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Abstract

We construct a covariant functor from a category of Abelian principal bundles over globally hyperbolic spacetimes to a category of $*$-algebras that describes quantized principal connections. We work within an appropriate differential geometric setting by using the bundle of connections and we study the full gauge group, namely the group of vertical principal bundle automorphisms. Properties of our functor are investigated in detail and, similar to earlier works, it is found that due to topological obstructions the locality property of locally covariant quantum field theory is violated. Furthermore, we prove that, for Abelian structure groups containing a nontrivial compact factor, the gauge invariant Borchers-Uhlmann algebra of the vector dual of the bundle of connections is not separating on gauge equivalence classes of principal connections. We introduce a topological generalization of the concept of locally covariant quantum fields. As examples, we construct for the full subcategory of principal $U(1)$-bundles two natural transformations from singular homology functors to the quantum field theory functor that can be interpreted as the Euler class and the electric charge. In this case we also prove that the electric charges can be consistently set to zero, which yields another quantum field theory functor that satisfies all axioms of locally covariant quantum field theory.

Keywords: locally covariant quantum field theory, quantum field theory on curved spacetimes, gauge theory on principal bundles

MSC 2010: 81T20, 81T05, 81T13, 53Cxx

1 Introduction

The algebraic theory of quantum fields on Lorentzian manifolds has made tremendous developments since the introduction of the principle of general local covariance by Brunetti, Fredenhagen and Verch [BFV03], see also [FV12]. Mathematically, this principle states that any reasonable quantum field theory has to be formulated by a covariant functor from a category of globally hyperbolic Lorentzian manifolds (spacetimes) to a category of unital $(C)^*$-algebras, subject to certain physical conditions. Many examples of linear quantum field theories satisfying the axioms of locally covariant quantum field theory have been constructed in the literature, see e.g. [BGP07, BG11] and references therein. The mathematical tools which are used in these constructions is the theory of normally hyperbolic and Dirac-type operators on vector bundles over spacetimes together with the $\mathfrak{CCR}$ and $\mathfrak{CAR}$ quantization functors. In our previous work [BDS12] we have generalized these constructions to classes of operators on affine bundles over spacetimes. In addition to these exactly tractable models, the techniques of locally covariant quantum field theory are essential for the perturbative construction of interacting quantum field theories, see for example [BDF09], and the generalization of the spin-statistics theorem from Minkowski spacetime to general spacetimes [Ver01].

One of the weak points of the current status of algebraic quantum field theory is our incomplete understanding of the formulation of gauge theories. Even though there exist important results on the quantization of electromagnetism [Dim92, Pfe09, DL12, DS13, SDH12], linearized general relativity [FH12] and generic linear gauge theories [HS12], as well as on the perturbative quantization of interacting gauge theories [Hol08, FR13], there are still open problems that deserve a detailed study. In particular, there is up to now no satisfactory formulation of quantized electromagnetism for the following two reasons: Firstly, applying canonical quantization techniques it has been found that electromagnetism violates the locality axiom of locally covariant quantum field theory. This has been shown for the field strength algebra in [DL12] and for the vector potential algebra in [SDH12]. The latter reference also gives an interpretation of this feature in terms of Gauss' law. Secondly, the differential geometric developments over the past decades indicate that the natural language for formulating gauge theories of Yang-Mills type is that of principal connections on principal G-bundles, which includes electromagnetism by choosing $G = U(1)$. Taking into account the principal bundle structure has far reaching consequences for the very principle of general local covariance: Since principal connections can not be associated to spacetimes, but only to principal bundles over spacetimes, the category of spacetimes in [BFV03] should be replaced by a category of principal bundles over spacetimes. This notion of general local covariance for gauge theories of Yang-Mills type appeared recently in the discussion of the locally covariant charged Dirac field [Zah12], where however the principal connections were assumed to be non-dynamical background fields. Besides this new notion of general local covariance in gauge theories of Yang-Mills type, the classical configuration space is different to the one used in previous works: The set of principal connections does not carry a vector space structure, but it is an affine space over the vector space of gauge potentials. The vector space structure employed in the works [Dim92, Pfe09, DS13, SDH12] comes from a (necessarily non-unique) fixing of some reference connection, which is unnatural in differential geometry and leads to the unnecessary question of independence of the theory on this choice [Hol08].

