Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 13/12

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July 2013

http://www.math.uni-wuppertal.de

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July 15, 2013

Abstract

The aim of this work is to complete our program on the quantization of connections on arbitrary principal $U(1)$-bundles over globally hyperbolic Lorentzian manifolds. In particular, we show that one can assign via a covariant functor to any such bundle an algebra of observables which separates gauge equivalence classes of connections. The C^*-algebra we construct generalizes the usual CCR-algebras since, contrary to the standard field-theoretic models, it is based on a presymplectic Abelian group instead of a symplectic vector space. We prove a no-go theorem according to which neither this functor, nor any of its quotients, satisfy the strict axioms of general local covariance. Yet, if we fix any principal $U(1)$-bundle, there exists a suitable category of sub-bundles for which a quotient of our functor yields a quantum field theory in the sense of Haag and Kastler. We shall provide a physical interpretation of this feature and we obtain some new insights concerning electric charges in locally covariant quantum field theory.

Keywords: locally covariant quantum field theory, quantum field theory on curved spacetimes, gauge theory on principal bundles

MSC 2010: 81T20, 81T05, 81T13, 53Cxx

1 Introduction

Although Maxwell's field is the simplest example of a Yang-Mills gauge theory, it is known since [AS80] that the construction and analysis of the associated algebra of observables and its representations can be complicated due to a non-trivial topology of the spacetime manifold. This peculiar feature is extremely relevant when one employs the algebraic framework in order to quantize such a theory on curved backgrounds. The first investigations along these lines are due to Dimock [Dim92], but a thorough analysis of topological effects started only recently, from both the perspective of the Faraday tensor [DL12] and, more generally, the quantization of linear gauge theories [Pfe09, DS13, FH13, HS13, SDH12, FS13]. The bottom line of some of these papers is the existence of a non-trivial center in the algebra of fields, provided certain topological, or more precisely cohomological, properties of the underlying background hold true. In [SDH12], it has been advocated that the elements of the center found in that paper could be interpreted in physical terms as being related to observables measuring electric charges. However, this leads unavoidably to a violation of the locality property (injectivity of the induced morphisms between the field algebras) of locally covariant quantum field theories, as formulated in [BFV03].

A complementary approach to the above ones has been introduced by some of us in [BDS12, BDS13] starting from the observation that, in the spirit of a Yang-Mills gauge theory, electromagnetism should be best described as a theory of connections on principal $U(1)$-bundles over globally hyperbolic Lorentzian manifolds. More properly, one starts from the characterization of connections as sections of an affine bundle, dubbed the *bundle of connections* [Ati57]. Subsequently the dynamics is implemented in terms of an affine equation of motion, the Maxwell equation. The system can be quantized in the algebraic framework following the prescription outlined in [BDS12]. This procedure is advantageous for three main reasons: First of all there is no need to fix any reference connection, as it is done (implicitly) elsewhere [Pfe09, DS13, SDH12, FS13]. As a useful consequence of this, we were able to construct in [BDS13] purely topological observables, resembling topological quantum fields, which can measure the Chern class of the underlying principal $U(1)$-bundle. Secondly, interactions between gauge and matter fields are modeled only in terms of connections, while an approach based on the Faraday tensor, as in [DL12], cannot account for this aspect. Thirdly, contrary to most of the previous approaches, the gauge group is completely determined geometrically by the underlying principal bundle, since it is the collection of vertical automorphisms.

By following this perspective, the algebra of fields for Abelian Yang-Mills theories has been constructed in [BDS13]. Yet, as explained in [BDS13, Remark 4.5], the latter fails to separate gauge equivalence classes of connections. The source of this obstruction can be traced back to the existence of disconnected components in the gauge group in the case of spacetimes with a non-trivial first de Rham cohomology group. From a physical point of view, this entails that those observables which are measuring the configurations tied to the Aharonov-Bohm effect, as discussed in [SDH12, FS13], are not contained in the algebra of observables.

The main goal of this paper is to fill this gap by elaborating on the proposal in [BDS13] to add Wilson-loop observables to the algebra of fields, as these new elements would solve the problem of separating all configurations. Following slavishly the original idea turned out to be rather cumbersome from a technical point of view. Yet, we found that it is more convenient to consider exponentiated versions of the affine observables constructed in [BDS13]. On the one hand, these observables resemble classical versions of Weyl operators, while, on the other hand, the requirement of gauge invariance leads to a weaker constraint – the exponent does not need to remain invariant under a gauge transformation, but it is allowed to change by any integer multiple of $2\pi i$.

After performing this construction, we shall prove that, contrary to what was shown in [BDS13, Section 7] for the non-exponentiated algebra of fields, in the complete framework it is not possible to restore general local covariance in the strict sense by singling out a suitable ideal. This no-go theorem holds true only if we consider all possible isometric embeddings allowed by the axioms of general local covariance devised in [BFV03]. If we restrict our category of principal $U(1)$-bundles to a suitable subcategory possessing a terminal object, a result similar to that of [BDS13, Section 7] can be shown to hold true. We will interpret this feature as a proof that we can construct a separating algebra of observables fulfilling the axioms of Haag and Kastler [HK63] generalized to an arbitrary but fixed globally hyperbolic spacetime. We shall further provide a physical interpretation for the impossibility to restore general local covariance in the strict sense on our category of all principal $U(1)$-bundles.

We present an outline of the paper: In Section 2 we fix the notations and preliminaries which should allow a reader with some experience in differential geometry to follow the rest of the article. For more details, explanations and proofs we refer to [BDS12, BDS13]. In Section 3 we provide a detailed study of the exponential observables mentioned above. We characterize explicitly the gauge invariant observables of exponential type and prove that they separate gauge equivalence classes of connections. This solves the problem explained in [BDS13, Remark 4.5] and captures the essence of what is called Aharonov-Bohm observables in [SDH12]. As a rather unexpected result, we find that the set of gauge invariant observables of exponential type can be labeled by a presymplectic Abelian group, which is not a vector space due to the disconnected components of the gauge group. We call this presymplectic Abelian group the phasespace of the theory and prove in Section 4 that those phasespaces naturally arise from a covariant functor from a category of principal $U(1)$-bundles over globally hyperbolic spacetimes to a category of presymplectic Abelian groups. The properties of this functor are carefully investigated and it is found that, in agreement with earlier results [SDH12, BDS13], the locality property is violated. Subsequently we study whether the phasespace functor allows for a quotient by 'electric charges' in order to overcome the failure of the locality property as it was done in [BDS13, Section 7]. We prove a no-go theorem: There exists no quotient such that the theory satisfies the locality property and we trace this feature back to Aharonov-Bohm observables, which were not present in [BDS13]. We end Section 4 by quantizing the phasespace functor in terms of the CCR-functor for presymplectic Abelian groups, which we develop in the Appendix A by applying and extending results of [M+73]. The resulting quantum field theory functor satisfies the quantum causality property and the time-slice axiom, and thus all axioms of locally covariant quantum field theory [BFV03] but the locality property. In Section 5 we consider suitable subcategories (possessing a terminal object) of the category of principal $U(1)$-bundles and prove that there exists a quotient which restores the locality property. The resulting theory is not a locally covariant quantum field theory in the strict definition of [BFV03], but rather a theory in the sense of Haag and Kastler [HK63] where a global spacetime manifold (not necessarily the Minkowski spacetime) is fixed at the very beginning and one takes into account only causally compatible open sub-regions. A physical interpretation of our results is given in Section 6.

2 Preliminaries and notation

Let us fix once and for all the Abelian Lie group $G = U(1)$. We denote its Lie algebra by $\mathfrak{g}$ and notice that $\mathfrak{g} = i\,\mathbb{R}$. The vector space dual of the Lie algebra $\mathfrak{g}$ is denoted by $\mathfrak{g}^*$ and we note that $\mathfrak{g}^* \simeq i\,\mathbb{R}$. For later convenience we introduce the Abelian subgroup $\mathfrak{g}_{\mathbb{Z}} := 2\pi\,i\,\mathbb{Z} \subset \mathfrak{g}$.

In [BDS13, Definition 2.4] we have defined a suitable category $G-\mathsf{PrBuGlobHyp}$ of principal G-bundles over globally hyperbolic spacetimes. An object is a tuple $\Xi = ((M,\mathfrak{o},g,\mathfrak{t}),(P,r))$, where $(M,\mathfrak{o},g,\mathfrak{t})$ is a globally hyperbolic spacetime (with M of finite type) and (P,r) is a principal G-bundle over M. A morphism $f:\Xi_1\to\Xi_2$ is a principal G-bundle map $f:P_1\to P_2$, such that the induced map $\underline{f}:M_1\to M_2$ is an orientation and time-orientation preserving isometric embedding with $\underline{f}[M_1]\subseteq M_2$ causally compatible and open.

To any object Ξ in $G-\mathsf{PrBuGlobHyp}$ we can associate (via a covariant functor) its bundle of connections $\mathcal{C}(\Xi)$, that is an affine bundle over M modeled on the homomorphism bundle $\mathrm{Hom}(TM,\mathrm{ad}(\Xi))$. Notice that the adjoint bundle is trivial, i.e. $\mathrm{ad}(\Xi) = M\times\mathfrak{g}$, since G is Abelian. The set of sections $\Gamma^\infty(\mathcal{C}(\Xi))$ of the bundle $\mathcal{C}(\Xi)$ is an (infinite-dimensional) affine space over $\Omega^1(M,\mathfrak{g})$. We denote the free and transitive action of $\Omega^1(M,\mathfrak{g})$ on $\Gamma^\infty(\mathcal{C}(\Xi))$ with the usual abuse of notation by $\lambda+\eta$, for all $\eta\in\Omega^1(M,\mathfrak{g})$ and $\lambda\in\Gamma^\infty(\mathcal{C}(\Xi))$. Let us denote by $\Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)$ the vector space of compactly supported sections of the vector dual bundle $\mathcal{C}(\Xi)^\dagger$. Every $\varphi\in\Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)$ defines a functional on the configuration space $\Gamma^\infty(\mathcal{C}(\Xi))$ by

$$\mathcal{O}_\varphi : \Gamma^\infty(\mathcal{C}(\Xi)) \to \mathbb{R}~,\quad \lambda\mapsto \mathcal{O}_\varphi(\lambda) = \int_M \mathrm{vol}\,\varphi(\lambda)~. \tag{2.1}$$

For any $\lambda\in\Gamma^\infty(\mathcal{C}(\Xi))$ and $\eta\in\Omega^1(M,\mathfrak{g})$ we obtain the affine property $\mathcal{O}_\varphi(\lambda+\eta) = \mathcal{O}_\varphi(\lambda) + \langle\varphi_V,\eta\rangle$, where

$$\langle\varphi_V,\eta\rangle := \int_M \varphi_V\wedge *(\eta)~. \tag{2.2}$$

We have denoted the Hodge operator by $*$ and the linear part of φ by $\varphi_V\in\Omega^1_0(M,\mathfrak{g}^*)$. The duality pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ is suppressed here and in the following. Let us define the vector subspace

$$\mathrm{Triv} := \Big\{a\,\mathbb{1} : a\in C^\infty_0(M)\text{ satisfies }\int_M \mathrm{vol}\,a = 0\Big\}\subseteq\Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)~, \tag{2.3}$$

where $\mathbb{1}\in\Gamma^\infty(\mathcal{C}(\Xi)^\dagger)$ denotes the canonical section which associates to any $x\in M$ the constant affine map $a\in\mathcal{C}(\Xi)|_x\mapsto 1$. Notice that any $\varphi\in\mathrm{Triv}$ defines the trivial functional $\mathcal{O}_\varphi\equiv 0$ and, vice versa, that for any trivial functional $\mathcal{O}_\varphi\equiv 0$ we have $\varphi\in\mathrm{Triv}$. Hence, we consider the quotient $\Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)/\mathrm{Triv}$. Elements in this quotient are equivalence classes that we simply denote by φ (suppressing square brackets).

The gauge group $\mathrm{Gau}(P)$, i.e. the group of vertical principal G-bundle automorphisms, is isomorphic to the group $C^\infty(M,G)$, which acts on $\Gamma^\infty(\mathcal{C}(\Xi))$ via

$$\Gamma^\infty(\mathcal{C}(\Xi)) \times C^\infty(M,G) \to \Gamma^\infty(\mathcal{C}(\Xi))\ , \quad (\lambda, \widehat{f}) \mapsto \lambda + \widehat{f}^*(\mu_G)\ , \tag{2.4}$$

where $\mu_G \in \Omega^1(G,\mathfrak{g})$ is the Maurer-Cartan form. Let us define the Abelian subgroup of $\Omega^1(M,\mathfrak{g})$ which is generated by gauge transformations,

$$B_G := \big\{\widehat{f}^*(\mu_G) : \widehat{f} \in C^\infty(M,G)\big\} \subseteq \Omega^1_{\mathrm{d}}(M,\mathfrak{g})\ . \tag{2.5}$$

The Abelian group B_G can be characterized by using cohomology: Let us denote by $H^1(M,\mathfrak{g}_\mathbb{Z})$ the first sheaf cohomology group of the locally constant sheaf with values in $\mathfrak{g}_\mathbb{Z} = 2\pi i\,\mathbb{Z}$. An explicit realization of this group is via Čech cohomology with values in $\mathfrak{g}_\mathbb{Z}$. Notice that the Abelian group $H^1(M,\mathfrak{g}_\mathbb{Z})$ is a free $\mathbb{Z}$-module, which is finitely generated because M is assumed to be of finite type. Furthermore, due to the embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$ and the Čech-de Rham isomorphism there exists a monomorphism of Abelian groups $H^1(M,\mathfrak{g}_\mathbb{Z}) \to H^1_{\mathrm{dR}}(M,\mathfrak{g})$, whose image we denote by $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})$. The Abelian subgroup $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})$ is a lattice in $H^1_{\mathrm{dR}}(M,\mathfrak{g})$, i.e. any $\mathbb{Z}$-module basis of $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})$ provides a vector space basis of $H^1_{\mathrm{dR}}(M,\mathfrak{g})$. As a consequence of [BDS13, Proposition 4.2] we obtain

$$B_G = \big\{\eta \in \Omega^1_{\mathrm{d}}(M,\mathfrak{g}) : [\eta] \in H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})\big\}\ , \tag{2.6}$$

where $[\eta] \in H^1_{\mathrm{dR}}(M,\mathfrak{g})$ is the element of the first de Rham cohomology group defined by $\eta \in \Omega^1_{\mathrm{d}}(M,\mathfrak{g})$.

