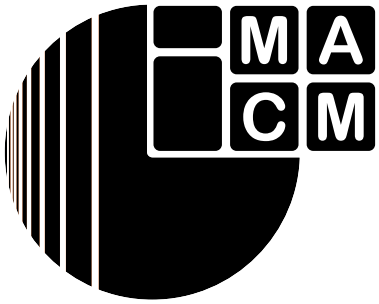

Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 13/05

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March 22, 2013

http://www.math.uni-wuppertal.de

On the convergence rate of dynamic iteration for coupled problems with multiple subsystem

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Abstract

For example in multiphysical modeling coupled problems naturally occur. Each subproblem is commonly represented by a system of partial differential-algebraic equations. Applying the method of lines, this results in coupled differential algebraic equations (DAEs). A standard technique for the transient simulation of such systems is dynamic iteration (or cosimulation). In contrast to the dynamic iteration of systems of ordinary differential equations, convergence for DAEs cannot be generally guaranteed unless some contraction condition is fulfilled. In the case of convergence, it is a linear one.

In this paper, we quantify the convergence rate, i.e., the slope of the contraction, in terms of the coupling structure for DAE and ODE systems and also for two and more subsystems. We find higher rates (for certain coupling structures) than known before and give sharp estimates for the rate. Furthermore it is revealed how the rate depends on the number of subsystems.

Keywords: Dynamic Iteration, Co-Simulation, PDAE, Coupling.
2000 MSC: 65L05, 65L80, 65L20.

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1. Introduction

Downscaling and advanced functionality require multiphysical models in many technical simulations. E.g. in electrical engineering, circuits are coupled to refined models (for networks, semiconductors [3]) or to magnetic field models (electric machines [10]). This results in multiphysical problems.

Simulator coupling is a standard technique for the transient simulation of coupled multiphysics problems. At synchronization times data between simulators is exchanged. Each simulator computes the solution for a dedicated subsystem only. Then iteration of this process ensures the consistency of the overall solution. In applications, this is referred to as cosimulation, whereas in mathematics it is usually called dynamic iteration or waveform relaxation.

Time integration of spatially discretized models is typically based on coupled system of differential algebraic equations (DAEs). Dynamic iteration of coupled ordinary differential equations (ODEs) always convergences linearly [5]. In contrast, the dynamic iteration of DAE systems will not always converge. Convergence can be guaranteed if contraction conditions are fulfilled, see e.g. [9, 8, 2].

To derive the convergence result, first an error recursion is set up, which covers the solution process of all subsystem once. To apply Banach's fixed-point theorem, a contraction condition is derived from the recursion. Then convergence and stability follows.

In case of convergence, one generally has linear convergence with rate $\mathcal{O}(\sqrt{H})$ (error reduction per iteration) where H denotes the time interval of interest. Convergence is not only influenced by the coupling structure, but also by the order of computation ([2, 3]) and by the actual dynamic iteration scheme employed.

A higher convergence rate can be obtained from certain coupling structures [4, 3]. Also in applications (field-circuit coupling [10]), a higher rate was numerically observed.

The aim of this work is to derive an analytical background for the higher convergence rates and to present coupling structures, which guarantee convergence rates up to $\mathcal{O}(H^2)$. To this end, we apply the strategy for the error recursions from previous work to systems with a refined structural analysis. Although Jacobi-type of dynamic iteration is quite popular (high potential of parallelization), we will investigate Gauss-Seidel-type iteration schemes that lead to faster convergence for particular DAE problems, which are free of contraction conditions.

The paper is organized as follows. Sec. 2 introduces the notation of coupled systems and dynamic iteration. Sec. 3 summarizes the known procedure to derive the recursion estimates, contraction and convergences. Our contribution of refined structural analysis for coupled DAEs follows in Sec. 4. We investigate two and multiple coupled systems and derive the convergence rate for different couplings. In Sec. 5 a academic test problem is presented that verifies the theoretical results by experiments and the application to real-world problems from electrical engineering is discussed. Conclusions form the final section.

2. Coupled DAE Systems and Dynamic Iteration Schemes

We consider coupled initial value problems (IVPs), which can be written in semi-explicit form (for each of the r subsystems):

$$\dot{\mathbf{y}}_i = \mathbf{f}_i(\mathbf{y}, \mathbf{z}), \qquad \mathbf{y} := (\mathbf{y}_1, \ldots, \mathbf{y}_r)^\top, \quad \mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{n_y} \tag{1a}$$

$$0 = \mathbf{g}_i(\mathbf{y}, \mathbf{z}), \qquad \mathbf{z} := (\mathbf{z}_1, \ldots, \mathbf{z}_r)^\top, \quad \mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{R}^{n_z} \tag{1b}$$

with $i = 1, \ldots, r$. Without loss of generality, this system is in autonomous form. This system is a split structure for an overall semi-explicit DAE system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{z})$, $0 = \mathbf{g}(\mathbf{y}, \mathbf{z})$ with $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r)^\top$ and $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_r)^\top$.

Assumption 1 *Given the coupled DAE-problem* (1).

a) Let $\mathbf{f}$ *and* $\mathbf{g}$ *be sufficiently often differentiable.*

b) Let consistent initial values be given.

c) Let $\partial\mathbf{g}/\partial\mathbf{z}$ *and* $\partial\mathbf{g}_i/\partial\mathbf{z}_i$ *be regular for each* i *with inverse* ϕ_i*.*

In other words, by Assumption 1 we have an index-1 problem for each subsystem and the overall system. Thus there is a unique solution $\mathbf{x} = (\mathbf{y}, \mathbf{z})^\top \in C^1([0,\, t_e], \mathbb{R}^{n_y}) \times C([0,\, t_e], \mathbb{R}^{n_z})$ with $\mathbf{y} : [0, t_e] \to \mathbb{R}^{n_y}$, $\mathbf{z} : [0, t_e] \to \mathbb{R}^{n_z}$. With the trivial constraint $\mathbf{g}_i \equiv 0$ and dimension $n_{z_i} = 0$ (i.e., $\mathbf{z}_i(t) \in \mathbb{R}^{n_{z_i}}$), the split system (1) may include also ODE subsystems.

We aim at computing a sufficiently accurate approximation $\tilde{\mathbf{x}} := (\tilde{\mathbf{y}}, \tilde{\mathbf{z}})^\top : [0, t_e] \to \mathbb{R}^{n_y} \times \mathbb{R}^{n_z}$ of the unique $\mathbf{x}$ for (1). This approximation is represented by a continuous waveform in our analysis. In practice it is represented by sampled points stemming from a sufficiently accurate numerical time stepping procedure.