We outline the structure of our paper: In Section 2 we fix the notations and review some aspects of the theory of Abelian principal bundles and principal connections. This material is essentially well-known in the differential geometry literature, but we require some details that go beyond standard textbook presentations and hence are worth for being discussed. In particular, we need a full-fledged study of the bundle of connections [Ati57] together with the action of principal bundle morphisms and the gauge group (the group of vertical principal bundle automorphisms) defined on it. Sections of the bundle of connections, that is an affine bundle over the base space, are in bijective correspondence with principal connection forms on the total space, but they have the advantage of being fields on the base space and not on the total space. This has far reaching consequences when one studies dynamical equations of connections and causality properties, since the total space is not equipped with a Lorentzian metric.

In Section 3 we associate to any Abelian principal bundle a gauge invariant phase space for its principal connections by extending ideas from [BDS12] and [HS12]. Our notion of gauge invariance is dictated by the principal bundle and in the general case differs from the one employed in [Dim92, Pfe09, DS13, SDH12]. The phase space is not symplectic, but only a presymplectic vector space, whose radical contains topological information to be discussed in Section 6.

We characterize explicitly the gauge invariant phase space and its radical in Section 4 by using Čech cohomology. This leads to two interesting observations: Firstly, the gauge invariant phase space and its radical for theories with a compact Abelian structure group exhibit a different structure with respect to their counterparts with a non-compact Abelian structure group. Secondly, if the Abelian structure group contains a compact factor, then the gauge invariant phase space is not separating on gauge equivalence classes of principal connections. In particular, gauge non-equivalent flat connections can not be resolved. The reason for this feature is that our gauge invariant phase space consists of affine functionals, but for Abelian structure groups with compact factors the set of gauge equivalence classes of principal connections is in general no longer an affine space. This shows that in these cases the standard phase space of sections of the vector dual of affine bundles introduced in [BDS12] has to be extended in order to be separating. Natural candidates for this extension are Wilson loops, which are however too singular for a straightforward description in algebraic quantum field theory. We hope to come back to this issue in future investigations.

The results above are combined in Section 5 to construct a covariant functor from a category of Abelian principal bundles over spacetimes to a category of presymplectic vector spaces. Composing this functor with the usual $\mathfrak{CCR}$-functor we obtain a quantum field theory functor that satisfies the causality property and the time-slice axiom. However, the locality property of [BFV03] is violated, confirming that the results of [DL12, SDH12] also hold true in our principal bundle geometric approach. This result was not obvious from the beginning, since our concept of morphisms and configuration space is different from the ones in earlier investigations.

In Section 6 we extend the concept of a locally covariant quantum field developed in [BFV03] to what we call a 'generally covariant topological quantum field'. By this we mean a natural transformation from a functor describing topological information to the quantum field theory functor. For the full subcategory of principal $U(1)$-bundles we provide two explicit examples where the functor describing topological information is a singular homology functor. The natural transformations are then the coherent association of observables that measure the Euler class of the principal bundle and the electric charge, that is a certain cohomology class.

Following the electric charge interpretation of the previous section (see also [SDH12] for an earlier account) we show in Section 7 that the electric charges can be consistently set to zero. This is physically motivated since in pure electromagnetism, without the presence of charged fields, there can not be electric charges. The resulting quantum field theory functor then satisfies in addition to the causality property and the time-slice axiom also the locality property. With this we succeed in constructing a locally covariant quantum field theory.

2 Geometric preliminaries

In this work all manifolds will be of class C^∞, Hausdorff and second-countable. If not stated otherwise, maps between manifolds are C^∞.

2.1 Spacetimes

We briefly review some standard notions of spacetimes, see [BGP07, BG11, Wal12] for a more detailed discussion.

