformation of AdS can be found in [17].

We focus on the Poincaré patch of 4-dimensional AdS, that is, $M = \mathbb{R}^3 \times (0,\infty)$ with coordinates $x^\mu = (x^i, z)$ and the vierbein $V = \gamma_a\, Rz^{-1}\,\delta^a_\mu\, dx^\mu$. The generalization to higher dimensions is straightforward and just amounts to using the higher dimensional Clifford algebras. In the following we fix the radius of curvature to $R = 1$, and for the gamma-matrices we denote the contraction with the (inverse) vielbein explicitly by a hat, e.g. $\hat{\gamma}^\mu := E^\mu_a\gamma^a = z\,\delta^\mu_a\gamma^a$. To deform this space we employ the twist (1) with the $2N$ mutually commuting vector fields X_α, $\alpha = 1,\dots,2N$, given by

$$X_{2n-1} = T^{\ i}_{2n-1}\partial_i \ , \qquad X_{2n} = \vartheta(z)\, T^{\ i}_{2n}\partial_i \ , \qquad n = 1,2,\dots,N \ . \tag{43}$$

In this expression the T^i_α are real numbers and $\vartheta(z) \in C^\infty(0,\infty)$ is a real valued function. Notice that this twist is semi-Killing, since all X_{2n-1} are Killing vector fields. The $\star$-commutation relations of the coordinate functions (x^i, z) read

$$[x^i \stackrel{\star}{,} x^j] = i\lambda\,\vartheta(z)\,\Theta^{\alpha\beta}T^i_\alpha\,T^j_\beta \quad , \qquad [x^i \stackrel{\star}{,} z] = 0 \ . \tag{44}$$

Hence, this model describes a z-dependent Moyal-Weyl deformation of the $\mathbb{R}^3$ hypersurfaces at constant z. The $\star$-torsion free spin connection for this model is $\Omega = -\frac{1}{2}V^i\,\gamma_{i3}$.

To compute the $\star$-covariant derivative $\mathrm{d}_\Omega\psi = V^a \star \nabla^\star_a\psi$ we first notice that $\mathcal{L}_{X_{2n-1}}(V^a) = 0$, $\mathcal{L}_{X_{2n}}(V^i) = \vartheta'(z)\,T^i_{2n}\,V^3$ and $\mathcal{L}_{X_{2n}}(V^3) = 0$. We then obtain

$$\nabla^\star_i = z\partial_i + \frac{\gamma_{i3}}{2} \ , \quad \nabla^\star_3 = z\partial_z + \frac{i\lambda}{2}\,z\,\vartheta'(z)\,\mathcal{T} \ , \tag{45}$$

where $\mathcal{T} := \mathcal{T}^{ij}\partial_i\partial_j := \sum_{n=1}^N T^i_{2n}\,T^j_{2n-1}\partial_i\partial_j$. Since the twist is semi-Killing and the noncommutative Dirac operators are therefore equivalent, we once again choose the technically most convenient one for the explicit evaluation. Choosing (15) we find

$$\not{D}_\star\psi = i\gamma^a\nabla^\star_a\psi = \not{D}_{(0)}\psi + \frac{\lambda}{2}z\,\vartheta'(z)\,\gamma_3\mathcal{T}\psi \ , \tag{46}$$

where $\not{D}_{(0)}$ is the commutative Dirac operator. Notice that $\not{D}_\star$ thus is a second-order differential operator. It is the equation of motion operator stemming from the action

$$S_\star = \langle\psi, \not{D}_\star\psi\rangle = \int_M \overline{\psi}\,\not{D}_\star\psi\,\mathrm{vol} \ , \tag{47}$$

where in the last equality we have used that for the present model the inner product (8) coincides with the undeformed one.

A crucial point for the construction of quantum fields on AdS is the existence of a finite inner product, which is closely related to the choice of boundary conditions, see e.g. [18] for an early reference. This issue is conveniently analyzed in terms of the hypersurface inner product on the space of solutions of the noncommutative Dirac equation, which we compute in the following. The resulting inner product space can be quantized by following the CAR-construction outlined in Section 5. Following the strategy developed in [19] we consider variations of the action functional (47) and derive a conserved current. Explicitly, we obtain for the current density of two solutions ψ_1 and ψ_2

$$i\,J^\mu = i\,\overline{\psi_1}\hat{\gamma}^\mu\psi_2\sqrt{|g|} + \delta^\mu_i\,\mathcal{T}^{ij}\,\frac{\lambda}{2}z\,\vartheta'(z)\,\sqrt{|g|}\,\Big(\overline{\psi_1}\gamma_3\partial_j\psi_2 - \overline{\partial_j\psi_1}\gamma_3\psi_2\Big) \ , \tag{48}$$

where $\sqrt{|g|} = z^{-4}$ is the square root of the metric determinant. Notice that $\nabla_\mu J^\mu = \partial_\mu J^\mu = 0$ whenever $\not{D}_\star\psi_1 = 0$ and $\not{D}_\star\psi_2 = 0$, i.e. J^μ is a conserved density. We integrate the current density J^μ over a fixed-time hypersurface Σ with normal vector field $n^\mu = (1,0,0,0)^\mu$ and obtain the hypersurface inner product

$$(\psi_1,\psi_2) = \int_\Sigma \Big(\psi_1^\dagger\psi_2 + i\mathcal{T}^{0j}\,\frac{\lambda}{2}z\,\vartheta'(z)\,\Big(\overline{\partial_j\psi_1}\gamma_3\psi_2 - \overline{\psi_1}\gamma_3\partial_j\psi_2\Big)\Big)\,\mathrm{vol}_\Sigma \ . \tag{49}$$

That inner product is conserved only up to boundary terms, which can not be assumed to vanish on AdS. Demanding actual conservation then yields the admissible boundary conditions. A well-motivated restriction on the deformation is to demand $\mathcal{T}^{0i} = 0$, in which case the deformation is purely in the spatial part and no higher-order time derivatives are introduced. In that case the hypersurface inner product (49) coincides with the undeformed one, i.e. $(\psi_1,\psi_2) = \int_\Sigma \psi_1^\dagger\,\psi_2\,\mathrm{vol}_\Sigma$. We would like to stress that the solutions of the noncommutative Dirac equation are of course still affected by the deformation, and it would hence be of interest to study the effect of different choices of T^i_α and $\vartheta(z)$. A natural choice would for example be such that $\mathcal{T} = \triangle$ is the spatial Laplacian on the hypersurfaces of constant z, as discussed in [11].

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A $D = 4$ Clifford algebra conventions

$$\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)_{ab} \ , \qquad \{\gamma_a,\gamma_b\} = 2\eta_{ab} \ , \qquad \gamma_{ab} := \frac{1}{2}[\gamma_a,\gamma_b] \ , \tag{50a}$$

$$\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3 \ , \qquad \gamma_5^2 = 1 \ , \qquad \epsilon_{0123} = -\epsilon^{0123} = 1 \ , \tag{50b}$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0 \ , \qquad \gamma_5^\dagger = \gamma_5 \ . \tag{50c}$$

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