Dynamic iteration schemes compute approximations to the subsystems separately, while an outer iteration loop can guarantee convergence towards

the desired unique solution. Thus each subsystem may invoke a dedicated solver to respect the corresponding structures like stiffness, activity, definiteness etc..

For the analysis, we formalize the iteration procedure. It is commonly performed on so called (time-)windows $[t_n, t_{n+1}]$ with $0 = t_0 < t_1 < t_2 < \ldots < t_N = t_e$ with window size $H_n := t_{n+1} - t_n$. Given a numerical approximation $\tilde{\mathbf{x}}$ on $[0, t_n]$, a dynamic iteration defines the approximations on the next window:

$$(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})|_{(t_n, t_{n+1}]} \in C_n^{1,0} \quad \text{with} \quad C_n^{1,0} := C^1((t_n, t_{n+1}], \mathbb{R}^{n_y}) \times C((t_n, t_{n+1}], \mathbb{R}^{n_z})$$

via an extrapolation $\Phi_n : C_{n-1}^{1,0} \to C_n^{1,0}$ and a number of iterations $\Psi_n : C_n^{1,0} \to C_n^{1,0}$: (see e.g. [2])

$$\Phi_n : \begin{pmatrix} \tilde{\mathbf{y}}|_{[t_{n-1}, t_n]} \\ \tilde{\mathbf{z}}|_{[t_{n-1}, t_n]} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\mathbf{y}}_n^{(0)} \\ \tilde{\mathbf{z}}_n^{(0)} \end{pmatrix} \qquad \Psi_n : \begin{pmatrix} \tilde{\mathbf{y}}_n^{(k-1)} \\ \tilde{\mathbf{z}}_n^{(k-1)} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\mathbf{y}}_n^{(k)} \\ \tilde{\mathbf{z}}_n^{(k)} \end{pmatrix} := \Psi_n \begin{pmatrix} \tilde{\mathbf{y}}_n^{(k-1)} \\ \tilde{\mathbf{z}}_n^{(k-1)} \end{pmatrix}.$$

Subscript n refers to the time window and superscript $k = 1, \ldots, k_n$ denotes the iteration count (with k_n finite). To solve the split DAE (1) for $\tilde{\mathbf{y}}_n = (\tilde{\mathbf{y}}_{1,n}, \ldots, \tilde{\mathbf{y}}_{r,n})^\top$, $\tilde{\mathbf{z}}_n = (\tilde{\mathbf{z}}_{1,n}, \ldots, \tilde{\mathbf{z}}_{r,n})^\top$, operator Ψ_n maps given approximations (index $(k-1)$) to new ones $\tilde{\mathbf{y}}^{(k)}, \tilde{\mathbf{z}}^{(k)}$ of the decoupled initial-values problems. To this end, the right-hand sides $\mathbf{f}, \mathbf{g}$ are formally replaced by corresponding splitting functions $\mathbf{F}$ and $\mathbf{G}$, which allow the simultaneous use of known, old iterates (index $(k-1)$) and updates (index (k)):

$$\begin{aligned} \dot{\tilde{\mathbf{y}}}_{i,n}^{(k)} &= \mathbf{F}_i(\tilde{\mathbf{y}}_n^{(k)}, \tilde{\mathbf{y}}_n^{(k-1)}, \tilde{\mathbf{z}}_n^{(k)}, \tilde{\mathbf{z}}_n^{(k-1)}), \quad \text{with} \quad \tilde{\mathbf{y}}_{i,n}^{(k)}(t_n) = \tilde{\mathbf{y}}_{i,n}^{(k-1)}(t_n), \quad \text{(2a)} \\ 0 &= \mathbf{G}_i(\tilde{\mathbf{y}}_n^{(k)}, \tilde{\mathbf{y}}_n^{(k-1)}, \tilde{\mathbf{z}}_n^{(k)}, \tilde{\mathbf{z}}_n^{(k-1)}) \quad \text{(2b)} \end{aligned}$$

(for $i = 1, \ldots, r$). The splitting functions $\mathbf{F}_i$ and $\mathbf{G}_i$, are arbitrary but smooth and satisfy compatibility:

$$\mathbf{F}_i(\mathbf{y}, \mathbf{y}, \mathbf{z}, \mathbf{z}) = \mathbf{f}_i(\mathbf{y}, \mathbf{z}), \qquad \mathbf{G}_i(\mathbf{y}, \mathbf{y}, \mathbf{z}, \mathbf{z}) = \mathbf{g}_i(\mathbf{y}, \mathbf{z}), \qquad i = 1, \ldots, r. \quad (3)$$

Hence the analytical solution $\mathbf{x} = (\mathbf{y}, \mathbf{z})^\top$ is a fixed-point of Ψ_n. For standard dynamic iterations schemes see [2]. In this paper we only employ Gauss-Seidel-type schemes and for simplicity of notation the computational sequence corresponds to the ordering of the subsystems, i.e., $1 \to 2 \to \ldots \to r$.

3. Error definition and basic convergence analysis

In [2, 4, 3] the convergence of the dynamic iteration schemes is investigated by studying error recursions within one window and the error transport from window to window. Here, we focus on the error recursion on one window. In contrast to previous works, we distinguish between the errors for each subsystem. This will allow for better error estimates. Contraction, general convergence and stability can be derived analogously as before. Therefore we briefly summarize the known results.

For given distance $d > 0$, we define a neighborhood $\mathcal{U}_{d,n}$ of the analytical solution $\mathbf{x}_n = (\mathbf{y}_n, \mathbf{z}_n)^\top$ in the n-th window:

$$\mathcal{U}_{d,n} = \Big\{ \mathbf{X} := (\mathbf{Y}, \mathbf{Z})^\top \in C_n^{1,0} : \big|\big|\mathbf{Y} - \mathbf{y}|_{[t_n,t_{n+1}]}\big|\big|, \big|\big|\mathbf{Z} - \mathbf{z}|_{[t_n,t_{n+1}]}\big|\big| \leq d \Big\},$$

where $||\mathbf{v}|| := \sup_{t_n < t \leq t_{n+1}} |\mathbf{v}(t)|$ using the the Euclidean norm $|\cdot|$.