Let M be a manifold that for later convenience we assume to be of **finite type**, i.e. M possesses a finite good cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{I}}$, with $\mathcal{I}$ finite. A **Lorentzian manifold** is a triple $(M,\mathfrak{o},g)$, where M is a manifold (of finite type), $\mathfrak{o}$ is an orientation on M and g is a Lorentzian metric on M of signature $(-,+,\dots,+)$. Given a time-orientation $\mathfrak{t}$ on a Lorentzian manifold $(M,\mathfrak{o},g)$, we call the quadruple $(M,\mathfrak{o},g,\mathfrak{t})$ a **spacetime**. Let $(M,\mathfrak{o},g,\mathfrak{t})$ be a spacetime and $S\subseteq M$ be a subset. We denote the **causal future/past** of S in M by $J^\pm_M(S)$. Furthermore, $J_M(S) := J^+_M(S)\cup J^-_M(S)$. The subset $S\subseteq M$ is called **causally compatible**, if $J^\pm_S(\{x\}) = J^\pm_M(\{x\})\cap S$, for all $x\in S$. A **Cauchy surface** in a spacetime $(M,\mathfrak{o},g,\mathfrak{t})$ is a subset $\Sigma\subseteq M$, which is met exactly once by every inextensible causal curve. A spacetime $(M,\mathfrak{o},g,\mathfrak{t})$ is called **globally hyperbolic**, if it contains a Cauchy surface.

2.2 Abelian principal bundles

We briefly review standard notions of principal bundles and refer to the textbook [KN96] for more details.

Definition 2.1. Let M be a manifold and G a Lie group. A **principal G-bundle over M** is a pair (P,r), where P is a manifold and $r: P\times G\to P\,,~(p,g)\mapsto r_g(p) =: p\,g$ is a smooth right G-action, such that

(i) the right G-action r is free,

(ii) $M = P/G$ is the quotient of the G-action r and the canonical projection $\pi : P \to M$ is smooth,

(iii) P is locally trivial, that is, there exists for every $x \in M$ an open neighborhood $U \subseteq M$ and a diffeomorphism $\psi : \pi^{-1}[U] \to U \times G$, which is G-equivariant, i.e., for all $p \in \pi^{-1}[U]$ and $g \in G$, $\psi(p\,g) = \psi(p)\,g$, and fibre preserving, i.e. $\mathrm{pr}_1 \circ \psi = \pi$. The right G-action on $U \times G$ is given by, for all $x \in U$ and $g, g' \in G$, $(x, g)\,g' := (x, g\,g')$ and $\mathrm{pr}_1 : U \times G \to U$ denotes the canonical projection on the first factor.

We call P the **total space**, M the **base space**, G the **structure group** and π the **projection**.

Definition 2.2. Let M_i be a manifold, G_i a Lie group and (P_i, r_i) a principal G_i-bundle over M_i, $i = 1, 2$. A **principal bundle map** is a pair of smooth maps $F = \big(f : P_1 \to P_2, \phi : G_1 \to G_2\big)$, such that ϕ is a homomorphism of Lie groups and f satisfies, for all $p \in P_1$ and $g \in G_1$, $f(p\,g) = f(p)\,\phi(g)$.

Remark 2.3. For every principal bundle map $F = \big(f : P_1 \to P_2, \phi : G_1 \to G_2\big)$ there exists a unique smooth map $\underline{f} : M_1 \to M_2$, such that the following diagram commutes:

$$\begin{array}{ccc} P_1 & \xrightarrow{\quad f \quad} & P_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\quad \underline{f} \quad} & M_2 \end{array} \tag{2.1}$$

We now define a suitable category of Abelian principal bundles over spacetimes.

Definition 2.4. The category $\mathsf{PrBuGlobHyp}$ consists of the following objects and morphisms:

- An object in $\mathsf{PrBuGlobHyp}$ is a triple $\Xi = \big((M, \mathfrak{o}, g, \mathfrak{t}), (G, h), (P, r)\big)$, where $(M, \mathfrak{o}, g, \mathfrak{t})$ is a globally hyperbolic spacetime, G is a connected Abelian Lie group with bi-invariant pseudo-Riemannian metric h and (P, r) is a principal G-bundle over M.