The gauge invariant functionals of affine type (2.1) have been characterized in [BDS13, Theorem 4.4]. It is found that these functionals are labeled by those $\varphi \in \Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)/\mathrm{Triv}$ which satisfy $\varphi_V \in \delta\Omega^2_0(M,\mathfrak{g}^*)$, where δ is the codifferential. As a consequence of [BDS13, Remark 4.5], these functionals in general do not separate gauge equivalence classes of connections. The goal of the present article is to resolve this issue by studying a set of observables different from (2.1).

3 Gauge invariant functionals of exponential type

Instead of (2.1), let us consider the functionals of exponential type, for all $\varphi \in \Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)$,

$$\mathcal{W}_\varphi : \Gamma^\infty(\mathcal{C}(\Xi)) \to \mathbb{C}\ , \quad \lambda \mapsto \mathcal{W}_\varphi(\lambda) = e^{2\pi i\, \mathcal{O}_\varphi(\lambda)}\ . \tag{3.1}$$

The affine property of $\mathcal{O}_\varphi$ implies that, for all $\lambda \in \Gamma^\infty(\mathcal{C}(\Xi))$ and $\eta \in \Omega^1(M,\mathfrak{g})$,

$$\mathcal{W}_\varphi(\lambda+\eta) = \mathcal{W}_\varphi(\lambda)\, e^{2\pi i\, \langle \varphi_V, \eta\rangle}\ . \tag{3.2}$$

We notice that the functional $\mathcal{W}_\varphi$ is trivial, i.e. $\mathcal{W}_\varphi \equiv 1$, if and only if φ is an element in the Abelian subgroup

$$\mathrm{Triv}_\mathbb{Z} := \Big\{a\,\mathbb{1} : a \in C^\infty_0(M) \text{ satisfies } \int_M \mathrm{vol}\, a \in \mathbb{Z}\Big\} \subseteq \Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)\ . \tag{3.3}$$

Hence, we consider the quotient $\mathcal{E}^{\mathrm{kin}} := \Gamma^\infty_0(\mathcal{C}(\Xi)^\dagger)/\mathrm{Triv}_\mathbb{Z}$. Elements in this quotient are equivalence classes that we simply denote by φ (suppressing square brackets). We say that a functional $\mathcal{W}_\varphi$, $\varphi \in \mathcal{E}^{\mathrm{kin}}$, is gauge invariant, if $\mathcal{W}_\varphi(\lambda+\eta) = \mathcal{W}_\varphi(\lambda)$, for all $\lambda \in \Gamma^\infty(\mathcal{C}(\Xi))$ and $\eta \in B_G$. Due to (3.2) this is equivalent to $\langle\varphi_V, B_G\rangle \subseteq \mathbb{Z}$. A necessary condition for $\mathcal{W}_\varphi$ to be gauge invariant is that $\delta\varphi_V = 0$, i.e. $\varphi_V \in \Omega^1_{0,\delta}(M,\mathfrak{g}^*)$. This can be seen by demanding invariance of $\mathcal{W}_\varphi$ under the gauge transformations $\lambda \mapsto \lambda + \mathrm{d}\chi$, $\chi \in C^\infty(M,\mathfrak{g})$, which are obtained by choosing $\widehat{f} = \exp\circ\chi \in C^\infty(M,G)$ in (2.4). We can associate to such φ_V an element $[\varphi_V]$ in the dual de Rham cohomology group $H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*) := \Omega^1_{0,\delta}(M,\mathfrak{g}^*)/\delta\Omega^2_0(M,\mathfrak{g}^*)$. Since any $\eta \in B_G$ is closed, the pairing in (3.2) depends only on the cohomology classes, i.e. $\langle\varphi_V,\eta\rangle = \langle[\varphi_V],[\eta]\rangle$. Notice that the pairing $\langle\ ,\ \rangle : H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*) \times H^1_{\mathrm{dR}}(M,\mathfrak{g}) \to \mathbb{R}$ is non-degenerate due to Poincaré duality, i.e. $H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*) \simeq H^1_{\mathrm{dR}}(M,\mathfrak{g})^* := \mathrm{Hom}_\mathbb{R}(H^1_{\mathrm{dR}}(M,\mathfrak{g}),\mathbb{R})$.

Let us denote the dual $\mathbb{Z}$-module of $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})$ by $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})^* := \mathrm{Hom}_\mathbb{Z}(H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z}),\mathbb{Z})$. Since $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})$ is a lattice in $H^1_{\mathrm{dR}}(M,\mathfrak{g})$ any element in $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})^*$ defines a unique element in $H^1_{\mathrm{dR}}(M,\mathfrak{g})^*$

by $\mathbb{R}$-linear extension. Thus, there is a monomorphism of Abelian groups $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})^* \to H^1_{\mathrm{dR}}(M,\mathfrak{g})^*$ which we shall suppress in the following. Composing this map with the isomorphism $H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*) \simeq H^1_{\mathrm{dR}}(M,\mathfrak{g})^*$ given by the pairing $\langle\ ,\ \rangle$ we can regard $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})^*$ as an Abelian subgroup of $H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)$, which we shall denote by $H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)_\mathbb{Z} \subseteq H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)$. With these preparations we can now provide a characterization of the gauge invariant functionals.

Proposition 3.1. *Let $\varphi \in \mathcal{E}^{\mathrm{kin}}$ be such that $\delta\varphi_V = 0$, i.e. φ satisfies the necessary condition for $\mathcal{W}_\varphi$ being gauge invariant. Then $\mathcal{W}_\varphi$ is a gauge invariant functional if and only if $[\varphi_V] \in H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)_\mathbb{Z}$.*

Proof. The functional (3.1) is gauge invariant if and only if $\langle \varphi_V, B_G\rangle = \langle [\varphi_V],[B_G]\rangle \subseteq \mathbb{Z}$. By (2.6) this is equivalent to the condition $\big\langle [\varphi_V], H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})\big\rangle \subseteq \mathbb{Z}$, which is satisfied if and only if $[\varphi_V] \in H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)_\mathbb{Z}$. □

Let us define the Abelian subgroup

$$\mathcal{E}^{\mathrm{inv}} := \big\{\varphi \in \mathcal{E}^{\mathrm{kin}} : \delta\varphi_V = 0 \text{ and } [\varphi_V] \in H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)_\mathbb{Z}\big\} \subseteq \mathcal{E}^{\mathrm{kin}}\ , \tag{3.4}$$

which labels the gauge invariant functionals $\mathcal{W}_\varphi$.

Theorem 3.2. *The set $\{\mathcal{W}_\varphi : \varphi \in \mathcal{E}^{\mathrm{inv}}\}$ of gauge invariant functionals of exponential type is separating on gauge equivalence classes of configurations. This means that, for any two $\lambda,\lambda' \in \Gamma^\infty(\mathcal{C}(\Xi))$ which are not gauge equivalent via (2.4), there exists $\varphi \in \mathcal{E}^{\mathrm{inv}}$, such that $\mathcal{W}_\varphi(\lambda') \neq \mathcal{W}_\varphi(\lambda)$.*

Proof. Let $\lambda,\lambda' \in \Gamma^\infty(\mathcal{C}(\Xi))$ be not gauge equivalent, i.e. $\lambda' = \lambda + \eta$ with $\eta \in \Omega^1(M,\mathfrak{g}) \setminus B_G$.

Let us first assume that η is not closed, $\mathrm{d}\eta \neq 0$. For all $\zeta \in \Omega^2_0(M,\mathfrak{g}^*)$ let us consider $\underline{\mathcal{F}}^*(\zeta) \in \mathcal{E}^{\mathrm{kin}}$, where $\underline{\mathcal{F}}^* : \Omega^2_0(M,\mathfrak{g}^*) \to \mathcal{E}^{\mathrm{kin}}$ is the formal adjoint of the curvature affine differential operator $\underline{\mathcal{F}} : \Gamma^\infty(\mathcal{C}(\Xi)) \to \Omega^2(M,\mathfrak{g})$ (cf. [BDS13, Lemma 2.14]). Notice that $\underline{\mathcal{F}}^*(\zeta)_V = -\delta\zeta$, hence $\underline{\mathcal{F}}^*(\zeta) \in \mathcal{E}^{\mathrm{inv}}$. We obtain for the corresponding functional

$$\mathcal{W}_{\underline{\mathcal{F}}^*(\zeta)}(\lambda') = \mathcal{W}_{\underline{\mathcal{F}}^*(\zeta)}(\lambda)\, e^{-2\pi i\,\langle\zeta,\mathrm{d}\eta\rangle}\ . \tag{3.5}$$

Since $\mathrm{d}\eta \neq 0$ there exists $\zeta \in \Omega^2_0(M,\mathfrak{g}^*)$ such that $\mathcal{W}_{\underline{\mathcal{F}}^*(\zeta)}(\lambda') \neq \mathcal{W}_{\underline{\mathcal{F}}^*(\zeta)}(\lambda)$.

Let us now assume that $\mathrm{d}\eta = 0$. By hypothesis, the corresponding cohomology class $[\eta] \in H^1_{\mathrm{dR}}(M,\mathfrak{g})$ is not included in the Abelian subgroup $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z}) \subseteq H^1_{\mathrm{dR}}(M,\mathfrak{g})$, since otherwise η would be an element in B_G. We prove the statement by contradiction: Assume that $\mathcal{W}_\varphi(\lambda') = \mathcal{W}_\varphi(\lambda)$, for all $\varphi \in \mathcal{E}^{\mathrm{inv}}$. As a consequence, $\big\langle H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)_\mathbb{Z}, [\eta]\big\rangle \subseteq \mathbb{Z}$, which implies that $[\eta]$ defines a homomorphism of Abelian groups $H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)_\mathbb{Z} \to \mathbb{Z}$. Notice that this is an element in the double dual $\mathbb{Z}$-module of $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})$, which is isomorphic to $H^1_{\mathrm{dR}}(M,\mathfrak{g}_\mathbb{Z})$ since the latter is finitely generated and free. This is a contradiction and hence there exists $\varphi \in \mathcal{E}^{\mathrm{inv}}$, such that $\mathcal{W}_\varphi(\lambda') \neq \mathcal{W}_\varphi(\lambda)$. □

Remark 3.3. There is the following relation to the usual Wilson loop observables: Given a smooth loop $\gamma : \mathbb{S}^1 \to M$ we can construct the pull-back bundle $\gamma^*(P)$, which is a principal $U(1)$-bundle over $\mathbb{S}^1$. By construction, we have the commuting diagram

$$\begin{tikzcd} \gamma^*(P) \arrow[r, "\overline{\gamma}"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ \mathbb{S}^1 \arrow[r, "\gamma"] & M \end{tikzcd} \tag{3.6}$$

Notice that $\gamma^*(P)$ is necessarily a trivial bundle and hence there exists a global section $\sigma : \mathbb{S}^1 \to \gamma^*(P)$ of π'. Given any $\lambda \in \Gamma^\infty(\mathcal{C}(\Xi))$, its associated connection form $\omega_\lambda \in \Omega^1(P,\mathfrak{g})$ pulls back to a connection form $\overline{\gamma}^*(\omega_\lambda) \in \Omega^1(\gamma^*(P),\mathfrak{g})$, which can be further pulled back via the section to a $\mathfrak{g}$-valued one-form on $\mathbb{S}^1$, $\sigma^*(\overline{\gamma}^*(\omega_\lambda)) \in \Omega^1(\mathbb{S}^1,\mathfrak{g})$. We call the functional

$$w_\gamma : \Gamma^\infty(\mathcal{C}(\Xi)) \to \mathbb{C}\ ,\quad \lambda \mapsto w_\gamma(\lambda) = e^{\int_{\mathbb{S}^1}\sigma^*(\overline{\gamma}^*(\omega_\lambda))} \tag{3.7}$$

a Wilson loop observable and notice that w_γ does not depend on the choice of trivialization σ. Wilson loop observables satisfy, for all $\lambda \in \Gamma^\infty(\mathcal{C}(\Xi))$ and $\eta \in \Omega^1(M,\mathfrak{g})$,

$$w_\gamma(\lambda+\eta) = w_\gamma(\lambda)\, e^{\int_{\mathbb{S}^1}\gamma^*(\eta)}\ , \tag{3.8}$$

which the reader should compare with (3.2). Hence, if we allow also for distributional sections of the vector dual bundle $\mathcal{C}(\Xi)^\dagger$, the usual Wilson loop observables (3.7) are contained in the class of functionals of exponential type (3.1). Notice, however, that the set of gauge invariant observables $\{\mathcal{W}_\varphi : \varphi \in \mathcal{E}^{\mathrm{inv}}\}$ is sufficiently large to separate gauge equivalence classes of connections, cf. Theorem 3.2. Thus, we do not need to enlarge this set of observables by distributional ones, which upon quantization would lead to singularities.