Assumption 2 *Given splitting functions* $\mathbf{F}$, $\mathbf{G}$ *for the split system* (1)*, there is* $d_0 > 0$ *such that on* $\mathcal{U}_{d_0,n}$*:*

$$(a)\ \textit{function } \mathbf{F} \textit{ is Lipschitz with constant } L_{\mathbf{F}} > 0, \tag{4}$$
$$(b)\ \mathbf{G} \textit{ is totally differentiable (i.e., derivatives are Lipschitz),} \tag{5}$$
$$(c)\ \mathbf{G}_{\mathbf{z}^{(k)}} \textit{ is invertible.} \tag{6}$$

Usually, contractivity is proven in the smaller set $\mathcal{U}_{d,n} \subset \mathcal{U}_{d_0,n}$ using Lipschitz estimates and a kind of homotpy for the algebraic variables. This is the basis for the following notations. Let $\mathbf{X}, \tilde{\mathbf{X}} \in \mathcal{U}_{d_0,n}$, for k dynamic iterations the n-th window, we use the shorthands:

$$(\mathbf{Y}_n^k,\, \mathbf{Z}_n^k)^\top := \Psi_n^k \mathbf{X}, \qquad (\tilde{\mathbf{Y}}_n^k,\, \tilde{\mathbf{Z}}_n^k)^\top := \Psi_n^k \tilde{\mathbf{X}}, \tag{7}$$

$$\begin{aligned} \Delta_{\mathbf{y}}^k(t) &:= \mathbf{Y}_n^k(t) - \tilde{\mathbf{Y}}_n^k(t), & \delta_{\mathbf{y}}^k &:= ||\Delta_{\mathbf{y}}^k|| = \sup_{t_n < t \leq t_{n+1}} \left(\left|\Delta_{\mathbf{y}}^k(t)\right|\right), \\ \Delta_{\mathbf{z}}^k(t) &:= \mathbf{Z}_n^k(t) - \tilde{\mathbf{Z}}_n^k(t), & \delta_{\mathbf{z}}^k n &:= ||\Delta_{\mathbf{z}}^k|| = \sup_{t_n < t \leq t_{n+1}} \left(\left|\Delta_{\mathbf{z}}^k(t)\right|\right). \end{aligned} \tag{8}$$

Based on the above definitions, one can deduce a basic error recursion (eg. [3]). It has the following structure. For sufficiently small H and small space $\mathcal{U}_{d,n}$, the iterates of the split DAE (1) fulfill the estimate:

$$\begin{pmatrix} \delta_{\mathbf{y}}^k \\ \delta_{\mathbf{z}}^k \end{pmatrix} \leq \mathbf{K} \begin{pmatrix} \delta_{\mathbf{y}}^{k-1} \\ \delta_{\mathbf{z}}^{k-1} \end{pmatrix} + \begin{pmatrix} 1 + CH \\ C \end{pmatrix} |\Delta_{\mathbf{y}}^{k-1}(t_n)| \tag{9}$$

with iteration matrix $\mathbf{K}$ and important constant α_n: (and constants $c, C > 0$)

$$\mathbf{K} := \begin{pmatrix} CH & CH \\ C & CH + \alpha_n \end{pmatrix}, \qquad \alpha_n := (1 + c\,d) \left|\left| \mathbf{G}^{-1}_{\mathbf{z}^{(k)}} \, \mathbf{G}_{\mathbf{z}^{(k-1)}} \right|\right| + Cd. \quad (10)$$

The second term in (9) describes the initial offset in a window. If the eigenvalues of $\mathbf{K}$ are strictly less than one, contraction is established, and the arguments in [2] yield convergence. The largest eigenvalue of $\mathbf{K}$ is responsible for the slowest reduction of errors (in the case of convergence).

Remark 3 (Simple Coupling) *If the subsystems are weakly coupled, i.e., $\mathbf{G}_{\mathbf{z}^{(k-1)}}$ which is due to our numeration the strict upper right triangle of $\mathbf{G} = \partial\mathbf{g}/\partial\mathbf{z}$ tends to zero then $\alpha_n \to 0$ and the maximum eigenvalue is dominated by terms in H. In the limit case $\alpha_n = 0$ ('simple coupling'), the algebraic constraints do not use old algebraic variables, [4, 3].*

We focus here on systems that are not constrained by α_n. This motivates:

Definition 4 *Let $\lambda_{\max}(\mathbf{K}) \in \mathcal{O}(H^p)$, then we refer to $\mathcal{O}(H^p)$ as rate of convergence where p is the order (per iteration).*

4. Refined Structural Analysis for DAE Coupling

The above strategy is employed to carry out a refined structural analysis of DAE problems with two and more subsystems w.r.t. the convergence rate.

4.1. Two Subsystems

First we consider the case of two coupled DAE systems. The convergence rate is generally α_n, see (9-10). This changes if we consider systems with less mutual interaction as discussed before. Therefore we compute the corresponding recursion estimates with a refined analysis, draw relations to the previously established results and show what can be gained in terms of convergence.

4.1.1. No Old Algebraic Iterates

Here we examine the coupled DAE system with special structure:

$$\begin{aligned} \dot{\mathbf{y}}_1 &= \mathbf{f}_1(\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2, \mathbf{z}_2), & \dot{\mathbf{y}}_2 &= \mathbf{f}_2(\mathbf{y}_1, \mathbf{z}_2, \mathbf{y}_2, \mathbf{z}_2), \\ \mathbf{0} &= \mathbf{g}_1(\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2,\ 0), & \mathbf{0} &= \mathbf{g}_2(\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2, \mathbf{z}_2). \end{aligned} \quad (11)$$

Notice $\mathbf{g}_1$ does not depend on $\mathbf{z}_2$. We use a Gauss-Seidel iteration scheme and start the iteration with subsystem 1. Thus — in contrast to the reversed computational sequence — there is no coupling via old algebraic iterates in algebraic equations, cf. Remark 3. Hence, for time windows small enough this leads to a convergent iteration scheme. To investigate the order of the convergence rate, we establish:

Lemma 5 (Error Recursion for Two Simply Coupled DAEs) *Given the system* (11) *with Ass. 1 fulfilled. Then exists $H_0 > 0$, such that for window size $H < H_0$ we obtain the error recursion (on the n-th window)*

$$\begin{pmatrix} \delta^k_{\mathbf{y_1}} \\ \delta^k_{\mathbf{y_2}} \\ \delta^k_{\mathbf{z_1}} \\ \delta^k_{\mathbf{z_2}} \end{pmatrix} \leq \mathbf{K} \begin{pmatrix} \delta^{k-1}_{\mathbf{y_1}} \\ \delta^{k-1}_{\mathbf{y_2}} \\ \delta^{k-1}_{\mathbf{z_1}} \\ \delta^{k-1}_{\mathbf{z_2}} \end{pmatrix} + \mathbf{M} \begin{pmatrix} |\Delta^{k-1}_{\mathbf{y_1}}(t_n)| \\ |\Delta^{k-1}_{\mathbf{y_2}}(t_n)| \end{pmatrix} \tag{12}$$

with matrices: (and constant $C > 0$)

$$\mathbf{K} = C \begin{pmatrix} 0 & H & 0 & H \\ 0 & H & 0 & H^2 \\ 0 & 1 & 0 & H \\ 0 & 1 & 0 & H \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1+CH & 0 \\ CH & 1+CH \\ C & 0 \\ C & C \end{pmatrix}. \tag{13}$$

Proof. We follow the line of the proof from [3], but we exploit the particular structure of (11) with less mutual dependency. First, we examine subsystem one with old iterates for $\mathbf{y}_2$ and $\mathbf{z}_2$. The Lipschitz continuity of $\mathbf{z}_1 = \Phi_1(\mathbf{y}_1, \mathbf{y}_2)$ implies:

$$|\Delta^k_{\mathbf{z_1}}| \leq L_\Phi \left(|\Delta^k_{\mathbf{y_1}}| + |\Delta^{k-1}_{\mathbf{y_2}}| \right). \tag{14}$$

where L_Φ is the maximum of the Lipschitz constants of Φ_i w.r.t. $\mathbf{y}_i$ and $\mathbf{z}_i$. Then applying Lipschitz continuity to the corresponding ODE-part yields:

$$|\Delta^k_{\mathbf{y_1}}(\tau)| \leq |\Delta^{k-1}_{\mathbf{y_1}}(t_n)| + L_f \int_{t_n}^{\tau} \left((1+L_\Phi) \left(|\Delta^k_{\mathbf{y_1}}| + |\Delta^{k-1}_{\mathbf{y_2}}| \right) + |\Delta^{k-1}_{\mathbf{z_2}}| \right) \mathrm{d}t.$$

where L_f is the maximum of the Lipschitz constants of $\mathbf{f}_i$. For $L_0 := L_f(1 + L_\Phi)$ this leads to the estimate for the maximum over the time interval

$$\delta^k_{\mathbf{y_1}} \leq |\Delta^{k-1}_{\mathbf{y_1}}(t_n)| + L_0 H \left(\delta^k_{\mathbf{y_1}} + \delta^{k-1}_{\mathbf{y_2}} + \delta^{k-1}_{\mathbf{z_2}} \right).$$

Let $H_0 < 1/L_0$, then for any $H < H_0$, we find

$$\delta^k_{\mathbf{y_1}} \leq (1+H')\,|\Delta^{k-1}_{\mathbf{y_1}}(t_n)| + H'\big(\delta^{k-1}_{\mathbf{y_2}} + \delta^{k-1}_{\mathbf{z_2}}\big) \tag{15}$$

with $H' := c_y H$ where $c_y := L_0/(1-L_0H_0)$. Then from (14) and (15) we get

$$\delta^k_{\mathbf{z_1}} \leq L_\Phi(1+H')|\Delta^{k-1}_{\mathbf{y_1}}(t_n)| + L_\Phi(1+H')\delta^{k-1}_{\mathbf{y_2}} + L_\Phi H'\delta^{k-1}_{\mathbf{z_2}}. \tag{16}$$

Analogously we get estimates for the second subsystem, where we use already known, actual iterates $\mathbf{y}_1^{(k)}, \mathbf{z}_1^{(k)}$. The corresponding estimates for (14) and (15) (by Lipschitz continuity) read:

$$|\Delta^k_{\mathbf{z_2}}| \leq L_\Phi\left(|\Delta^k_{\mathbf{y_1}}| + |\Delta^k_{\mathbf{y_2}}| + |\Delta^k_{\mathbf{z_1}}|\right), \tag{17}$$

$$\delta^k_{\mathbf{y_2}} \leq (1+H')|\Delta^{k-1}_{\mathbf{y_2}}(t_n)| + H'\left(\delta^k_{\mathbf{y_1}} + \delta^k_{\mathbf{z_1}}\right). \tag{18}$$

Regarding (18), we insert for the new iterates (15) and (16) and obtain

$$\begin{aligned}\delta^k_{\mathbf{y_2}} \leq\,& (1+L_\Phi)(1+H')H'|\Delta^{k-1}_{\mathbf{y_1}}(t_n)| + (1+H')|\Delta^{k-1}_{\mathbf{y_2}}(t_n)| \\ &+ (H' + L_\Phi(1+H'))\,H'\delta^{k-1}_{\mathbf{y_2}} + (1+L_\Phi)(H')^2\delta^{k-1}_{\mathbf{z_2}}.\end{aligned} \tag{19}$$

Similar we treat the algebraic estimate (17), where we additionally use (19):

$$\begin{aligned}\delta^k_{\mathbf{z_2}} \leq\,& C_0(1+H')^2|\Delta^{k-1}_{\mathbf{y_1}}(t_n)| + L_\Phi(1+H')|\Delta^{k-1}_{\mathbf{y_2}}(t_n)| \\ &+ C_0(1+H')^2\delta^{k-1}_{\mathbf{y_2}} + C_0(1+H')H'\delta^{k-1}_{\mathbf{z_2}}\end{aligned} \tag{20}$$

with $C_0 := L_\Phi(1+L_\Phi)$. Equations (15), (19), (16) and (20) together with the constant $C := C_0 \cdot (1 + 2H_0 + H_0^2) \cdot \max\{1,\, c_y,\, c_y^2\}$ conclude the proof. □

The just established error recursion yields for k iterations:

Theorem 6 (Recursion Estimate) *Let the assumptions of Lemma 5 hold. Let C and H_0 be the constants for that lemma. Then a constant $\hat{C}$ exists such that for all $k \geq 1$ and for all $H \leq H_0$ it holds:*

$$\begin{aligned}\begin{pmatrix}\delta^k_{\mathbf{y_1}}\\ \delta^k_{\mathbf{y_2}}\\ \delta^k_{\mathbf{z_1}}\\ \delta^k_{\mathbf{z_2}}\end{pmatrix} \leq\,& 2^{k-2}H^{k-1}C^k \begin{pmatrix}0 & H + H^{\max(2-k,0)} & 0 & 2^{\max(2-k,0)}(H+H^2)\\ 0 & 2H & 0 & 2H^2\\ 0 & 2 & 0 & H\\ 0 & 2 & 0 & H\end{pmatrix}\begin{pmatrix}\delta^0_{\mathbf{y_1}}\\ \delta^0_{\mathbf{y_2}}\\ \delta^0_{\mathbf{z_1}}\\ \delta^0_{\mathbf{z_2}}\end{pmatrix} \\ &+ \begin{pmatrix}1+\hat{C}H & \hat{C}H\\ \hat{C}H & 1+\hat{C}H\\ \hat{C} & \hat{C}\\ \hat{C} & \hat{C}\end{pmatrix}\begin{pmatrix}\delta^0_{\mathbf{y_1}}(t_n)\\ \delta^0_{\mathbf{y_2}}(t_n)\end{pmatrix}.\end{aligned}$$

Proof. We follow the technique from [2]. By (13) and induction holds:

$$\mathbf{K}^j = 2^{j-1} C^j H^{j-1} \begin{pmatrix} 0 & H+1 & 0 & H^2+H \\ 0 & 2H & 0 & 2H^2 \\ 0 & 2 & 0 & 2H \\ 0 & 2 & 0 & 2H \end{pmatrix} \tag{21}$$

given $j \geq 2$. Then, for $CH_0 < \frac{1}{2}$ we have

$$\sum_{i=0}^{j-1} \mathbf{K}^i \leq \sum_{i=0}^{\infty} \mathbf{K}^i = \mathbf{I} + \mathbf{K} + C^\star H \begin{pmatrix} 0 & H+1 & 0 & H^2+H \\ 0 & 2H & 0 & 2H^2 \\ 0 & 2 & 0 & 2H \\ 0 & 2 & 0 & 2H \end{pmatrix}$$

with identity matrix $\mathbf{I}$ and $C^\star := \frac{C^2}{1-2CH_0}$. Thus we get

$$\sum_{i=0}^{j-1} \mathbf{K}^i \mathbf{M} \leq \left(\sum_{i=0}^{\infty} \mathbf{K}^i \right) \mathbf{M} = \begin{pmatrix} 1+\hat{C}H & \hat{C}H \\ \hat{C}H & 1+\hat{C}H \\ \hat{C} & \hat{C} \\ \hat{C} & \hat{C} \end{pmatrix}. \tag{22}$$

The equations (21) and (22) conclude the proof. □

Corollary 7 *(Convergence Rate) The single, non-zero eigenvalue of* $\mathbf{K}$ *in* (13) *is* $\lambda = 2CH$. *Hence the convergence rate is* $\mathcal{O}(H)$ *with* $p = 1$.

Remark 8 *(Improved Estimate) Notice the perspective of these estimates:* $\mathbf{K}$ *expresses the contraction in terms of* $\mathbf{y}_2$ *and* $\mathbf{z}_2$. *These are the latest updated variables in a Gauss-Seidel scheme.*

The global estimates in [2, 3] suggested a rate of convergence $\mathcal{O}(\sqrt{H})$ *(for Gauss-Seidel iteration). This is due to the fact that there the error recursion did not exploit the particular structure. In contrast our detailed analysis reveals a rate of* $\mathcal{O}(H)$. *For Jacobi iteration the analysis reveals a rate of only* $\mathcal{O}(\alpha_n)$.

4.1.2. Coupled Systems with Higher Order Convergence

We have shown above that system (11) has a convergence rate $\mathcal{O}(H)$. Here, we give examples of coupling structures (of two subsystems), where even higher rates of convergence are expected, i.e., previous results were too pessimistic for those systems.

Example 9 *We consider the coupled DAE system*

$$\begin{aligned} \dot{\mathbf{y}}_1 &= \mathbf{f}_1(\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2,\ 0), & \dot{\mathbf{y}}_2 &= \mathbf{f}_2(\mathbf{y}_1, 0,\ \mathbf{y}_2, \mathbf{z}_2), \\ \mathbf{0} &= \mathbf{g}_1(\mathbf{y}_1, \mathbf{z}_1,\ 0,\ 0), & \mathbf{0} &= \mathbf{g}_2(\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2, \mathbf{z}_2). \end{aligned} \tag{23}$$

The Gauss-Seidel dynamic iteration scheme yields the iteration matrix $\mathbf{K}$ *(starting with subsystem with subscript* 1*) with some constant* $\hat{C}$*:*

$$\mathbf{K} = \hat{C}H \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & H & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The non-zero eigenvalue $\lambda = \hat{C}H^2$ *gives convergence rate* $\mathcal{O}(H^2)$*. The analysis in [3] would suggest a convergence rate* $\mathcal{O}(H)$*.*

Example 10 *The dynamic iteration scheme applied to the coupled system*

$$\begin{aligned} \dot{\mathbf{y}}_1 &= \mathbf{f}_1(\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2, \mathbf{z}_2), & \dot{\mathbf{y}}_2 &= \mathbf{f}_2(\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2, \mathbf{z}_2), \\ \mathbf{0} &= \mathbf{g}_1(\mathbf{y}_1, \mathbf{z}_1,\ 0,\ 0), & \mathbf{0} &= \mathbf{g}_2(0,\ 0,\ \mathbf{y}_2, \mathbf{z}_2) \end{aligned} \tag{24}$$

leads to the iteration matrix (independent of the starting subsystem)

$$\mathbf{K} = \hat{C}H \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & H & 0 & H \\ 0 & 1 & 0 & 1 \\ 0 & H & 0 & H \end{pmatrix}.$$

The non-zero eigenvalue $\lambda = 2\hat{C}H^2$ *gives convergence rate* $\mathcal{O}(H^2)$*. Notice, this is more or less a coupled ODE-system, since the algebraic equations are not mutually coupled (coupling is done by the differential variables only).*

In summary, for two systems we have at least $\mathcal{O}(H)$ without old algebraic variables. The rate can reach up to $\mathcal{O}(H^2)$, since for each iteration two integrations are performed (one for $\mathbf{y}_1$ and one for $\mathbf{y}_2$).