- A morphism between two objects Ξ_i, $i = 1, 2$, in $\mathsf{PrBuGlobHyp}$ is a principal bundle map $F = \big(f : P_1 \to P_2, \phi : G_1 \to G_2\big)$, such that $\phi : G_1 \to G_2$ is an isometry and $\underline{f} : M_1 \to M_2$ is an orientation and time-orientation preserving isometric embedding with $\underline{f}[M_1] \subseteq M_2$ causally compatible and open.

Remark 2.5. The category $\mathsf{PrBuGlobHyp}$ is quite big in the sense that it contains principal bundles for all possible connected Abelian structure groups. In physics it might be of interest to study only the case $G = U(1)$ which corresponds to electromagnetism. This can be achieved by restricting all functors that we will construct in this paper to the full subcategory $\mathsf{PrBuGlobHyp}^G$ defined by the subcollection of objects $\Xi = \big((M, \mathfrak{o}, g, \mathfrak{t}), (G, h), (P, r)\big)$ where G is fixed. We have decided to include a bi-invariant pseudo-Riemannian metric h on the structure group G in the data of the category. This datum is equivalent to an ad-invariant inner product (possibly indefinite) on the Lie algebra $\mathfrak{g}$ of G, which is required to specify the action functional and therewith a covariant Poisson bracket for the gauge theory.

Let M be a manifold, G a Lie group and (P, r) a principal G-bundle over M. For every manifold N with a smooth left G-action $\rho : G \times N \to N\,,\ (g, \xi) \mapsto g\,\xi$ there exists a fibre bundle over M associated to (P, r) with N as typical fibre: Consider the Cartesian product $P \times N$ and define the following right G-action $P \times N \times G \to P \times N\,,\ (p, \xi, g) \mapsto (p\,g, g^{-1}\,\xi)$. Denote by $P_N := (P \times N)/G$ the quotient of this right G-action and define the map $\pi_N : P_N \to M\,,\ [p, \xi] \mapsto \pi(p)$, which is well-defined since $\pi(p\,g) = \pi(p)$, for all $p \in P$ and $g \in G$. The data (P_N, M, π_N, N) specifies a fibre bundle (the local trivialization is shown to exist in [KN96]), which we call the **(N, ρ)-associated bundle to** (P, r).

Of particular relevance for us is the case where N is the Lie algebra $\mathfrak{g}$ of the Lie group G and ρ is the adjoint action $\mathrm{ad} : G \times \mathfrak{g} \to \mathfrak{g}$. The $(\mathfrak{g}, \mathrm{ad})$-associated bundle is called the **adjoint bundle** of the principal bundle (P, r) and we denote it also by $\big(\mathrm{ad}(P), M, \pi_{\mathfrak{g}}, \mathfrak{g}\big)$. We notice that the metric h on the Lie group G specifies a fibre metric on the adjoint bundle

$$\mathrm{ad}(P) \times_M \mathrm{ad}(P) \to M \times \mathbb{R}\,,\ \ ([p, \xi], [p', \xi']) \mapsto (\pi(p), h(\xi, \xi'))\,. \tag{2.2}$$

Lemma 2.6. *Let M be a manifold, G an Abelian Lie group and (P, r) a principal G-bundle over M. Then $\mathrm{ad}(P) = M \times \mathfrak{g}$, i.e. the adjoint bundle is trivial.*

Proof. Since G is Abelian the adjoint action is trivial, which implies $\mathrm{ad}(P) = (P \times \mathfrak{g})/G = P/G \times \mathfrak{g} = M \times \mathfrak{g}$. □

Any principal bundle map $F = \big(f : P_1 \to P_2, \phi : G_1 \to G_2\big)$ induces a vector bundle map $\big(F_{\mathrm{ad}(P)} : \mathrm{ad}(P_1) \to \mathrm{ad}(P_2), \underline{f} : M_1 \to M_2\big)$ between the corresponding adjoint bundles, where

$$F_{\mathrm{ad}(P)} : \mathrm{ad}(P_1) \to \mathrm{ad}(P_2)\ , \quad [p, \xi] \mapsto [f(p), \phi_*(\xi)] \tag{2.3}$$

and $\phi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ denotes the push-forward. Since ϕ is an isometry this vector bundle map preserves the fibre metrics. By Lemma 2.6 we have that for Abelian structure groups $\mathrm{ad}(P_i) = M_i \times \mathfrak{g}_i$, $i = 1, 2$, and thus (2.3) reads

$$F_{M\times\mathfrak{g}} : M_1 \times \mathfrak{g}_1 \to M_2 \times \mathfrak{g}_2\ , \quad (x, \xi) \mapsto \big(\underline{f}(x), \phi_*(\xi)\big)\ . \tag{2.4}$$

We will now show that the association of the adjoint bundle is functorial.