Let us denote by $G_{(k)} : \Omega^k_0(M,\mathfrak{g}^*) \to \Omega^k(M,\mathfrak{g}^*)$ the causal propagator of the Hodge-d'Alembert operator $\square_{(k)} := \delta\circ\mathrm{d} + \mathrm{d}\circ\delta : \Omega^k(M,\mathfrak{g}^*) \to \Omega^k(M,\mathfrak{g}^*)$, where $k = 0,\dots,\dim(M)$. Given further a bi-invariant pseudo-Riemannian metric h on the structure group G, we can define a presymplectic structure $\tau : \mathcal{E}^{\mathrm{inv}}\times\mathcal{E}^{\mathrm{inv}} \to \mathbb{R}$ on the Abelian group $\mathcal{E}^{\mathrm{inv}}$ by, for all $\varphi,\psi\in\mathcal{E}^{\mathrm{inv}}$,

$$\tau(\varphi,\psi) := \big\langle \varphi_V, G_{(1)}(\psi_V)\big\rangle_h := \int_M \varphi_V\wedge *\big(h^{-1}\big(G_{(1)}(\psi_V)\big)\big)\ , \tag{3.9}$$

where $h^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ is the inverse of h. This presymplectic structure can be derived from the Lagrangian density $\mathcal{L}[\lambda] = -\frac{1}{2}\, h(\underline{\mathcal{F}}(\lambda))\wedge *(\underline{\mathcal{F}}(\lambda))$ by slightly adapting Peierls' method [BDS13, Remark 3.5]. The metric h plays the role of an electric charge constant and it will be fixed throughout this work.

Before we take the quotient of $\mathcal{E}^{\mathrm{inv}}$ by an Abelian subgroup containing the equation of motion, let us study the elements $\psi\in\mathcal{E}^{\mathrm{inv}}$ which lead to central Weyl symbols in the quantum field theory. The Weyl relations (A.1) read $W(\varphi)\,W(\psi) = e^{-i\,\tau(\varphi,\psi)/2}\,W(\varphi+\psi)$. $W(\psi)$ commutes with all other Weyl symbols if and only if $\tau(\mathcal{E}^{\mathrm{inv}},\psi)\subseteq 2\pi\,\mathbb{Z}$. We denote by $\mathcal{N}\subseteq\mathcal{E}^{\mathrm{inv}}$ the Abelian subgroup of all elements $\psi\in\mathcal{E}^{\mathrm{inv}}$ satisfying this condition.

Proposition 3.4. *$\mathcal{N} = \big\{\psi\in\mathcal{E}^{\mathrm{inv}} : \psi_V\in\delta\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)$ and $\big[h^{-1}\big(G_{(1)}(\psi_V)\big)\big]\in 2\pi\, H^1_{\mathrm{dR}}(M,\mathfrak{g}_{\mathbb{Z}})\big\}$.*

Proof. We first prove the inclusion $\supseteq$. Assume that $\psi\in\mathcal{E}^{\mathrm{inv}}$ satisfies the first condition of the Abelian group specified on the right hand side above, i.e. $\psi_V = \delta\zeta$ for some $\zeta\in\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)$. Then $\mathrm{d}(h^{-1}(G_{(1)}(\psi_V))) = h^{-1}(G_{(2)}(\mathrm{d}\delta\zeta)) = h^{-1}(G_{(2)}(\square_{(2)}(\zeta))) = 0$, thus the second condition is well-posed. Using Proposition 3.1 the following holds true,

$$\tau(\mathcal{E}^{\mathrm{inv}},\psi) = \big\langle H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)_{\mathbb{Z}}, \big[h^{-1}\big(G_{(1)}(\psi_V)\big)\big]\big\rangle \subseteq 2\pi\,\mathbb{Z}\ . \tag{3.10}$$

To prove the inclusion $\subseteq$, suppose that $\psi\in\mathcal{E}^{\mathrm{inv}}$ is such that $\tau(\mathcal{E}^{\mathrm{inv}},\psi)\subseteq 2\pi\,\mathbb{Z}$. Since $\mathcal{E}^{\mathrm{inv}}$ contains the Abelian subgroup $\{\varphi\in\mathcal{E}^{\mathrm{kin}} : \varphi_V\in\delta\Omega^2_0(M,\mathfrak{g}^*)\}$, we obtain that $\mathrm{d}(h^{-1}(G_{(1)}(\psi_V))) = 0$. As a consequence of global hyperbolicity and $\square_{(2)}$ being normally hyperbolic, $\mathrm{d}\psi_V = \square_{(2)}(\zeta)$ for some $\zeta\in\Omega^2_0(M,\mathfrak{g}^*)$. Applying d to this equation shows that $\zeta\in\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)$. Applying δ and using that $\delta\psi_V = 0$ we find $\psi_V = \delta\zeta$. The condition $\tau(\mathcal{E}^{\mathrm{inv}},\psi)\subseteq 2\pi\,\mathbb{Z}$ then reads as in (3.10), which implies that $[h^{-1}(G_{(1)}(\psi_V))]\in 2\pi\, H^1_{\mathrm{dR}}(M,\mathfrak{g}_{\mathbb{Z}})$. □

With this characterization it is easy to see that the equation of motion is contained in $\mathcal{N}$.

Lemma 3.5. *$\mathsf{MW}^*\big[\Omega^1_0(M,\mathfrak{g}^*)\big]\subseteq\mathcal{N}$, where $\mathsf{MW}^* = \underline{\mathcal{F}}^*\circ\mathrm{d} : \Omega^1_0(M,\mathfrak{g}^*)\to\mathcal{E}^{\mathrm{kin}}$ is the formal adjoint of Maxwell's affine differential operator $\mathsf{MW} := \delta\circ\underline{\mathcal{F}} : \Gamma^\infty(\mathcal{C}(\Xi))\to\Omega^1(M,\mathfrak{g})$.*

Proof. For any $\zeta\in\Omega^1_0(M,\mathfrak{g}^*)$, $\mathsf{MW}^*(\zeta)_V = -\delta\mathrm{d}\zeta$. As a consequence, $G_{(1)}(\mathsf{MW}^*(\zeta)_V) = -G_{(1)}(\delta\mathrm{d}\zeta) = -G_{(1)}((\square_{(1)}-\mathrm{d}\delta)(\zeta)) = \mathrm{d}\delta G_{(1)}(\zeta)$ and thus $[h^{-1}(G_{(1)}(\mathsf{MW}^*(\zeta)_V))] = 0$. □

The characterization of $\mathcal{N}$ given in Proposition 3.4 is still rather abstract. In particular, it is pretty hard to control the second condition since it involves the causal propagator and hence the equation of motion together with its solution theory. Fortunately, it will be sufficient for us to characterize explicitly only the Abelian subgroup of $\mathcal{N}$ given by

$$\mathcal{N}^0 := \big\{\psi\in\mathcal{E}^{\mathrm{inv}} : \psi_V\in\delta\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)\text{ and }\big[h^{-1}\big(G_{(1)}(\psi_V)\big)\big] = 0\big\}\subseteq\mathcal{N}\ . \tag{3.11}$$

Notice that $\mathcal{N}^0$ can be defined as the set of all $\psi\in\mathcal{E}^{\mathrm{inv}}$ satisfying $\tau(\mathcal{E}^{\mathrm{inv}},\psi) = \{0\}$.

Proposition 3.6. $\mathcal{N}^0 = \big\{\psi \in \mathcal{E}^{\mathrm{inv}} : \psi_V \in \delta\big(\Omega^2_0(M,\mathfrak{g}^*) \cap \mathrm{d}\Omega^1_{\mathrm{tc}}(M,\mathfrak{g}^*)\big)\big\}$, *where the subscript* $_{\mathrm{tc}}$ *stands for forms with timelike compact support.*

Proof. We first show the inclusion $\subseteq$: Let $\psi_V = \delta\zeta$, $\zeta \in \Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)$, be the linear part of $\psi \in \mathcal{N}^0$. The second condition in (3.11) implies that there exists a $\chi' \in C^\infty(M,\mathfrak{g})$, such that $h^{-1}(G_{(1)}(\psi_V)) = \mathrm{d}\chi'$. Absorbing h^{-1} into χ' we obtain the equivalent equation $G_{(1)}(\psi_V) = \mathrm{d}\chi$, for some $\chi \in C^\infty(M,\mathfrak{g}^*)$. Applying δ to both sides leads to $\Box_{(0)}(\chi) = 0$, hence there exists an $\alpha \in C^\infty_{\mathrm{tc}}(M,\mathfrak{g}^*)$ such that $\chi = G_{(0)}(\alpha)$. The original equation implies that $\psi_V = \mathrm{d}\alpha + \Box_{(1)}(\beta)$ for some $\beta \in \Omega^1_{\mathrm{tc}}(M,\mathfrak{g}^*)$. Applying δ gives $\alpha = -\delta\beta$ and the equation simplifies to $\delta\zeta = \psi_V = \delta\mathrm{d}\beta$. Applying d shows that $\zeta = \mathrm{d}\beta$ and hence $\psi_V = \delta\mathrm{d}\beta$, $\beta \in \Omega^1_{\mathrm{tc}}(M,\mathfrak{g}^*)$. The other inclusion $\supseteq$ is easily shown, for all $\mathrm{d}\beta \in \Omega^2_0(M,\mathfrak{g}^*) \cap \mathrm{d}\Omega^1_{\mathrm{tc}}(M,\mathfrak{g}^*)$,

$$G_{(1)}(\delta\mathrm{d}\beta) = \delta\mathrm{d}G_{(1)}(\beta) = (\Box_{(1)} - \mathrm{d}\delta)\big(G_{(1)}(\beta)\big) = -\mathrm{d}\delta G_{(1)}(\beta) \tag{3.12}$$

and hence $[h^{-1}(G_{(1)}(\delta\mathrm{d}\beta))] = 0$. □

Corollary 3.7. $\mathsf{MW}^*[\Omega^1_0(M,\mathfrak{g}^*)] \subseteq \mathcal{N}^0$.

Proof. Follows immediately from the proof of Lemma 3.5. □

There is another interesting Abelian subgroup of $\mathcal{N}$. It is specified by

$$\mathcal{N}^c := \big\{\psi \in \mathcal{E}^{\mathrm{inv}} : \psi_V = 0\big\} \subseteq \mathcal{N}^0 \subseteq \mathcal{N}\ . \tag{3.13}$$

Elements $\psi \in \mathcal{N}^c$ can be identified with those (non-trivial) functionals that give the same result on any two configurations, for all $\lambda, \lambda' \in \Gamma^\infty(\mathcal{C}(\Xi))$, $\mathcal{W}_\psi(\lambda') = \mathcal{W}_\psi(\lambda)$. Similar to [BDS13, Section 6] one can show that $\mathcal{N}^c$ contains the Abelian subgroup $\underline{\mathcal{F}}^*[\Omega^2_{0,\delta}(M,\mathfrak{g}^*)]$, which defines via (3.1) functionals that measure the Chern class of the underlying principal G-bundle, cf. Remark 4.11.

4 The phasespace functor and its quotients

We associate a phasespace, i.e. a presymplectic Abelian group, to any object Ξ in $G-\mathsf{PrBuGlobHyp}$ and study how morphisms in $G-\mathsf{PrBuGlobHyp}$ induce morphisms between phasespaces. Our strategy is to construct first an off-shell phasespace functor, i.e. a functor which does not encode the equation of motion operator MW, and then to characterize consistent quotients of the off-shell phasespaces which contain at least the equation of motion. For the definition of the category PAG of presymplectic Abelian groups we refer to the Appendix, Definition A.1.

Proposition 4.1. *There exists a covariant functor* $\mathfrak{PhSpOff} : G-\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$. *It associates to any object* Ξ *the presymplectic Abelian group* $\mathfrak{PhSpOff}(\Xi) = (\mathcal{E}^{\mathrm{inv}},\tau)$, *where* $\mathcal{E}^{\mathrm{inv}}$ *is given in (3.4) and* τ *in (3.9). To any morphism* $f : \Xi_1 \to \Xi_2$ *the functor associates a morphism in* PAG *via*

$$\mathfrak{PhSpOff}(f) : \mathfrak{PhSpOff}(\Xi_1) \to \mathfrak{PhSpOff}(\Xi_2)\ ,\quad \varphi \mapsto f_*(\varphi)\ , \tag{4.1}$$

where f_* *is the push-forward given in [BDS13, Definition 5.4].*

Proof. The proof can be obtained by following the same steps as in the proof of [BDS13, Theorem 5.5]. □

This covariant functor is not yet the one required in physics since it does not encode the equation of motion. We will address the question of taking quotients of the objects $\mathfrak{PhSpOff}(\Xi)$ by Abelian subgroups $\mathfrak{Q}(\Xi) \subseteq \mathfrak{PhSpOff}(\Xi)$ from a more abstract point of view. This is required to understand if we can take in our present model a quotient by the equation of motion and also certain "electric charges", cf. [BDS13, Section 7]. Eventually, this will decide whether the covariant functor resulting from taking quotients satisfies the locality property (i.e. injectivity of the induced morphisms in PAG) or not.