4.1.3. Several Subsystems

Next, we investigate the rate of convergence for multiple subsystems ($r > 2$). Again the general system (1) with all mutual dependencies has

convergence rate α_n. First we assume a coupling without old algebraic variables such that α_n vanishes, i.e.,

$$\frac{\partial \mathbf{g}_i}{\partial \mathbf{z}_j} = 0 \quad \text{for} \quad j > i. \tag{25}$$

All other dependencies via differential variables $\mathbf{y}_j$ are allowed. Then we obtain:

Lemma 11 *Given r DAE-subsystems with the coupling structure* (25)*, we obtain an error recursion (for the Gauss-Seidel type-scheme) with*

$$\mathbf{K} = C \begin{pmatrix} \mathcal{H} & \mathcal{P} \\ \mathcal{I} & \mathcal{H} \end{pmatrix} \quad \textit{with } \mathcal{H} = \begin{pmatrix} 0 & H & \dots & H \\ \vdots & \vdots & \ddots & \vdots \\ 0 & H & \dots & H \end{pmatrix} \in \mathbb{R}^{r\times r},$$

$$\mathcal{P} = \begin{pmatrix} 0 & H & H & \dots & H \\ 0 & H^2 & H & \dots & H \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & H \\ 0 & H^2 & \dots & \dots & H^2 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{r\times r}. \tag{26}$$

Furthermore, $\lambda_{1,2} = (r-1)H \pm \sqrt{\binom{r}{2}H^2 + \binom{r-1}{2}H}$ are the non-zero eigenvalues of $\mathbf{K}$. Thus we have a convergence rate of $\mathcal{O}(\sqrt{H})$.

Proof. Applying the technique of Lemma 5, the recursion matrix $\mathbf{K}$ is deduced. The characteristic polynomial can be obtained by using block Gauss elimination: first $\mathcal{I}$ is reduced to one row (by a row transformation), and then the lower block $\mathcal{H} - \lambda I$ is reduced to diagonal block plus one row (by the inverse transformation from the right). Secondly, the corresponding transformations are made for the first block row. See Appendix A for details. □

Remark 12 *For more than 2 subsystems, the rate $\mathcal{O}(\sqrt{H})$ for Gauss-Seidel-type iteration coincides with the general result in [2]. Only for 2 subsystem we can proof higher convergence rates for certain fine structured problem classes.*

Secondly, we restrict the coupling further to an ODE-like coupling:

$$\dot{\mathbf{y}}_i = \mathbf{f}_i(\mathbf{y}, \mathbf{z}), \qquad 0 = \mathbf{g}_i(\mathbf{y}_i, \mathbf{z}_i) \qquad (i = 1, \dots r), \tag{27}$$

that is, the i-th algebraic constraint depends only on the local variables of the i-th system.

Lemma 13 *Given $r \geq 2$ DAE-subsystems with the coupling structure* (27)*, we obtain an error recursion (for the Gauss-Seidel type-scheme) with*

$$\mathbf{K} = C \begin{pmatrix} \mathcal{P} & \mathcal{P} \\ \mathcal{P} & \mathcal{P} \end{pmatrix} \tag{28}$$

where $\mathcal{P}$ is given in (26)*. Thus we have a convergence rate of $\mathcal{O}(H^{\frac{r}{r-1}})$.*

Proof. Again applying the technique of Lemma 5, one deduces the recursion matrix $\mathbf{K}$. The characteristic polynomial is obtained by first applying a block Gaussian elimination (with a matrix $\mathcal{B}$, see below) and than expanding w.r.t. the columns/rows which have only one non-zero entry:

$$\begin{aligned} \chi(\mathbf{K}) &= \det\left(\mathcal{B}\left(\mathbf{K} - \lambda I_{2r}\right)\mathcal{B}\right) \qquad \text{with } \mathcal{B} = \begin{pmatrix} I_r & 0 \\ -I_r & I_r \end{pmatrix} \\ &= (-\lambda)^{r+1} \det\left(2H\mathcal{M}_{r-1} - \lambda I_{r-1}\right) \quad \text{with } \mathcal{M}_{r-1} := \begin{pmatrix} H & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ H & \dots & \dots & H \end{pmatrix} \end{aligned}$$

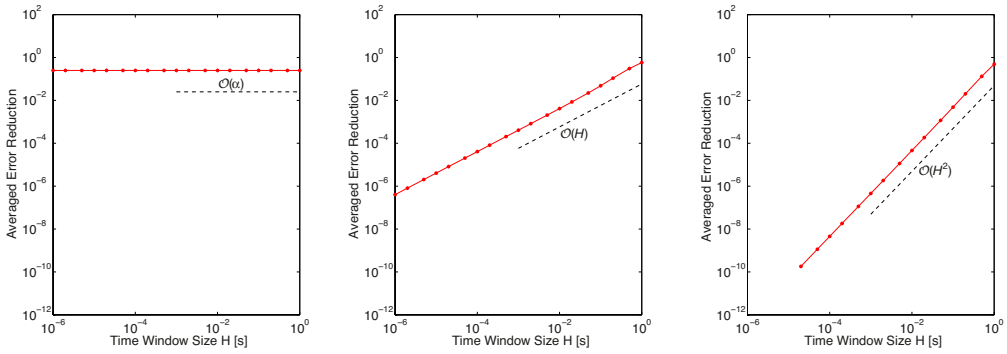
(and identity matrix $I_m \in \mathbb{R}^{m\times m}$). For $\mathcal{M}_{r-1}$, one finds by induction:

$$\chi\left(\mathcal{M}_{r-1}\right) = \lambda^{r-1} + \mathcal{O}(H)\sum_{k=0}^{r-2}\lambda^k = \lambda^{r-1} + \mathcal{O}(H), \quad \text{for } H \to 0.$$

(expanding the determinant along the first column). Here we use that $\lambda_{\mathcal{M}_{r-1}}$ is at least of order $\mathcal{O}(H^0)$. Thus, for the non-zero eigenvalues it holds that $\lambda_{(2H\mathcal{M}_{r-1})} = \mathcal{O}(H^{\frac{r}{r-1}})$ for $H \to 0$. □

Thus, it is shown that the convergence rate approaches $\mathcal{O}(H)$ as the number of subsystems approaches infinity.

Remark 14 *(Coupling of Systems of ODEs) Lemma 13 includes the case of multiple ODE systems since the reasoning remains valid if one performs an index-reduction for each subproblem.*



(a) System of type (1) yields rate $\mathcal{O}(\alpha)$

(b) System of type (11) yields rate $\mathcal{O}(H)$

(c) System of type (23) yields rate $\mathcal{O}(H^2)$

Figure 1: Experiments for $r = 2$ coupled Prothero-Robinson DAE systems. Convergence rate versus window size H computed according to the error norm (8).