Definition 2.7. The category $\mathsf{VeBuGlobHyp}$ consists of the following objects and morphisms:

- An object in $\mathsf{VeBuGlobHyp}$ is a pair $\big((M, \mathfrak{o}, g, \mathfrak{t}), (\mathsf{V}, M, \pi_{\mathsf{V}}, V)\big)$, where $(M, \mathfrak{o}, g, \mathfrak{t})$ is a globally hyperbolic spacetime and $(\mathsf{V}, M, \pi_{\mathsf{V}}, V)$ is a vector bundle over M.
- A morphism between two objects $\big((M_i, \mathfrak{o}_i, g_i, \mathfrak{t}_i), (\mathsf{V}_i, M_i, \pi_{i\mathsf{V}}, V_i)\big)$, $i = 1, 2$, in $\mathsf{VeBuGlobHyp}$ is a vector bundle map $\big(f : \mathsf{V}_1 \to \mathsf{V}_2, \underline{f} : M_1 \to M_2\big)$, such that $f|_x : \mathsf{V}_1|_x \to \mathsf{V}_2|_{\underline{f}(x)}$ is a vector space isomorphism, for all $x \in M_1$, and $\underline{f} : M_1 \to M_2$ is an orientation and time-orientation preserving isometric embedding with $\underline{f}[M_1] \subseteq M_2$ causally compatible and open.

Lemma 2.8. *There is a covariant functor $\mathfrak{Ad} : \mathsf{PrBuGlobHyp} \to \mathsf{VeBuGlobHyp}$. It is specified on objects by $\mathfrak{Ad}(\Xi) = \big((M, \mathfrak{o}, g, \mathfrak{t}), (\mathrm{ad}(P), M, \pi_{\mathfrak{g}}, \mathfrak{g})\big)$ and on morphisms by $\mathfrak{Ad}(F) = (F_{\mathrm{ad}(P)}, \underline{f})$, with $F_{\mathrm{ad}(P)}$ given in (2.3).*

Proof. Let Ξ be an object in $\mathsf{PrBuGlobHyp}$, then $\mathfrak{Ad}(\Xi) = \big((M, \mathfrak{o}, g, \mathfrak{t}), (\mathrm{ad}(P), M, \pi_{\mathfrak{g}}, \mathfrak{g})\big)$ is an object in $\mathsf{VeBuGlobHyp}$. Let F be a morphism in $\mathsf{PrBuGlobHyp}$, then $\mathfrak{Ad}(F) = (F_{\mathrm{ad}(P)}, \underline{f})$ is a morphism in $\mathsf{VeBuGlobHyp}$, since the push-forward ϕ_* of the isometry ϕ is a vector space isomorphism.

For the identity $\mathrm{id}_\Xi = \big(\mathrm{id}_P : P \to P, \mathrm{id}_G : G \to G\big)$ we obtain $\underline{\mathrm{id}_P} = \mathrm{id}_M$, $\phi_* = \mathrm{id}_{\mathfrak{g}}$ and hence by (2.3) it holds $\mathfrak{Ad}(\mathrm{id}_\Xi) = (\mathrm{id}_{\mathrm{ad}(P)}, \mathrm{id}_M)$. For two morphisms $F : \Xi_1 \to \Xi_2$ and $F' : \Xi_2 \to \Xi_3$ in $\mathsf{PrBuGlobHyp}$ we obtain $\mathfrak{Ad}(F' \circ F) = \big((F' \circ F)_{\mathrm{ad}(P)}, \underline{(f' \circ f)}\big) = \big(F'_{\mathrm{ad}(P)} \circ F_{\mathrm{ad}(P)}, \underline{f'} \circ \underline{f}\big) = \mathfrak{Ad}(F') \circ \mathfrak{Ad}(F)$. □