There are the following restrictions on the choice of the collection $\mathfrak{Q}(\Xi) \subseteq \mathfrak{PhSpOff}(\Xi)$ of Abelian subgroups. First, for $\mathfrak{PhSpOff}(\Xi)/\mathfrak{Q}(\Xi)$ to be an object in PAG (with the induced presymplectic structure) it is necessary and sufficient that $\mathfrak{Q}(\Xi)$ is an Abelian subgroup of $\mathcal{N}^0 \subseteq \mathfrak{PhSpOff}(\Xi)$. Second, for $\mathfrak{PhSpOff}(f) : \mathfrak{PhSpOff}(\Xi_1) \to \mathfrak{PhSpOff}(\Xi_2)$ to induce a morphism on the quotients it is necessary and sufficient that it maps $\mathfrak{Q}(\Xi_1)$ to $\mathfrak{Q}(\Xi_2)$. These conditions can be abstractly phrased as follows.

Definition 4.2. Let C be any category. A **quotient of a covariant functor** $\mathfrak{F} : \mathsf{C} \to \mathsf{PAG}$ is a pair $(\mathfrak{Q}, \iota)$, where $\mathfrak{Q} : \mathsf{C} \to \mathsf{PAG}$ is a covariant functor and $\iota : \mathfrak{Q} \Rightarrow \mathfrak{F}$ is a natural transformation, such that for any object A in C the morphism $\iota_A : \mathfrak{Q}(A) \to \mathfrak{F}(A)$ is injective and its image is contained in the radical of the presymplectic structure in $\mathfrak{F}(A)$.

Remark 4.3. Notice that if the category C is $G{-}\mathsf{PrBuGlobHyp}$ and the functor $\mathfrak{F}$ is $\mathfrak{PhSpOff}$ we recover exactly the situation explained before Definition 4.2. We have formulated the definition in this generality, since in Section 5 we encounter also the case of a category C different from $G{-}\mathsf{PrBuGlobHyp}$.

Remark 4.4. Let $(\mathfrak{Q}, \iota)$ be any quotient of a covariant functor $\mathfrak{F} : \mathsf{C} \to \mathsf{PAG}$. Then the presymplectic structure in $\mathfrak{Q}(A)$ is trivial for all objects A in C. For any morphism $f : A_1 \to A_2$ in C the morphism $\mathfrak{Q}(f)$ is uniquely determined by $\mathfrak{F}(f)$, since ι is a natural transformation with all ι_A injective. We will suppress in the following the injections ι_A and consider $\mathfrak{Q}(A)$ as a presymplectic Abelian subgroup of $\mathfrak{F}(A)$, for all A.

Proposition 4.5. *Let $(\mathfrak{Q}, \iota)$ be a quotient of a covariant functor $\mathfrak{F} : \mathsf{C} \to \mathsf{PAG}$. Then there exists a covariant functor $\mathfrak{F}/\mathfrak{Q} : \mathsf{C} \to \mathsf{PAG}$. It associates to any object A in C the object $\mathfrak{F}(A)/\mathfrak{Q}(A)$ in PAG. To any morphism $f : A_1 \to A_2$ in C the functor associates the morphism $\mathfrak{F}(A_1)/\mathfrak{Q}(A_1) \to \mathfrak{F}(A_2)/\mathfrak{Q}(A_2)$ in PAG that is canonically induced by $\mathfrak{F}(f)$.*

Proof. For any object A in C the quotient $\mathfrak{F}(A)/\mathfrak{Q}(A)$ is an object in PAG, since $\mathfrak{Q}(A)$ is a presymplectic Abelian subgroup of the radical of the presymplectic structure in $\mathfrak{F}(A)$ (remember that we suppress the injections ι_A). For any morphism $f : A_1 \to A_2$ in C the morphism $\mathfrak{F}(f) : \mathfrak{F}(A_1) \to \mathfrak{F}(A_2)$ induces a well-defined morphism between the quotients, since by hypothesis (i.e. the natural transformation is injective) $\mathfrak{Q}(A_1)$ is mapped to $\mathfrak{Q}(A_2)$. $\square$

It remains to provide explicit examples of quotients of $\mathfrak{PhSpOff} : G{-}\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$. The following example is standard, since it describes within the terminology developed above the quotient by the equation of motion.

Proposition 4.6. *Let $\mathfrak{MW} : G{-}\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$ be the covariant functor defined as follows: To any object Ξ in $G{-}\mathsf{PrBuGlobHyp}$ it associates the object $\mathfrak{MW}(\Xi) = (\mathsf{MW}^*[\Omega^1_0(M, \mathfrak{g}^*)], 0)$ in PAG, where the presymplectic structure is trivial. To any morphism $f : \Xi_1 \to \Xi_2$ in $G{-}\mathsf{PrBuGlobHyp}$ the functor associates the morphism in PAG defined by*

$$\mathfrak{MW}(f) : \mathfrak{MW}(\Xi_1) \to \mathfrak{MW}(\Xi_2)\ , \quad \mathsf{MW}^*_1(\eta) \mapsto f_*\big(\mathsf{MW}^*_1(\eta)\big)\ . \tag{4.2}$$

Let $\iota = \{\iota_\Xi\}$ be the canonical injections $\iota_\Xi : \mathfrak{MW}(\Xi) \to \mathfrak{PhSpOff}(\Xi)$. Then $(\mathfrak{MW}, \iota)$ is a quotient of $\mathfrak{PhSpOff} : G{-}\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$.

Proof. The morphism (4.2) is well-defined, since $f_*[\mathsf{MW}^*_1[\Omega^1_0(M_1, \mathfrak{g}^*)]] \subseteq \mathsf{MW}^*_2[\Omega^1_0(M_2, \mathfrak{g}^*)]$, cf. the proof of [BDS13, Theorem 5.5]. The canonical injections ι_Ξ are injective and they are morphisms in PAG, since $\mathsf{MW}^*[\Omega^1_0(M, \mathfrak{g}^*)]$ lies in $\mathcal{N}^0 \subseteq \mathfrak{PhSpOff}(\Xi)$, cf. Corollary 3.7. $\square$

Using Proposition 4.5 we construct a covariant functor $\mathfrak{PhSp} := \mathfrak{PhSpOff}/\mathfrak{MW} : G{-}\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$. This functor describes exactly the usual gauge invariant on-shell phasespaces, i.e. for any object Ξ in $G{-}\mathsf{PrBuGlobHyp}$ we have $\mathfrak{PhSp}(\Xi) = (\mathcal{E}^{\mathrm{inv}}/\mathsf{MW}^*[\Omega^1_0(M, \mathfrak{g}^*)], \tau)$ with $\mathcal{E}^{\mathrm{inv}}$ given in (3.4) and τ in (3.9). As we will comment below, the functor $\mathfrak{PhSp}$ has many desired properties of a locally covariant field theory (the causality property and the time-slice axiom), however it does not satisfy the locality property.

Definition 4.7. Let C be any category. A covariant functor $\mathfrak{F} : \mathsf{C} \to \mathsf{PAG}$ is said to satisfy the **locality property**, if it is a functor to the subcategory $\mathsf{PAG}^{\mathrm{inj}}$ where all morphisms are injective, cf. Definition A.1.

Proposition 4.8. *The covariant functor $\mathfrak{PhSp} := \mathfrak{PhSpOff}/\mathfrak{MW} : G{-}\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$ does not satisfy the locality property.*

Proof. We design a suitable counterexample following [BDS13, Remark 5.6]: Let us take any object Ξ_2 in $G-\mathsf{PrBuGlobHyp}$ such that $(M_2, \mathfrak{o}_2, g_2, \mathfrak{t}_2)$ is the Minkowski spacetime of dimension $m = \dim(M_2) \geq 4$. Let us denote by Ξ_1 the object in $G-\mathsf{PrBuGlobHyp}$ obtained by restricting all geometric data to the causally compatible and globally hyperbolic open subset $M_1 := M_2 \backslash J_{M_2}(\{0\})$, where $\{0\}$ is the set consisting of some point 0 in M_2. The canonical embedding provides us with a morphism $f : \Xi_1 \to \Xi_2$ in $G-\mathsf{PrBuGlobHyp}$.

We will now construct an element $\zeta = \mathrm{d}\beta \in \Omega^2_0(M_1, \mathfrak{g}^*) \cap \mathrm{d}\Omega^1_{\mathrm{tc}}(M_1, \mathfrak{g}^*)$ that is not contained in $\mathrm{d}\Omega^1_0(M_1, \mathfrak{g}^*)$. This element is used later to show that $\mathfrak{PhSp}(f)$ is not injective. Notice that M_1 is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^{m-2}$. This allows us to introduce a time coordinate t and a space coordinate x on the $\mathbb{R}^2$ factor of M_1. Let $0 \neq [\alpha] \in H^1_{0\,\mathrm{dR}}(\mathbb{R})$ and $\alpha \in \Omega^1_{0,\mathrm{d}}(\mathbb{R})$ any representative. We pull α back via t and x and define an element $\zeta := i\,\alpha_t \wedge \alpha_x \in \Omega^2_{0,\mathrm{d}}(M_1, \mathfrak{g}^*)$, where the complex unit i is used to make ζ valued in $\mathfrak{g}^* \simeq i\,\mathbb{R}$. Since $H^1_{\mathrm{dR}}(M_1) = \{0\}$ and $\mathrm{d}\alpha_x = 0$ there exists a $\gamma \in C^\infty(M_1)$, such that $\alpha_x = -\mathrm{d}\gamma$. Then $\zeta = \mathrm{d}(i\gamma\,\alpha_t) =: \mathrm{d}\beta$, where $\beta := i\gamma\,\alpha_t \in \Omega^1_{\mathrm{tc}}(M_1, \mathfrak{g}^*)$ is the desired element. Indeed, $\zeta = \mathrm{d}\beta \notin \mathrm{d}\Omega^1_0(M_1, \mathfrak{g}^*)$ since $\int_M \zeta \wedge \mathrm{pr}^*(\nu_{\mathbb{S}^{m-2}}) = i(\int_{\mathbb{R}} \alpha)^2 \neq 0$, where $\mathrm{pr} : M_1 \to \mathbb{S}^{m-2}$ is the projection to the sphere factor and $\nu_{\mathbb{S}^{m-2}}$ is the normalized volume form on $\mathbb{S}^{m-2}$. Notice further that $H^2_{0\,\mathrm{dR}}(M_1, \mathfrak{g}^*) \simeq H^{m-2}_{\mathrm{dR}}(M_1, \mathfrak{g}^*) \simeq \mathfrak{g}^* \simeq i\,\mathbb{R}$ is one-dimensional and hence $\Omega^2_0(M_1, \mathfrak{g}^*) \cap \mathrm{d}\Omega^1_{\mathrm{tc}}(M_1, \mathfrak{g}^*) = \Omega^2_{0,\mathrm{d}}(M_1, \mathfrak{g}^*)$ for our choice of M_1.