5. Applications

First we discuss an academic test example, which is derived from Prothero-Robinson's test equation. Then we investigate a particular coupling in circuit simulation in more detail.

5.1. Academic Test Case

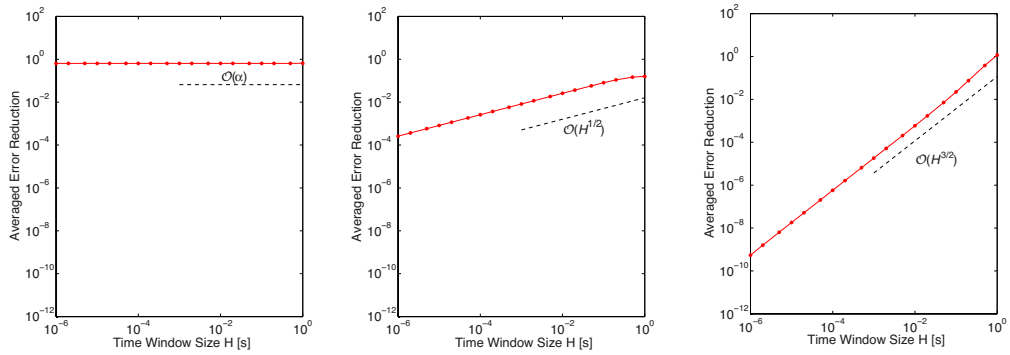
To analyze the dynamic iteration scheme analytically, we derive a new test case. To this end, we extend the classical Prothero-Robinson test equation (for stiff ODEs) [7] to coupled DAEs. For the unknowns $(\mathbf{y}, \mathbf{z})^\top = (y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_r)^\top : \mathbb{R} \to \mathbb{R}^{2r}$ it reads:

$$\mathbf{y}' = \mathbf{A}(\mathbf{y}, \mathbf{z}, t)\,(\mathbf{y} - \eta(t)) + \mathbf{B}(\mathbf{y}, \mathbf{z}, t)\,(\mathbf{z} - \Phi(\mathbf{y}, t)) + \eta'(t) \tag{29a}$$

$$0 = \mathbf{C}(\mathbf{y}, t)\,(\mathbf{y} - \eta(t)) + \mathbf{D}(\mathbf{y}, t)\,(\mathbf{z} - \Phi(\mathbf{y}, t)) \tag{29b}$$

for given $\eta : \mathbb{R} \to \mathbb{R}^r$ continuous differentiable, $\Phi : \mathbb{R}^{r+1} \to \mathbb{R}^r$ continuous, as well as $\mathbf{A}, \mathbf{B} : \mathbb{R}^{2r+1} \to \mathbb{R}^{r\times r}$, and $\mathbf{C}, \mathbf{D} : \mathbb{R}^{r+1} \to \mathbb{R}^{r\times r}$ continuously differentiable, and initial values $\mathbf{y}(0) = \big(y_1(0), \ldots, y_r(0)\big)^\top = \eta(0)$. To have an index-1 system, we assume $\mathbf{D}$ to be a regular (for any $\mathbf{y}$ and t). Thus the solution to (29) is:

$$\mathbf{y}(t) = \eta(t) \qquad \text{and} \qquad \mathbf{z}(t) = \Phi(\mathbf{y}, t).$$



(a) System of type (1) yields rate $\mathcal{O}(\alpha)$

(b) System of type (25) yields rate $\mathcal{O}(\sqrt{H})$

(c) System of type (27) yields rate $\mathcal{O}(H^{\frac{3}{2}})$

Figure 2: Experiments for $r = 3$ coupled Prothero-Robinson DAE systems. Convergence rate versus window size H computed according to the error norm (8).

A linear version using constant matrices and $\Phi(\mathbf{y}, t) = \mathbf{F}\mathbf{y} + \zeta(t)$ reads:

$$\begin{pmatrix} \mathbf{y}' \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} - \mathbf{B}\mathbf{F} & \mathbf{B} \\ \mathbf{C} - \mathbf{D}\mathbf{F} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} + \begin{pmatrix} -\mathbf{A}\eta(t) - \mathbf{B}\zeta(t) + \eta'(t) \\ -\mathbf{C}\eta(t) - \mathbf{D}\zeta(t). \end{pmatrix} \tag{30}$$

The simplicity of the linear test equation (30) allows us to solve the differential equations analytically by a computer algebra system and thus perform a dynamic iteration with continuous waveforms. We solve r scalar DAE subsystems in the obvious sequence $(y_1, z_1)^\top \to (y_2, z_2)^\top \to \ldots \to (y_r, z_r)^\top$ using the Gauss-Seidel-type scheme with $k = 4$ iterations. The computations have been carried out with Mathematica using simple choices of coefficients such that analytical time integration was feasible. For example the results of Fig. 1c) are obtained by

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with right-hand-side functions $\eta(t) = (\sin(2\pi 10^6 t),\ 2\cos(2\pi 10^7 t) + 100)^\top$ and $\zeta(t) = (2\cos(t),\ 7t)^\top$. The experiments were found to be insensitive with respect to the choice of coefficients.

The error reduction per iteration (8) is estimated by averaging the quotient of the errors after k iterations:

$$\bar{\delta} := \sqrt[k]{\frac{\delta_{y,z}^k}{\delta_{y,z}^0}}.$$

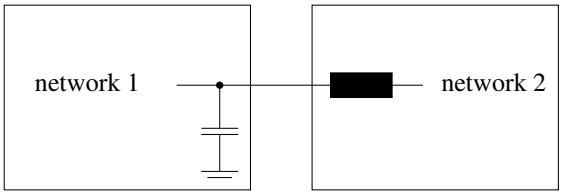


Figure 3: Coupling via an LC-link.

The dependency of $\bar{\delta}$ on the window size H is investigated in Fig. 1 and 2. The figures depict the results for two and three coupled subsystems, respectively. The experiments match the theoretical results very well and indicate that the theoretical estimates for $\mathcal{O}(H^p)$ are sharp and not pessimistic: there is an example for each coupling type that gives the predicted convergence rate.