Remark 2.9. We can also associate functorially to any object Ξ in $\mathsf{PrBuGlobHyp}$ a vector bundle as in $\mathsf{VeBuGlobHyp}$ equipped with the fibre metric (2.2) and to any morphism F in $\mathsf{PrBuGlobHyp}$ a vector bundle map as in $\mathsf{VeBuGlobHyp}$ which preserves the fibre metrics. We refrain from introducing yet another notation for a category of vector bundles with fibre metrics and remember this fact when necessary.

2.3 Principal connections

Connections on principal bundles constitute the fundamental degrees of freedom in gauge theories of Yang-Mills type. In this subsection we will review the relevant definitions and properties following [KN96].

Definition 2.10. Let M be a manifold, G a Lie group and (P, r) a principal G-bundle over M. A **connection form** on (P, r) is a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying:

(i) $\omega(X_p^\xi) = \xi$, for all $\xi \in \mathfrak{g}$ and $p \in P$, where $X_p^\xi \in T_pP$ is the fundamental vector at p corresponding to ξ.

(ii) $r_g^*(\omega) = \mathrm{ad}_{g^{-1}}(\omega)$, for all $g \in G$.

We denote the set of all connection forms by $\mathrm{Con}(P)$.

Remark 2.11. Due to [KN96, Chapter II, Theorem 2.1] there exists a connection form, i.e. $\mathrm{Con}(P) \neq \emptyset$.

Definition 2.12. Let $\Omega^k(P, \mathfrak{g})$ be the vector space of $\mathfrak{g}$-valued k-forms, $k = 0, \dots, \dim(P)$.

(i) We call $\eta \in \Omega^k(P, \mathfrak{g})$ G-**equivariant**, if $r_g^*(\eta) = \mathrm{ad}_{g^{-1}}(\eta)$, for all $g \in G$.

(ii) We call $\eta \in \Omega^k(P, \mathfrak{g})$ **horizontal**, if $\eta(Y_1, \dots, Y_k) = 0$ whenever at least one $Y_i \in T_pP$ is vertical, i.e. $\pi_*(Y_i) = 0$.

The vector space of G-equivariant and horizontal $\mathfrak{g}$-valued k-forms is denoted by $\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{eqv}}$.

Proposition 2.13. *Let M be a manifold, G an Abelian Lie group and (P, r) a principal G-bundle over M. Then there exists a linear isomorphism between $\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{eqv}}$ and $\Omega^k(M, \mathfrak{g})$, for all $k = 0, \dots, \dim(M)$.*

Proof. Let $\eta \in \Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{eqv}}$ be arbitrary. We define an element $\underline{\eta} \in \Omega^k(M, \mathfrak{g})$ by, for all $X_1, \dots, X_k \in T_xM$, $x \in M$,

$$\underline{\eta}(X_1, \dots, X_k) := \eta(Y_1, \dots, Y_k) \;, \tag{2.5}$$

where $Y_1, \dots, Y_k \in T_pP$ are tangent vectors at $p \in \pi^{-1}[\{x\}]$, such that $\pi_*(Y_i) = X_i$, for all i. Since η is horizontal, $\underline{\eta}$ does not depend on the choice of such Y_i. (We can in particular set $Y_i = X_{i\,p}^{\uparrow_\omega}$ as the horizontal lift of X_i with respect to some connection.) Due to G-equivariance the construction does not depend on the choice of $p \in \pi^{-1}[\{x\}]$.