Let us consider $\underline{\mathcal{F}}^*_1(\zeta)$, which is an element in $\mathcal{N}^0_1 \subseteq \mathcal{E}^{\mathrm{inv}}_1$ since $\underline{\mathcal{F}}^*_1(\zeta)_V = -\delta\mathrm{d}\beta$, cf. Proposition 3.6. Notice that the class $[\underline{\mathcal{F}}^*_1(\zeta)] \in \mathfrak{PhSp}(\Xi_1)$ is not trivial: If there would exist an $\eta \in \Omega^1_0(M_1, \mathfrak{g}^*)$ such that $\underline{\mathcal{F}}^*_1(\zeta) = \mathsf{MW}^*_1(\eta)$, then taking the linear part gives $\delta\mathrm{d}\beta = \delta\mathrm{d}\eta$. Applying d implies $\mathrm{d}\beta = \mathrm{d}\eta$ which is a contradiction since by construction $[\zeta] = [\mathrm{d}\beta] \neq 0$ in $H^2_{0\,\mathrm{dR}}(M_1, \mathfrak{g}^*)$. Applying the morphism (4.1) corresponding to $f : \Xi_1 \to \Xi_2$ we obtain

$$f_*\big(\underline{\mathcal{F}}^*_1(\zeta)\big) = \underline{\mathcal{F}}^*_2\big(\underline{f}_*(\zeta)\big) = \underline{\mathcal{F}}^*_2(\mathrm{d}\eta) = \mathsf{MW}^*_2(\eta)\;. \tag{4.3}$$

We have used that the cohomology group $H^2_{0\,\mathrm{dR}}(M_2, \mathfrak{g}^*)$ is trivial and hence for $\underline{f}_*(\zeta) \in \Omega^2_{0,\mathrm{d}}(M_2, \mathfrak{g}^*)$ there exists an $\eta \in \Omega^1_0(M_2, \mathfrak{g}^*)$ such that $\mathrm{d}\eta = \underline{f}_*(\zeta)$. As a consequence, $\mathfrak{PhSp}(f)([\underline{\mathcal{F}}^*_1(\zeta)]) = 0$ and $\mathfrak{PhSp}(f)$ is not injective. □

This proposition shows that the usual on-shell phasespace functor $\mathfrak{PhSp} : G-\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$ is not locally covariant in the sense of [BFV03], since it violates the locality property. We have traced back this failure to the existence of non-trivial elements in $\mathcal{N}^0_1/\mathsf{MW}^*_1[\Omega^1_0(M_1, \mathfrak{g}^*)] \subseteq \mathcal{E}^{\mathrm{inv}}_1/\mathsf{MW}^*_1[\Omega^1_0(M_1, \mathfrak{g}^*)]$ which map via a suitably designed morphism $\mathfrak{PhSp}(f)$ to the trivial class in $\mathcal{E}^{\mathrm{inv}}_2/\mathsf{MW}^*_2[\Omega^1_0(M_2, \mathfrak{g}^*)]$. This result might suggest that we can restore injectivity by taking quotients larger than $(\mathfrak{MW}, \iota)$, cf. [BDS13, Section 7] for a similar strategy. Notice that any physically reasonable quotient $(\mathfrak{Q}, \iota)$ should be such that for any object Ξ in $G-\mathsf{PrBuGlobHyp}$ we have $\mathfrak{MW}(\Xi) \subseteq \mathfrak{Q}(\Xi)$, since otherwise the equation of motion is not encoded. This turns out to be impossible due to the following

Theorem 4.9. *There exists no quotient $(\mathfrak{Q}, \iota)$ of $\mathfrak{PhSpOff} : G-\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$, such that*

1.) For any object Ξ in $G-\mathsf{PrBuGlobHyp}$ we have $\mathfrak{MW}(\Xi) \subseteq \mathfrak{Q}(\Xi)$,

2.) $\mathfrak{PhSpOff}/\mathfrak{Q} : G-\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$ satisfies the locality property.

Proof. The strategy for the proof is as follows: We will construct three objects Ξ_i, $i = 1, 2, 3$, and two morphisms $f_j : \Xi_3 \to \Xi_j$, $j = 1, 2$, in $G-\mathsf{PrBuGlobHyp}$, such that injectivity of $\mathfrak{PhSpOff}(f_2)$ on the quotients requires $\mathfrak{Q}(\Xi_3)$ in such a way that $\mathfrak{PhSpOff}(f_1)$ is not well-defined on the quotients.

To this avail, we consider $M_1 := \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^{m-2}$ with $m \geq 4$ and $g_1 := -dt \otimes dt + d\phi \otimes d\phi + g_{\mathbb{S}^{m-2}}$, where t is a (time) coordinate on $\mathbb{R}$, $\phi \in (0, 2\pi)$ is an angle coordinate on $\mathbb{S}^1$ and $g_{\mathbb{S}^{m-2}}$ is the canonical metric on the unit $\mathbb{S}^{m-2}$-sphere. Let us further consider $M_2 := \mathbb{R} \times \mathbb{R}^{m-1}$ with $g_2 := -dt \otimes dt + \alpha(r^2)\, \sum_{i=1}^{m-1} dx^i \otimes dx^i + \beta(r^2)\, dr \otimes dr$, where x^i are Cartesian coordinates, $r = \sqrt{\sum_{i=1}^{m-1} x^i\, x^i}$ is the radius, $\alpha : \mathbb{R} \to \mathbb{R}$ is a strictly positive smooth function, such that $\alpha(\xi) = 1$ for $\xi < 1$ and $\alpha(\xi) = \xi^{-1}$ for $\xi > 2$, and $\beta : \mathbb{R} \to \mathbb{R}$ is a positive smooth function, such that $\beta(\xi) = 0$ for $\xi < 1$ and $\beta(\xi) = 1 - \xi^{-1}$ for $\xi > 2$. Notice that, for $r^2 < 1$, $g_2 = -dt \otimes dt + \sum_{i=1}^{m-1} dx^i \otimes dx^i$ is the Minkowski metric and that, for $r^2 > 2$, $g_2 = -dt \otimes dt + dr \otimes dr + g_{\mathbb{S}^{m-2}}$ formally looks like g_1.

Let us define M_3 as the Cauchy development of $\{0\} \times I \times \mathbb{S}^{m-2}$ in (M_1, g_1), where I is some open interval in $(0, 2\pi)$. Notice that $(M_3, g_3 := g_1|_{M_3})$ is a causally compatible and globally hyperbolic open subset of (M_1, g_1). We denote by $\underline{f_1} : (M_3, g_3) \to (M_1, g_1)$ the isometric embedding. Furthermore, there exists an isometric embedding $\underline{f_2} : (M_3, g_3) \to (M_2, g_2)$ into the region of M_2 specified by $r^2 > 2$, such that the image is causally compatible and open. We can equip M_i, $i = 1, 2, 3$, with orientations and time-orientations, such that the isometric embeddings preserve those. Furthermore, taking the trivial principal G-bundles $P_i = M_i \times G$, $i = 1, 2, 3$, we can construct three objects Ξ_i, $i = 1, 2, 3$, and two morphisms $f_j := (\underline{f_j}, \mathrm{id}) : M_3 \times G \to M_j \times G$, $j = 1, 2$, in $G-\mathsf{PrBuGlobHyp}$.

The arguments in the proof of Proposition 4.8 show that for $\mathfrak{PhSpOff}(f_2)$ to be injective on the quotients (with $\mathfrak{MW}(\Xi_2) \subseteq \mathfrak{Q}(\Xi_2)$) it is necessary that $\underline{\mathcal{F}}_3^*[\Omega^2_{0,\mathrm{d}}(M_3, \mathfrak{g}^*)] \subseteq \mathfrak{Q}(\Xi_3)$. We finish the proof by showing that $\underline{\mathcal{F}}_3^*[\Omega^2_{0,\mathrm{d}}(M_3, \mathfrak{g}^*)]$ is not mapped to $\mathcal{N}_1$ (and in particular not to $\mathcal{N}_1^0$) via $\mathfrak{PhSpOff}(f_1)$. Remember that $\mathrm{d}\Omega^1_0(M_3, \mathfrak{g}^*) \neq \Omega^2_{0,\mathrm{d}}(M_3, \mathfrak{g}^*) = \Omega^2_0(M_3, \mathfrak{g}^*) \cap \mathrm{d}\Omega^1_{\mathrm{tc}}(M_3, \mathfrak{g}^*)$ on our specific choice of M_3, cf. proof of Proposition 4.8.

Let $\zeta \in \Omega^2_{0,\mathrm{d}}(M_3, \mathfrak{g}^*)$ be any representative of a non-trivial class $[\zeta] \in H^2_{0\,\mathrm{dR}}(M_3, \mathfrak{g}^*)$. Under the morphism $\mathfrak{PhSpOff}(f_1)$ the element $\underline{\mathcal{F}}_3^*(\zeta)$ is mapped to $\underline{\mathcal{F}}_1^*(\tilde{\zeta})$, where $\tilde{\zeta} := \underline{f_1}_*(\zeta) \in \Omega^2_{0,\mathrm{d}}(M_1, \mathfrak{g}^*)$. If we could show that $\tilde{\zeta} \notin \mathrm{d}\Omega^1_{\mathrm{tc}}(M_1, \mathfrak{g}^*) = \mathrm{d}\Omega^1_0(M_1, \mathfrak{g}^*)$ (the last equality holds since M_1 has compact Cauchy surfaces), then $[h^{-1}(G_{(1)\,1}(\delta\tilde{\zeta}))] \neq 0$ would be non-trivial. Using the possibility to rescale ζ by any element in $\mathbb{R}$ this would show that the discrete condition in Proposition 3.4 is violated and hence that $\mathfrak{PhSpOff}(f_1)$ does not map $\underline{\mathcal{F}}_3^*[\Omega^2_{0,\mathrm{d}}(M_3, \mathfrak{g}^*)]$ to $\mathcal{N}_1$.

Thus, it remains to show that $\tilde{\zeta} \notin \mathrm{d}\Omega^1_0(M_1, \mathfrak{g}^*)$. Let us consider the closed form $\mathrm{pr}_1^*(\nu_{\mathbb{S}^{m-2}}) \in \Omega^{m-2}_{\mathrm{d}}(M_1)$, where $\mathrm{pr}_1 : M_1 \to \mathbb{S}^{m-2}$ denotes the projection on the sphere factor and $\nu_{\mathbb{S}^{m-2}}$ is the normalized volume form on the sphere. Introducing also the projection $\mathrm{pr}_3 : M_3 \to \mathbb{S}^{m-2}$ we obtain

$$\int_{M_1} \tilde{\zeta} \wedge \mathrm{pr}_1^*(\nu_{\mathbb{S}^{m-2}}) = \int_{M_3} \zeta \wedge \mathrm{pr}_3^*(\nu_{\mathbb{S}^{m-2}}) \neq 0 \,, \tag{4.4}$$

since ζ has been constructed in the proof of Proposition 4.8 such that it has a non-trivial integral with $\mathrm{pr}_3^*(\nu_{\mathbb{S}^{m-2}})$. □

We consider for the remaining part of this section the usual on-shell phasespace functor $\mathfrak{PhSp} = \mathfrak{PhSpOff}/\mathfrak{MW} : G-\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$, with the quotient $(\mathfrak{MW}, \iota)$ given in Proposition 4.6. Analogously to [BDS13, Theorem 5.7], the fact that the presymplectic structure τ is given by the causal propagator (cf. (3.9)) implies that the covariant functor $\mathfrak{PhSp}$ satisfies the classical causality property. Following the steps in the proof of [BDS13, Theorem 5.8] one also obtains that $\mathfrak{PhSp}$ satisfies the classical time-slice axiom.

Let us finally comment on the quantization of the phasespace functor $\mathfrak{PhSp}$. Due to Theorem A.5 we can construct a covariant functor $\mathfrak{A} : G-\mathsf{PrBuGlobHyp} \to C^*\mathsf{Alg}$ by composing the covariant functors $\mathfrak{PhSp}$ and $\mathfrak{CCR}$, i.e. $\mathfrak{A} := \mathfrak{CCR} \circ \mathfrak{PhSp}$. The functor $\mathfrak{A}$ describes the association of C^*-algebras of observables $\mathfrak{A}(\Xi)$ to objects Ξ in $G-\mathsf{PrBuGlobHyp}$. The validity of the classical causality property and of the classical time-slice axiom for the phasespace functor $\mathfrak{PhSp}$ implies the quantum causality property and the quantum time-slice axiom due to the construction of the functor $\mathfrak{CCR}$. We can summarize this construction as follows:

Theorem 4.10. *There exists a covariant functor $\mathfrak{A} := \mathfrak{CCR} \circ \mathfrak{PhSp} : G-\mathsf{PrBuGlobHyp} \to C^*\mathsf{Alg}$ describing the C^*-algebras of observables for quantized principal G-connections. The covariant functor $\mathfrak{A}$ satisfies the quantum causality property and the quantum time-slice axiom. Furthermore, for each object Ξ in $G-\mathsf{PrBuGlobHyp}$ the C^*-algebra $\mathfrak{A}(\Xi)$ is a quantization of an algebra of functionals on $\Gamma^\infty(\mathcal{C}(\Xi))$ that separates gauge equivalence classes of connections (cf. Theorem 3.2). However, $\mathfrak{A}$ does not satisfy the locality property, i.e. $\mathfrak{A}$ is not a covariant functor to the subcategory $C^*\mathsf{Alg}^{\mathrm{inj}}$.*

Remark 4.11. In analogy to [BDS13, Section 6] we can construct a natural transformation Ψ^{mag} from the singular homology functor $\mathfrak{H}_2$ to the functor $\mathfrak{A}$. The interpretation of this natural transformation is that of a topological quantum field measuring the Chern class of the principal G-bundle. To be precise, let us denote by $\mathfrak{H}_2 : G-\mathsf{PrBuGlobHyp} \to \mathsf{Monoid}$ the covariant functor associating to any object Ξ in $G-\mathsf{PrBuGlobHyp}$ the singular homology group $H_2(M, \mathfrak{g}^*)$ (considered as a monoid with respect to $+$) of the base space. To any morphism $f : \Xi_1 \to \Xi_2$ in $G-\mathsf{PrBuGlobHyp}$ the functor associates the usual morphism of singular homology groups, considered as a morphism in the category Monoid. We further use the forgetful functor $C^*\mathsf{Alg} \to \mathsf{Monoid}$, which forgets all structures of C^*-algebras (but the multiplication) and turns the multiplication into a monoid structure. With a slight abuse of notation we use the same symbol $\mathfrak{A}$ to denote the covariant functor $\mathfrak{A} : G-\mathsf{PrBuGlobHyp} \to \mathsf{Monoid}$. With $\mathcal{K} : H_2(M, \mathfrak{g}^*) \to H^2_{0\,\mathrm{dR}^*}(M, \mathfrak{g}^*)$ denoting the natural isomorphism described in [BDS13, Section 6] we can define for each object Ξ in $G-\mathsf{PrBuGlobHyp}$ a map

$$\Psi^{\mathrm{mag}}_{\Xi} : \mathfrak{H}_2(\Xi) \to \mathfrak{A}(\Xi)\ , \quad \sigma \mapsto W\big([\underline{\mathcal{F}}^*(\mathcal{K}(\sigma))]\big)\ . \tag{4.5}$$

Notice that $\Psi^{\mathrm{mag}}_{\Xi}$ is a morphism of monoids, since, for all $\sigma, \sigma' \in \mathfrak{H}_2(\Xi)$,

$$\begin{aligned} W\big([\underline{\mathcal{F}}^*(\mathcal{K}(\sigma+\sigma'))]\big) &= W\big([\underline{\mathcal{F}}^*(\mathcal{K}(\sigma))] + [\underline{\mathcal{F}}^*(\mathcal{K}(\sigma'))]\big) \\ &= W\big([\underline{\mathcal{F}}^*(\mathcal{K}(\sigma))]\big)\, W\big([\underline{\mathcal{F}}^*(\mathcal{K}(\sigma'))]\big)\ , \end{aligned} \tag{4.6}$$

where in the last equality we have used the Weyl relation (A.1) and the fact that $\tau([\underline{\mathcal{F}}^*(\mathcal{K}(\sigma))], [\underline{\mathcal{F}}^*(\mathcal{K}(\sigma'))]) = 0$, which follows from $\underline{\mathcal{F}}^*(\mathcal{K}(\sigma))_V = 0$. The collection $\Psi^{\mathrm{mag}} = \{\Psi^{\mathrm{mag}}_{\Xi}\}$ is then a natural transformation from $\mathfrak{H}_2$ to $\mathfrak{A}$ that associates to elements in the second singular homology group observables that can measure the Chern class of the principal G-bundle.