5.2. Examples of Simple Coupling in Electrical Engineering

The problems with reduced mutual interdependence, e.g. (23) are not pathological and can be observed in practice. In [3], we have analyzed the convergence order of a Gauss-Seidel based dynamic iteration scheme for circuit-semiconductor coupling and could show a convergence rate of $\mathcal{O}(H)$ (independent of the sequence of the subsystems). Furthermore in the field circuit coupling of magnetoquasistatics and electric networks, a coupled system with two subsystems was investigated in [10]. There a second order convergence rate $\mathcal{O}(H^2)$ of the dynamic iteration was found, which is now explainable with our refined theory.

Here we propose another application from circuit simulation: let be given an index-1 electric network model (for modeling of circuits see [6]). If such a circuit is composed of two sub-networks of index-1, where the sub-networks are coupled by LC-links, see Fig. 3, then we have differential coupling. The decoupling can be realized by a source coupling, where the sub-network on the inductor side (network 2) is represented by a current source and vice versa the other sub-network by a voltage source, see Fig. 4. The voltage at the capacitance is a differential variable (index-1) and the current through the inductor is differential (index-1). Hence all coupling variables are differentiable. In fact, one can also have a resistance in serial to the inductor, such that the coupling is via an RLC-link. Since transmission lines can be modeled by RLC-links, this is an immediate application. Therefore the results of Example 10 apply. Moreover we can have multiple coupled systems in this way

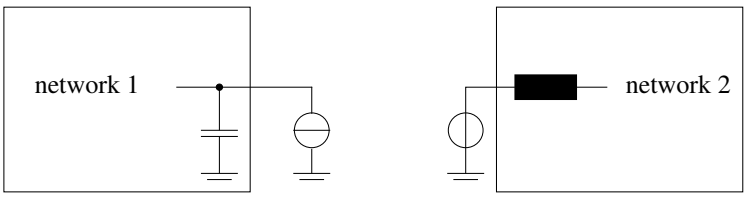


Figure 4: Decoupled networks.

and thus realize the differential-like coupling in Lemma 13 (Section 4.1.3). This underlines the practical importance of differential couplings.

6. Conclusions

We employed the line of arguments from [2, 3], but with refined structure. This allowed us to prove a higher convergence rate for particular coupled systems of two DAE systems. As a partial result we have proven a convergence rate of $\mathcal{O}(H^2)$ for coupled systems of two ODE systems. For more than two coupled DAE subsystems (except for ODE-like coupling) we have shown that the convergence rate drops to $\mathcal{O}(\sqrt{H})$. For coupled systems of $r > 2$ ODE systems (or $r > 2$ coupled DAE systems with ODE-like coupling), we have proven a different behaviour. Those systems observe a convergence rate of f $\mathcal{O}(H^{\frac{r}{r-1}})$. Thus for these system the convergence rate decreases with increasing number r.

By an analytic example based on Prothero-Robinson, we were able to prove that our estimations are sharp in the limit of the window size ($H \to 0$).

There are also other factors, which influence the rate of convergence. One aspect is the sequence of computation. Obviously it can modify the coupling structure and thus the rate of convergences. E.g. in [1], it was shown, that computational sequence of subsystems can be crucial for convergence or divergence. Regarding the derivation of the error estimates, the Lipschitz constants comprise the constants of our estimates in the end. Thus the rate of convergences can also be influenced by these values (see also [3] for further discussion and an example).

Furthermore the new results on the convergence rate have an important impact on window size selection procedures. This will be the topic of future research.

Appendix A. Characteristic Polynomial for r DAE Subsystems

We derive the characteristic polynomial of $\mathbf{K}$ from Lemma 11 with corresponding eigenvalues λ. Let $\hat{\mathbf{K}} := \frac{1}{\tilde{C}}\mathbf{K}$ with eigenvalues $\hat{\lambda} = \frac{1}{\tilde{C}}\lambda$. By Laplacian expansion for the first and $r+1$-th column of $\hat{\mathbf{K}}$, we obtain

$$\chi(\hat{\mathbf{K}}) = \left(-\hat{\lambda}\right)^2 \det \begin{pmatrix} H-\hat{\lambda} & \dots & \dots & H & H^2 & H & \dots & H \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & H \\ H & \dots & \dots & H-\hat{\lambda} & H^2 & \dots & \dots & H^2 \\ 1 & \dots & \dots & 1 & H-\hat{\lambda} & \dots & \dots & H \\ \vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 1 & \dots & \dots & 1 & H & \dots & \dots & H-\hat{\lambda} \end{pmatrix}$$
$$=: \hat{\lambda}^2 \det\left(\mathbf{M}_{r-1}\right)$$

We perform a block-Gaussian elimination which diagonalizes the last $r-2$ equations:

$$\det(\mathbf{M}_{r-1}) = \det\left(\mathbf{B}\mathbf{M}_{r-1}\mathbf{B}^{-1}\right), \qquad \text{with } \mathbf{B} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -1 & \ddots & & \\ & & \vdots & & \ddots & \\ & & -1 & & & 1 \end{pmatrix},$$

where modification from identity in $\mathbf{B}$ is located in the r-th column,

$$= \left(-\hat{\lambda}\right)^{n-2} \det \begin{pmatrix} H-\hat{\lambda} & \dots & \dots & H & 1H^2+(n-2)H \\ \vdots & \ddots & & \vdots & 2H^2+(n-3)H \\ \vdots & & \ddots & \vdots & \vdots \\ H & \dots & \dots & H-\hat{\lambda} & (n-1)H^2+0H \\ 1 & \dots & \dots & 1 & (n-1)H-\lambda \end{pmatrix}.$$

Applying the same procedure again as

$$\det(\mathbf{M}_{n-1}) = \det(\mathbf{D}\mathbf{B}\mathbf{M}_{n-1}\mathbf{B}^{-1}\mathbf{D}^{-1}) \quad \text{with } \mathbf{D} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ -1 & \dots & -1 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

(modification from the identity is located in the $(r-1)$-th row), which transfers the first $r-1$ row to an upper triangular form. Hence we obtain:

$$\chi\left(\hat{\mathbf{K}}_1\right) = \left(-\hat{\lambda}\right)^{2r-2}\left(\left((r-1)H - \hat{\lambda}\right)^2 - \tbinom{r}{2}H^2 - \tbinom{r-1}{2}H\right).$$

Acknowledgements

The third author is supported by the 'Excellence Initiative' of the German Federal and State Governments and the Graduate School of Computational Engineering at Technische Universitaet Darmstadt.

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