Let now $\eta \in \Omega^k(M, \mathfrak{g})$ be arbitrary and consider the pull-back $\overline{\eta} := \pi^*(\eta) \in \Omega^k(P, \mathfrak{g})$. This element is G-equivariant, since, for all $g \in G$, $r_g^*(\overline{\eta}) = (\pi \circ r_g)^*(\eta) = \pi^*(\eta) = \overline{\eta}$. It is also horizontal, since for all $Y_1, \dots, Y_k \in T_pP$ with at least one vector vertical (this vector is annihilated by π_*) we have

$$\overline{\eta}(Y_1, \dots, Y_k) = \eta\big(\pi_*(Y_1), \dots, \pi_*(Y_k)\big) = 0 \;. \tag{2.6}$$

These two identifications provide the desired vector space isomorphism. □

Lemma 2.14. *Let M be a manifold, G an Abelian Lie group and (P, r) a principal G-bundle over M. Let us define the map*

$$\Phi : \mathrm{Con}(P) \times \Omega^1(M, \mathfrak{g}) \to \mathrm{Con}(P) \;, \quad (\omega, \eta) \mapsto \Phi(\omega, \eta) = \omega + \overline{\eta} \;. \tag{2.7}$$

Then $\big(\mathrm{Con}(P), \Omega^1(M, \mathfrak{g}), \Phi\big)$ is an affine space.

Proof. The one-form $\omega + \overline{\eta} \in \Omega^1(P, \mathfrak{g})$ is an element in $\mathrm{Con}(P)$, since $\overline{\eta}$ is horizontal and G-equivariant. The action (2.7) is free and transitive. □

Definition 2.15. Let M be a manifold, G a Lie group and (P, r) a principal G-bundle over M. The **curvature** is given by the following map

$$\mathcal{F} : \mathrm{Con}(P) \to \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{eqv}} \;, \quad \omega \mapsto \mathcal{F}(\omega) = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{g}} \;, \tag{2.8}$$

where d is the exterior differential and $[\cdot, \cdot]_{\mathfrak{g}}$ denotes the Lie bracket on $\mathfrak{g}$. In case G is Abelian, the curvature reads, for all $\omega \in \mathrm{Con}(P)$, $\mathcal{F}(\omega) = \mathrm{d}\omega$, since the Lie bracket is trivial.

Remark 2.16. Let G be an Abelian Lie group. Applying Proposition 2.13 we can consider equivalently the curvature as a map

$$\underline{\mathcal{F}} : \mathrm{Con}(P) \to \Omega^2(M, \mathfrak{g}) \;, \quad \omega \mapsto \underline{\mathcal{F}}(\omega) = \underline{\mathcal{F}(\omega)} = \underline{\mathrm{d}\omega} \;. \tag{2.9}$$

As a consequence of the (Abelian) Bianchi identity $\mathrm{d}\mathcal{F}(\omega) = \mathrm{dd}\omega = 0$, for all $\omega \in \mathrm{Con}(P)$, we obtain that $\underline{\mathcal{F}}(\omega) \in \Omega^2(M, \mathfrak{g})$ is closed, for all $\omega \in \mathrm{Con}(P)$.

Lemma 2.17. *Let M be a manifold, G an Abelian Lie group and (P,r) a principal G-bundle over M. The map $\underline{\mathcal{F}} : \mathrm{Con}(P) \to \Omega^2(M,\mathfrak{g})$ is an affine map with linear part $\underline{\mathcal{F}}_V : \Omega^1(M,\mathfrak{g}) \to \Omega^2(M,\mathfrak{g})\,,\ \eta \mapsto \mathrm{d}\eta$.*

Proof. Let $\omega \in \mathrm{Con}(P)$ and $\eta \in \Omega^1(M,\mathfrak{g})$ be arbitrary, then

$$\underline{\mathcal{F}}(\omega+\overline{\eta}) = \underline{\mathrm{d}\omega + \mathrm{d}\overline{\eta}} = \underline{\mathcal{F}}(\omega) + \underline{\mathrm{d}\pi^*(\eta)} = \underline{\mathcal{F}}(\omega) + \underline{\pi^*(\mathrm{d}\eta)} = \underline{\mathcal{F}}(\omega) + \mathrm{d}\eta\,. \tag{2.10}$$