5 The locality property in Haag-Kastler-type quantum field theories

We have shown in Theorem 4.9 that there exists no quotient $(\mathfrak{Q}, \iota)$ (containing Maxwell's equations) of the off-shell phasespace functor $\mathfrak{PhSpOff} : G-\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$, such that the covariant functor $\mathfrak{PhSpOff}/\mathfrak{Q}$ satisfies the locality property. In this section we shall prove that if we fix any object $\widehat{\Xi} = ((\widehat{M}, \widehat{\mathfrak{o}}, \widehat{g}, \widehat{\mathfrak{t}}), (\widehat{P}, \widehat{r}))$ of the category $G-\mathsf{PrBuGlobHyp}$ and consider a suitable category of subsets of $\widehat{M}$, then there exists a quotient such that the corresponding phasespace functor satisfies the locality property. This setting does of course not cover the full generality of locally covariant quantum field theory, however, it provides us with a quantum field theory in the sense of Haag and Kastler (generalized to curved spacetimes), where the focus is on associating algebras to suitable subsets of a fixed spacetime in a coherent way.

Let us fix any object $\widehat{\Xi} = ((\widehat{M}, \widehat{\mathfrak{o}}, \widehat{g}, \widehat{\mathfrak{t}}), (\widehat{P}, \widehat{r}))$ of the category $G-\mathsf{PrBuGlobHyp}$. We denote by $\mathsf{Sub}_{\widehat{\Xi}}$ the following category: The objects in $\mathsf{Sub}_{\widehat{\Xi}}$ are causally compatible and globally hyperbolic open subsets of $\widehat{M}$. The morphisms in $\mathsf{Sub}_{\widehat{\Xi}}$ are given by the subset relation $\subseteq$, i.e. for any two objects M_1, M_2 in $\mathsf{Sub}_{\widehat{\Xi}}$ there is a unique morphism $M_1 \to M_2$ if and only if $M_1 \subseteq M_2$. Notice that by definition there exists for any object M in $\mathsf{Sub}_{\widehat{\Xi}}$ a unique morphism $M \to \widehat{M}$, i.e. $\widehat{M}$ is a terminal object in $\mathsf{Sub}_{\widehat{\Xi}}$. We interpret $\widehat{M}$ physically as the whole spacetime (the universe). There exists further a covariant functor $\mathfrak{Pull}_{\widehat{\Xi}} : \mathsf{Sub}_{\widehat{\Xi}} \to G-\mathsf{PrBuGlobHyp}$: To any object M in $\mathsf{Sub}_{\widehat{\Xi}}$ the functor associates the object $\mathfrak{Pull}_{\widehat{\Xi}}(M)$ in $G-\mathsf{PrBuGlobHyp}$ obtained by pulling back all the geometric data of $\widehat{\Xi}$ to M. To any morphism $M_1 \to M_2$ in $\mathsf{Sub}_{\widehat{\Xi}}$ the functor associates the canonical embedding $\mathfrak{Pull}_{\widehat{\Xi}}(M_1) \to \mathfrak{Pull}_{\widehat{\Xi}}(M_2)$, which is a morphism in $G-\mathsf{PrBuGlobHyp}$. We can compose the covariant functor $\mathfrak{Pull}_{\widehat{\Xi}} : \mathsf{Sub}_{\widehat{\Xi}} \to G-\mathsf{PrBuGlobHyp}$ with the *on-shell* phasespace functor $\mathfrak{PhSp} := \mathfrak{PhSpOff}/\mathfrak{MW} : G-\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$ and obtain a covariant functor $\mathfrak{PhSp}_{\widehat{\Xi}} := \mathfrak{PhSp} \circ \mathfrak{Pull}_{\widehat{\Xi}} : \mathsf{Sub}_{\widehat{\Xi}} \to \mathsf{PAG}$. Physically, $\mathfrak{PhSp}_{\widehat{\Xi}}$ associates the on-shell phasespace to any causally compatible and globally hyperbolic open subset M of the terminal object $\widehat{M}$.

We shall make heavy use of the following fact: Let $M_1 \to M_2$ be any morphism in $\mathsf{Sub}_{\widehat{\Xi}}$, then by definition of the category $\mathsf{Sub}_{\widehat{\Xi}}$ there exists a commutative diagram:

$$\begin{tikzcd} & \widehat{M} & \\ M_1 \arrow[ur] \arrow[rr] & & M_2 \arrow[ul] \end{tikzcd} \tag{5.1}$$

Due to functoriality this induces the commutative diagram:

$$\begin{tikzcd} & \mathfrak{PhSp}_{\hat{\equiv}}(\widehat{M}) & \\ \mathfrak{PhSp}_{\hat{\equiv}}(M_1) \arrow[ur] \arrow[rr] & & \mathfrak{PhSp}_{\hat{\equiv}}(M_2) \arrow[ul] \end{tikzcd} \tag{5.2}$$

Lemma 5.1. *For any object M in $\mathsf{Sub}_{\hat{\equiv}}$ define $\mathfrak{Ker}_{\hat{\equiv}}(M)$ to be the object in PAG given by the kernel of the canonical map $\mathfrak{PhSp}_{\hat{\equiv}}(M) \to \mathfrak{PhSp}_{\hat{\equiv}}(\widehat{M})$. For any morphism $M_1 \to M_2$ in $\mathsf{Sub}_{\hat{\equiv}}$ define the morphism $\mathfrak{Ker}_{\hat{\equiv}}(M_1) \to \mathfrak{Ker}_{\hat{\equiv}}(M_2)$ in PAG by restriction of $\mathfrak{PhSp}_{\hat{\equiv}}(M_1) \to \mathfrak{PhSp}_{\hat{\equiv}}(M_2)$ to $\mathfrak{Ker}_{\hat{\equiv}}(M_1)$. Then $\mathfrak{Ker}_{\hat{\equiv}} : \mathsf{Sub}_{\hat{\equiv}} \to \mathsf{PAG}$ is a covariant functor. Let further $\iota = \{\iota_M\}$ be the canonical injections $\iota_M : \mathfrak{Ker}_{\hat{\equiv}}(M) \to \mathfrak{PhSp}_{\hat{\equiv}}(M)$, then $(\mathfrak{Ker}_{\hat{\equiv}}, \iota)$ is a quotient of $\mathfrak{PhSp}_{\hat{\equiv}} : \mathsf{Sub}_{\hat{\equiv}} \to \mathsf{PAG}$.*

Proof. For any object M in $\mathsf{Sub}_{\hat{\equiv}}$, the kernel $\mathfrak{Ker}_{\hat{\equiv}}(M)$ of the canonical map $\mathfrak{PhSp}_{\hat{\equiv}}(M) \to \mathfrak{PhSp}_{\hat{\equiv}}(\widehat{M})$ is an object in PAG with the presymplectic structure induced from $\mathfrak{PhSp}_{\hat{\equiv}}(M)$, that becomes trivial in $\mathfrak{Ker}_{\hat{\equiv}}(M)$. Furthermore, due to the commutative diagram (5.2), the restriction of any morphism $\mathfrak{PhSp}_{\hat{\equiv}}(M_1) \to \mathfrak{PhSp}_{\hat{\equiv}}(M_2)$ to $\mathfrak{Ker}_{\hat{\equiv}}(M_1)$ induces a morphism $\mathfrak{Ker}_{\hat{\equiv}}(M_1) \to \mathfrak{Ker}_{\hat{\equiv}}(M_2)$ in PAG. The composition property of $\mathfrak{PhSp}_{\hat{\equiv}}$ is preserved and hence $\mathfrak{Ker}_{\hat{\equiv}} : \mathsf{Sub}_{\hat{\equiv}} \to \mathsf{PAG}$ is a covariant functor. The family $\iota = \{\iota_M\}$ is obviously a natural transformation, such that ι_M is injective and maps to the radical $\mathcal{N}^0 \subseteq \mathfrak{PhSp}_{\hat{\equiv}}(M)$. Hence, $(\mathfrak{Ker}_{\hat{\equiv}}, \iota)$ is a quotient of $\mathfrak{PhSp}_{\hat{\equiv}} : \mathsf{Sub}_{\hat{\equiv}} \to \mathsf{PAG}$. $\square$

We can now prove the main statement of this section.

Theorem 5.2. *The covariant functor $\mathfrak{PhSp}^0_{\hat{\equiv}} := \mathfrak{PhSp}_{\hat{\equiv}} / \mathfrak{Ker}_{\hat{\equiv}} : \mathsf{Sub}_{\hat{\equiv}} \to \mathsf{PAG}$ satisfies the locality property, the causality property and the time-slice axiom.*

Proof. The causality property and the time-slice axiom are induced since $\mathfrak{PhSp} : G\text{--}\mathsf{PrBuGlobHyp} \to \mathsf{PAG}$ satisfies these properties. To prove the locality property we have to check if all morphism $\mathfrak{PhSp}^0_{\hat{\equiv}}(M_1) \to \mathfrak{PhSp}^0_{\hat{\equiv}}(M_2)$ are injective. Since by definition $\mathfrak{PhSp}^0_{\hat{\equiv}}(M_i) = \mathfrak{PhSp}_{\hat{\equiv}}(M_i)/\mathfrak{Ker}_{\hat{\equiv}}(M_i)$, for $i = 1, 2$, and due to the commutative diagram (5.2) we obtain the commutative diagram

$$\begin{tikzcd} & \mathfrak{PhSp}^0_{\hat{\equiv}}(\widehat{M}) = \mathfrak{PhSp}_{\hat{\equiv}}(\widehat{M}) & \\ \mathfrak{PhSp}^0_{\hat{\equiv}}(M_1) \arrow[ur] \arrow[rr] & & \mathfrak{PhSp}^0_{\hat{\equiv}}(M_2) \arrow[ul] \end{tikzcd} \tag{5.3}$$

where both upwards going arrows are injective (by construction). As a consequence, the horizontal arrow has to be injective too, which proves the locality property. $\square$

As a consequence of this theorem we can consider $\mathfrak{PhSp}^0_{\hat{\equiv}}$ as a covariant functor $\mathfrak{PhSp}^0_{\hat{\equiv}} : \mathsf{Sub}_{\hat{\equiv}} \to \mathsf{PAG}^{\mathrm{inj}}$, where the latter category is defined in Definition A.1. Using further Theorem A.8 we obtain a covariant functor $\mathfrak{A}^0 := \mathfrak{CCR} \circ \mathfrak{PhSp}^0_{\hat{\equiv}} : \mathsf{Sub}_{\hat{\equiv}} \to C^*\mathsf{Alg}^{\mathrm{inj}}$.

Corollary 5.3. *The covariant functor $\mathfrak{A}^0 : \mathsf{Sub}_{\hat{\equiv}} \to C^*\mathsf{Alg}^{\mathrm{inj}}$ satisfies the causality property, the time-slice axiom and the locality property.*

Remark 5.4. Following the ideas presented in [BFV03, Proposition 2.3] we can construct a theory in the sense of Haag and Kastler from the covariant functor $\mathfrak{A}^0 : \mathsf{Sub}_{\hat{\equiv}} \to C^*\mathsf{Alg}^{\mathrm{inj}}$: Let us consider the set $\mathrm{Obj}(\mathsf{Sub}_{\hat{\equiv}})$ of objects in $\mathsf{Sub}_{\hat{\equiv}}$, i.e. the set of causally compatible and globally hyperbolic open subsets of the reference spacetime $\widehat{M}$. The functor $\mathfrak{A}^0$ associates to the terminal object $\widehat{M}$ a C^*-algebra $\mathfrak{A}^0(\widehat{M})$, which we shall interpret as the global algebra of observables. To any element $M \in \mathrm{Obj}(\mathsf{Sub}_{\hat{\equiv}})$ the functor associates a C^*-algebra $\mathfrak{A}^0(M)$, which can be mapped with an injective unital C^*-algebra homomorphism into $\mathfrak{A}^0(\widehat{M})$.

With a slight abuse of notation we denote the image of $\mathfrak{A}^0(M)$ under this map by the same symbol. Hence, we have an association

$$\mathsf{Obj}(\mathsf{Sub}_{\widehat{\Xi}}) \ni M \mapsto \mathfrak{A}^0(M) \subseteq \mathfrak{A}^0(\widehat{M}) \ . \tag{5.4}$$

Following the proof of [BFV03, Proposition 2.3] one can show that this association satisfies isotony, causality and the time-slice axiom. Furthermore, if there is a group of orientation and time-orientation preserving isometries acting on $\widehat{M}$, the association (5.4) is covariant. Hence, it is a quantum field theory in the sense of Haag and Kastler [HK63], generalized to an arbitrary but fixed spacetime $\widehat{M}$.

The considerations in this section make heavy use of a terminal object in the category $\mathsf{Sub}_{\widehat{\Xi}}$. Indeed, the existence of this object has provided us with commutative diagrams of the form (5.1), which are essential for constructing a suitable quotient. With this quotient we could construct quantum field theories in the sense of Haag and Kastler. Notice that the category $G-\mathsf{PrBuGlobHyp}$ has no terminal object, hence the techniques developed in this section do not apply to this case. This is of course already clear from Theorem 4.9, where it is shown that there exists no quotient which leads to a theory obeying the strict axioms of general local covariance [BFV03].

6 Concluding physical remarks

The theory of electromagnetism contains several features which are connected to the topology of a region M of an m-dimensional spacetime $\widehat{M}$ in an algebraic description – the Aharonov-Bohm effect, related to $H^1_{\mathrm{dR}}(M)$, as well as electric and magnetic charges, related to $H^{m-2}_{\mathrm{dR}}(M)$ and $H^2_{\mathrm{dR}}(M)$, respectively. Describing electromagnetism as a theory of principal $U(1)$-connections in its entirety, we have been able to provide a quantum framework which describes all of these topological features in a coherent manner. In the following we briefly comment on the relation of our constructions and results to these physical aspects of electromagnetism.

To discuss the Aharonov-Bohm effect, we recall the main aspects of [SDH12, Example 3.1] (see also [LRT78] for an early account of the Aharonov-Bohm effect in the algebraic framework): Consider as a globally hyperbolic spacetime M the Cauchy development in 4-dimensional Minkowski spacetime of the time-zero hypersurface $\{0\} \times \mathbb{R}^3$ with the z-axis removed. The (necessarily trivial) principal $U(1)$-bundle over $\widehat{M}$ pulls back to a trivial principal $U(1)$-bundle over M. One has $H^1_{\mathrm{dR}}(M, \mathfrak{g}) \neq \{0\}$ and, choosing the trivial connection as a reference, $H^1_{\mathrm{dR}}(M, \mathfrak{g})$ can be spanned by the on-shell vector potential $i\, \mathrm{d}\varphi \in \Omega^1(M, \mathfrak{g})$, with φ being the azimuthal angle around the z-axis in cylindrical coordinates (t, ρ, φ, z) on M. Here the z-axis represents an infinitely thin coil whose magnetic flux Φ through the plane perpendicular to the coil can be encoded in the vector potential $i\, \frac{\Phi}{2\pi} \mathrm{d}\varphi$. The gauge invariant affine functionals (2.1) introduced in [BDS13] can not distinguish connections with the different gauge potentials $i\, \frac{\Phi}{2\pi} \mathrm{d}\varphi$, $\Phi \in \mathbb{R}$, and are thus not sufficient to measure this flux, cf. [BDS13, Remark 4.5]. The exponential observables (3.1) solve this problem and further shortcomings in previous treatments of the subject. On the one hand, they contain Aharonov-Bohm observables which in contrast to the ones in [Dim92, Pfe09, SDH12, FS13] are fully gauge invariant and measure the phase $\exp i\Phi$ rather than the flux Φ itself. This is consistent with and, indeed, reproduces the Aharonov-Bohm experiment. On the other hand, they are regular enough for quantization, in contrast to Aharonov-Bohm observables of Wilson-loop type. In fact, they can be considered as regularized Wilson loops, cf. Remark 3.3.

In [DL12] it has been found that the sensitivity of electromagnetism to $H^2_{\mathrm{dR}}(M)$ (and $H^{m-2}_{\mathrm{dR}}(M)$) leads to a failure of the locality axiom in locally covariant quantum field theory as introduced in [BFV03]. This has been confirmed in [SDH12, BDS13] and in Proposition 4.8 of this work and ascribed to the Gauss law in [SDH12]. To understand this in view of our results, we introduce a few notions.

Definition 6.1. An **electric charge observable** is a gauge invariant functional of exponential type $\mathcal{W}_\varphi$, with $\varphi = \underline{\mathcal{F}}^*(\zeta)$, $\zeta \in \Omega^2_{0,\mathrm{d}}(M, \mathfrak{g}^*)$ and $0 \neq [\zeta] \in H^2_{0\,\mathrm{dR}}(M, \mathfrak{g}^*)$. An **electrically charged configuration** is an on-shell connection, i.e. $\lambda \in \Gamma^\infty(\mathcal{C}(\Xi))$ and $\mathsf{MW}(\lambda) = 0$, such that there exists an electric charge observable with $\mathcal{W}_\varphi(\lambda) \neq 1$. In the notation of Section 5, for any object M in $\mathsf{Sub}_{\widehat{\Xi}}$ a **material electric charge observable** is an electric charge observable $\mathcal{W}_\varphi$, such that $0 \neq [\varphi] \in \mathfrak{Ker}_{\widehat{\Xi}}(M)$. A **materially electric**

charged configuration is an on-shell connection λ, such that for some material electric charge observable $\mathcal{W}_\varphi(\lambda) \neq 1$.

It is easy to prove that no materially electric charged configuration on M can be extended from M to $\widehat{M}$. Thus, all of these configurations are unphysical in pure electromagnetism and have to be discarded. In an interacting theory including charged matter fields, these configurations can be interpreted as the connections sourced by an electric current density located in $\widehat{M} \setminus M$, cf. [SDH12, Example 3.7]. Consequently, all material electric charge observables have to be discarded by taking an appropriate quotient as in Theorem 5.2, since they do not measure anything in pure electromagnetism. In view of Remark 5.4, one may say that *this quotient gives for each region M of an arbitrary but fixed spacetime $\widehat{M}$ the correct, full algebra of observables of this region in pure electromagnetism.* In this respect Theorem 4.9 can be interpreted as to imply that *it is mathematically impossible to construct the correct algebra of observables of a region of spacetime in pure electromagnetism without knowing a priori the whole spacetime.* In the counterexample constructed in the proof of this theorem, considering three objects Ξ_i, $i = 1, 2, 3$, and two morphisms $f_j : \Xi_3 \to \Xi_j$, $j = 1, 2$, in $G-\mathsf{PrBuGlobHyp}$, the electric charge observables on M_3 are material with respect to M_2 but not with respect to M_1. Thus, they might or might not belong to the correct algebra of observables depending on whether M_2 or M_3 is the whole spacetime. We expect that this problem disappears in an interacting theory containing matter field currents, see also [SDH12, Remark 4.15].

By starting from a smaller algebra which does not contain Aharonov-Bohm observables, it is possible to construct a quantum field theory functor on the full category $G-\mathsf{PrBuGlobHyp}$ that satisfies the locality property, cf. [BDS13, Theorem 7.3]. In the context of the counterexample constructed in the proof of Theorem 4.9, this implies to discard all electric charge observables on M_3, even if they might be indispensable observables in pure electromagnetism depending on the nature of the whole spacetime. Hence, the result in [BDS13, Theorem 7.3] seems mathematically very pleasing, but it is not satisfactory from the physical point of view.

Acknowledgements

We would like to thank Hanno Gottschalk for comments on this work and especially for his help in preparing the appendix. We are also grateful to Klaus Fredenhagen, Daniel Paulino and Katarzyna Rejzner for useful discussions. The work of C.D. has been supported partly by the University of Pavia and partly by the Indam-GNFM project "Influenza della materia quantistica sulle fluttuazioni gravitazionali". The work of M.B. has been supported partly by a DAAD scholarship. M.B. is grateful to the II. Institute for Theoretical Physics of the University of Hamburg for the kind hospitality. The work of T.-P. H. is supported by a research fellowship of the Deutsche Forschungsgemeinschaft (DFG).

A CCR-representations of generic presymplectic Abelian groups

In this appendix we discuss the generalization of the theory of Weyl systems and $\mathfrak{CCR}$-representations from symplectic vector spaces to generic presymplectic Abelian groups. For the case of symplectic vector spaces the theory of $\mathfrak{CCR}$-representations is well understood and details can be found in the textbooks [BGP07, BR96]. The generalization to presymplectic vector spaces has been studied in [BHR04] and $\mathfrak{CCR}$-representations of presymplectic Abelian groups appeared in [M+73]. We will first review the results of Manuceau et al. [M+73] and afterwards provide some further constructions which are essential for locally covariant quantum field theory.

Definition A.1. (i) A **presymplectic Abelian group** is a tuple (B, τ), where B is an Abelian group[1] and $\tau : B \times B \to \mathbb{R}$ is an antisymmetric map, such that $\tau(b, \cdot\,) : B \to \mathbb{R}\,,\ b' \mapsto \tau(b, b')$ is a homomorphism of Abelian groups, for all $b \in B$.

[1] We denote the group operation by $+$, the identity element by 0 and the inverse of $b \in B$ by $-b$

(ii) The category PAG consists of the following objects and morphisms: An object is a presymplectic Abelian group (B, τ). A morphism $\phi : (B_1, \tau_1) \to (B_2, \tau_2)$ is a group homomorphism (not necessarily injective) that preserves the presymplectic structures, i.e. $\tau_2 \circ (\phi \times \phi) = \tau_1$.

(iii) The category $\mathsf{PAG}^{\mathrm{inj}}$ is the subcategory of PAG where all morphisms are injective.

To any object (B, τ) in PAG we can associate a unital $*$-algebra over $\mathbb{C}$ as follows: Consider the $\mathbb{C}$-vector space $\Delta(B, \tau)$ that is spanned by a basis $W(b)$, $b \in B$. Any element in $\Delta(B, \tau)$ is of the form $a = \sum_{i=1}^{N} \alpha_i \, W(b_i)$, where $\alpha_i \in \mathbb{C}$ and $b_i \in B$, for all $i = 1, \dots, N$. We can assume without loss of generality that all b_i's are different in expressions like this one. We define the structure of an associative unital algebra on $\Delta(B, \tau)$ by, for all $b, c \in B$,

$$W(b)\, W(c) := e^{-i\,\tau(b,c)/2}\, W(b + c)\ . \tag{A.1}$$

Notice that $W(0) = \mathbf{1}$ is the unit element. We further define a $*$-structure on $\Delta(B, \tau)$ by $W(b)^* := W(-b)$, for all $b \in B$, and we notice that this turns $\Delta(B, \tau)$ into a unital $*$-algebra over $\mathbb{C}$. All $W(b)$ are unitary.

Given a morphism $\phi : (B_1, \tau_1) \to (B_2, \tau_2)$ in PAG we can construct a unital $*$-algebra homomorphism between $\Delta(B_1, \tau_1)$ and $\Delta(B_2, \tau_2)$ as follows: Define for any element $a = \sum_{i=1}^{N} \alpha_i \, W_1(b_i) \in \Delta(B_1, \tau_1)$, $\Delta(\phi)(a) := \sum_{i=1}^{N} \alpha_i \, W_2(\phi(b_i))$. Then $\Delta(\phi) : \Delta(B_1, \tau_1) \to \Delta(B_2, \tau_2)$ is clearly a $\mathbb{C}$-linear map and also a unital $*$-algebra homomorphism, since ϕ is a group homomorphism preserving the presymplectic structures. Notice that for the identity morphism $\mathrm{id}_{(B,\tau)}$ in PAG we have that $\Delta(\mathrm{id}_{(B,\tau)}) = \mathrm{id}_{\Delta(B,\tau)}$. Furthermore, given two composable morphisms $\phi_1 : (B_1, \tau_1) \to (B_2, \tau_2)$ and $\phi_2 : (B_2, \tau_2) \to (B_3, \tau_3)$ in PAG it is easy to check that $\Delta(\phi_2 \circ \phi_1) = \Delta(\phi_2) \circ \Delta(\phi_1)$. Hence, $\Delta : \mathsf{PAG} \to {}^*\mathsf{Alg}$ is a covariant functor, where the category ${}^*\mathsf{Alg}$ consists of unital $*$-algebras over $\mathbb{C}$ as objects and unital $*$-algebra homomorphisms (not necessarily injective) as morphisms. It is easy to see that Δ restricts to a covariant functor $\Delta : \mathsf{PAG}^{\mathrm{inj}} \to {}^*\mathsf{Alg}^{\mathrm{inj}}$, where ${}^*\mathsf{Alg}^{\mathrm{inj}}$ is the subcategory of ${}^*\mathsf{Alg}$ where all morphisms are injective.

For constructing a suitable C^*-completion of $\Delta(B, \tau)$ we follow the strategy of [M+73] and introduce as an intermediate step a $*$-Banach algebra. Let us consider the $*$-norm $\|\cdot\|^{\mathrm{Ban}} : \Delta(B, \tau) \to \mathbb{R}^+$ defined by

$$\Big\| \sum_{i=1}^{N} \alpha_i \, W(b_i) \Big\|^{\mathrm{Ban}} := \sum_{i=1}^{N} |\alpha_i|\ . \tag{A.2}$$

We denote the completion of $\Delta(B, \tau)$ by $\Delta^{\mathrm{Ban}}(B, \tau)$ and notice that it is a unital $*$-Banach algebra. A generic element in $\Delta^{\mathrm{Ban}}(B, \tau)$ is of the form $a = \sum_{i=1}^{\infty} \alpha_i \, W(b_i)$, with $\alpha_i \in \mathbb{C}$ and $b_i \in B$, such that $\sum_{i=1}^{\infty} |\alpha_i| < \infty$.

Given a morphism $\phi : (B_1, \tau_1) \to (B_2, \tau_2)$ in PAG we note that $\Delta(\phi) : \Delta(B_1, \tau_1) \to \Delta(B_2, \tau_2)$ is bounded by 1, i.e. $\|\Delta(\phi)(a)\|_2^{\mathrm{Ban}} \leq \|a\|_1^{\mathrm{Ban}}$, for all $a \in \Delta(B_1, \tau_1)$. Hence, there exists a unique continuous extension of $\Delta(\phi)$ to the completions, which we denote by the symbol $\Delta^{\mathrm{Ban}}(\phi) : \Delta^{\mathrm{Ban}}(B_1, \tau_1) \to \Delta^{\mathrm{Ban}}(B_2, \tau_2)$. If the morphism ϕ is in $\mathsf{PAG}^{\mathrm{inj}}$, then $\Delta(\phi)$ is an isometry, i.e. $\|\Delta(\phi)(a)\|_2^{\mathrm{Ban}} = \|a\|_1^{\mathrm{Ban}}$ for all $a \in \Delta(B_1, \tau_1)$. In this case $\Delta^{\mathrm{Ban}}(\phi)$ is an isometry and hence in particular injective. Furthermore, given two composable morphisms $\phi_1 : (B_1, \tau_1) \to (B_2, \tau_2)$ and $\phi_2 : (B_2, \tau_2) \to (B_3, \tau_3)$ in PAG it is easy to check that $\Delta^{\mathrm{Ban}}(\phi_2 \circ \phi_1) = \Delta^{\mathrm{Ban}}(\phi_2) \circ \Delta^{\mathrm{Ban}}(\phi_1)$. Hence, $\Delta^{\mathrm{Ban}} : \mathsf{PAG} \to B^*\mathsf{Alg}$ is a covariant functor, where the category $B^*\mathsf{Alg}$ consists of unital $*$-Banach algebras as objects and unital $*$-Banach algebra homomorphisms (not necessarily injective) as morphisms. Notice that Δ^{Ban} restricts to a covariant functor $\Delta^{\mathrm{Ban}} : \mathsf{PAG}^{\mathrm{inj}} \to B^*\mathsf{Alg}^{\mathrm{inj}}$, where $B^*\mathsf{Alg}^{\mathrm{inj}}$ is the subcategory of $B^*\mathsf{Alg}$ where all morphisms are injective.

In the following we shall require states on the $*$-Banach algebras $\Delta^{\mathrm{Ban}}(B, \tau)$, i.e. continuous positive linear functionals $\omega : \Delta^{\mathrm{Ban}}(B, \tau) \to \mathbb{C}$ satisfying $\omega(\mathbf{1}) = 1$. The following proposition, which is proven in [M+73, Proposition (2.17)], will be very helpful in constructing such states:

Proposition A.2. *Any positive linear functional on $\Delta(B, \tau)$ extends to a continuous positive linear functional on $\Delta^{\mathrm{Ban}}(B, \tau)$.*

There exists a faithful state on $\Delta^{\mathrm{Ban}}(B, \tau)$, which can be seen as follows: Let us define a positive linear functional $\omega : \Delta(B, \tau) \to \mathbb{C}$ by $\omega(W(b)) = 0$, if $b \neq 0$, and $\omega(W(0)) = \omega(\mathbf{1}) = 1$. By Proposition A.2 we can extend ω to a continuous positive linear functional on $\Delta^{\mathrm{Ban}}(B, \tau)$ (denoted by the same symbol), which satisfies $\omega(\mathbf{1}) = 1$, hence it is a state. This state is faithful, i.e. $\omega(a^* a) > 0$ for any $a \in \Delta^{\mathrm{Ban}}(B, \tau)$, $a \neq 0$. The existence of a faithful state allows us to define the following C^*-norm on $\Delta^{\mathrm{Ban}}(B, \tau)$.

Definition A.3. Let $\mathcal{F}$ be the set of states on $\Delta^{\mathrm{Ban}}(B,\tau)$. The **minimal regular norm** on $\Delta^{\mathrm{Ban}}(B,\tau)$ is defined by, for all $a \in \Delta^{\mathrm{Ban}}(B,\tau)$,

$$\|a\| := \sup_{\omega\in\mathcal{F}} \sqrt{\omega(a^*\,a)}\ . \tag{A.3}$$

The completion of $\Delta^{\mathrm{Ban}}(B,\tau)$ (or equivalently $\Delta(B,\tau)$) with respect to the minimal regular norm is denoted by $\mathfrak{CCR}(B,\tau)$. Then $\mathfrak{CCR}(B,\tau)$ is a unital C^*-algebra (cf. [M+73]).

Proposition A.4. *Let $\phi : (B_1,\tau_2) \to (B_2,\tau_2)$ be a morphism in* PAG. *Then there exists a unique continuous extension $\mathfrak{CCR}(\phi) : \mathfrak{CCR}(B_1,\tau_1) \to \mathfrak{CCR}(B_2,\tau_2)$ of $\Delta^{\mathrm{Ban}}(\phi)$ (and hence also of $\Delta(\phi)$).*

Proof. We have to prove that there exists $C \in \mathbb{R}$, such that $\|\Delta^{\mathrm{Ban}}(\phi)(a)\|_2 \leq C\,\|a\|_1$, for all $a \in \Delta^{\mathrm{Ban}}(B_1,\tau_1)$. The existence and uniqueness of a continuous extension then follows by standard extension theorems. We obtain by a straightforward calculation

$$\|\Delta^{\mathrm{Ban}}(\phi)(a)\|_2 = \sup_{\omega\in\mathcal{F}_2} \sqrt{\omega\big(\Delta^{\mathrm{Ban}}(\phi)(a^*\,a)\big)} \leq \sup_{\omega'\in\mathcal{F}_1} \sqrt{\omega'(a^*\,a)} = \|a\|_1\ , \tag{A.4}$$

where in the second step we have used that $\omega\circ\Delta^{\mathrm{Ban}}(\phi) \in \mathcal{F}_1$. Hence, $C = 1$. $\square$

Let us denote by $C^*\mathsf{Alg}$ the category whose objects are unital C^*-algebras and whose morphisms are unital C^*-algebra homomorphisms (not necessarily injective). The first main result of this appendix is summarized in the following

Theorem A.5. $\mathfrak{CCR} : \mathsf{PAG} \to C^*\mathsf{Alg}$ *is a covariant functor.*

It remains to show that $\mathfrak{CCR}$ restricts to a covariant functor $\mathfrak{CCR} : \mathsf{PAG}^{\mathrm{inj}} \to C^*\mathsf{Alg}^{\mathrm{inj}}$, where $C^*\mathsf{Alg}^{\mathrm{inj}}$ is the subcategory of $C^*\mathsf{Alg}$ where all morphisms are injective. Notice that for a morphism $\phi : (B_1,\tau_1) \to (B_2,\tau_2)$ in $\mathsf{PAG}^{\mathrm{inj}}$ the morphism $\mathfrak{CCR}(\phi) : \mathfrak{CCR}(B_1,\tau_1) \to \mathfrak{CCR}(B_2,\tau_2)$ would be an isometry (in particular injective) if we could prove that for any $\omega' \in \mathcal{F}_1$ there exists a $\omega \in \mathcal{F}_2$, such that $\omega\circ\Delta^{\mathrm{Ban}}(\phi) = \omega'$. Due to Lemma A.2 it is sufficient to prove that for any normalized positive linear functional ω' on $\Delta(B_1,\tau_1)$ there exists a normalized positive linear functional ω on $\Delta(B_2,\tau_2)$, such that $\omega\circ\Delta(\phi) = \omega'$. On the image $\Delta(\phi)[\Delta(B_1,\tau_1)] \subseteq \Delta(B_2,\tau_2)$ we can invert $\Delta(\phi)$ since it is injective and hence arrive at the following extension problem: Does there exist a positive linear functional $\omega : \Delta(B_2,\tau_2) \to \mathbb{C}$ extending $\omega'\circ\Delta(\phi)^{-1} : \Delta(\phi)[\Delta(B_1,\tau_1)] \to \mathbb{C}$? Indeed, such an extension can be found by applying the positive-cone version of the Hahn-Banach Theorem, see e.g. [Edw65, Theorem 2.6.2].

Proposition A.6. *Let $\phi : (B_1,\tau_1) \to (B_2,\tau_2)$ be a morphism in* $\mathsf{PAG}^{\mathrm{inj}}$. *Then there exists for any positive linear functional $\tilde{\omega} : \Delta(\phi)[\Delta(B_1,\tau_1)] \to \mathbb{C}$ an extension $\omega : \Delta(B_2,\tau_2) \to \mathbb{C}$ that is a positive linear functional on $\Delta(B_2,\tau_2)$.*

Proof. Let us denote by $H := \{a \in \Delta(B_2,\tau_2) : a^* = a\}$ and $\tilde{H} := \{a \in \Delta(\phi)[\Delta(B_1,\tau_1)] : a^* = a\}$ the $\mathbb{R}$-vector spaces of hermitian elements. Notice that $\mathbf{1} \in \tilde{H} \subseteq H$. The given positive linear functional $\tilde{\omega}$ restricts to a positive $\mathbb{R}$-linear functional (denoted by the same symbol) $\tilde{\omega} : \tilde{H} \to \mathbb{R}$. By [Edw65, Theorem 2.6.2] we can extend $\tilde{\omega}$ to a positive linear functional $\omega : H \to \mathbb{R}$, provided that for each element $h \in H$ there exists at least one $\tilde{h} \in \tilde{H}$, such that $\tilde{h} - h$ is in the positive cone K [2]. This condition is satisfied for the following reason: Any $h \in H$ can be expressed as a finite sum of the basic hermitian elements $h_{\alpha,b} := \alpha\,W_2(b) + \overline{\alpha}\,W_2(-b)$, with $\alpha \in \mathbb{C}$, $b \in B_2$ and $\bar{\cdot}$ denotes complex conjugation. Hence, it is sufficient to prove that for any $\alpha \in \mathbb{C}$ and $b \in B$ there exists $\tilde{h} \in \tilde{H}$, such that $\tilde{h} - h_{\alpha,b} \in K$. Defining $a := \mathbf{1} - \alpha\,W_2(b)$ we find $a^*a = (1+\overline{\alpha}\alpha)\,\mathbf{1} - h_{\alpha,b}$ and thus $\tilde{h} - h_{\alpha,\beta} \in K$ for $\tilde{h} = (1+\overline{\alpha}\alpha)\,\mathbf{1}$.

The positive linear functional $\omega : H \to \mathbb{R}$ which is obtained by this extension procedure is further extended to $\Delta(B_2,\tau_2)$ as follows: For any $a \in \Delta(B_2,\tau_2)$ we define the real and imaginary part by $a_R := (a+a^*)/2$ and $a_I := (a-a^*)/2i$. Notice that $a_R, a_I \in H$. We then extend ω to a $\mathbb{C}$-linear map on all of $\Delta(B_2,\tau_2)$ by defining $\omega(a) := \omega(a_R) + i\,\omega(a_I)$. It is easy to see that this is an extension of $\tilde{\omega}$, which completes the proof. $\square$

[2] The positive cone here is the subset $K \subset H$ consisting of finite sums of elements $\beta\,a^*a$, with $\beta > 0$ and $a \in \Delta(B_2,\tau_2)$.

Corollary A.7. *For any morphism $\phi : (B_1, \tau_1) \to (B_2, \tau_2)$ in $\mathsf{PAG}^{\mathrm{inj}}$ the map $\mathfrak{CCR}(\phi) : \mathfrak{CCR}(B_1, \tau_1) \to \mathfrak{CCR}(B_2, \tau_2)$ of Proposition A.4 is an isometry. In particular, $\mathfrak{CCR}(\phi)$ is an injective unital C^*-algebra homomorphism.*

The second main result of this appendix is summarized in the following

Theorem A.8. $\mathfrak{CCR} : \mathsf{PAG}^{\mathrm{inj}} \to C^*\mathsf{Alg}^{\mathrm{inj}}$ *is a covariant functor.*

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