□

2.4 The Atiyah sequence

We present the Atiyah sequence only for Abelian principal bundles and refer to [Ati57] for the general case. Let us consider the tangent bundle $\big(TP,P,\pi_{TP},\mathbb{R}^{\dim(P)}\big)$ over P. On the total space TP there is a right G-action in terms of the push-forward of tangent vectors

$$r_* : TP \times G \to TP\,,\quad (Y,g) \mapsto r_{g\,*}(Y)\,. \tag{2.11}$$

For any $Y \in T_pP$ we have $r_{g\,*}(Y) \in T_{p\,g}P$ and hence $\pi_{TP} \circ r_{g\,*} = r_g \circ \pi_{TP}$, for all $g \in G$. In other words, $\pi_{TP} : TP \to P$ is G-equivariant. As a consequence, we can define the quotient bundle $\big(TP/G, P/G, \pi\circ\pi_{TP}, \mathbb{R}^{\dim(P)}\big)$, which is a vector bundle over $M = P/G$. We denote the projection of this vector bundle by $\pi_{TP/G} := \pi \circ \pi_{TP}$.

The push-forward of $\pi : P \to M$ gives a vector bundle map from the tangent bundle over P to the tangent bundle $\big(TM, M, \pi_{TM}, \mathbb{R}^{\dim(M)}\big)$ over M, i.e. the following diagram commutes:

$$\begin{array}{ccc} TP & \xrightarrow{\ \pi_*\ } & TM \\ {\scriptstyle \pi_{TP}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{TM}} \\ P & \xrightarrow{\ \pi\ } & M \end{array} \tag{2.12}$$

Since $\pi \circ r_g = \pi$, for all $g \in G$, and thus also $\pi_* \circ r_{g\,*} = (\pi \circ r_g)_* = \pi_*$, for all $g \in G$, we can perform the quotient by G and obtain the vector bundle map (denoted with a slight abuse of notation by the same symbol):

$$\begin{array}{ccc} TP/G & \xrightarrow{\ \pi_*\ } & TM \\ {\scriptstyle \pi_{TP/G}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{TM}} \\ M & \xrightarrow{\ \mathrm{id}_M\ } & M \end{array} \tag{2.13}$$

There is also a vector bundle map from the adjoint bundle (remember that $\mathrm{ad}(P) = M \times \mathfrak{g}$ since G is Abelian, cf. Lemma 2.6) $\big(M\times\mathfrak{g}, M, \mathrm{pr}_1, \mathfrak{g}\big)$ to $\big(TP/G, M, \pi_{TP/G}, \mathbb{R}^{\dim(P)}\big)$:

$$\begin{array}{ccc} M\times\mathfrak{g} & \xrightarrow{\ \iota\ } & TP/G \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{TP/G}} \\ M & \xrightarrow{\ \mathrm{id}_M\ } & M \end{array} \tag{2.14}$$

The map ι is defined by, for all $(x,\xi) \in M\times\mathfrak{g}$, $\iota(x,\xi) := \big[X_p^\xi\big]$, where $X_p^\xi \in T_pP$ is the fundamental vector corresponding to ξ and $p \in \pi^{-1}[\{x\}]$ is arbitrary. Indeed, the map ι is well-defined, since for any other $p' \in \pi^{-1}[\{x\}]$ there exists a $g \in G$, such that $p' = p\,g$ and hence $\big[X_{p'}^\xi\big] = \big[X_{p\,g}^\xi\big] = \big[r_{g\,*}(X_p^{\mathrm{ad}_g(\xi)})\big] = \big[r_{g\,*}(X_p^\xi)\big] = \big[X_p^\xi\big]$, where we have again used that G is Abelian.

Consider now also the trivial vector bundle $\big(M\times\{0\}, M, \mathrm{pr}_1, \{0\}\big)$ and the following two vector bundle maps

$$\begin{array}{ccc} M\times\{0\} & \xrightarrow{\ \alpha\ } & M\times\mathfrak{g} \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ M & \xrightarrow{\ \mathrm{id}_M\ } & M \end{array} \tag{2.15a}$$

with $\alpha(x,0) = (x,0)$, for all $x \in M$, and

$$\begin{array}{ccc} TM & \xrightarrow{\quad\beta\quad} & M \times \{0\} \\ {\scriptstyle \pi_{TM}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ M & \xrightarrow{\quad\mathrm{id}_M\quad} & M \end{array} \tag{2.15b}$$

with $\beta(X) = (\pi_{TM}(X), 0)$, for all $X \in TM$.

Composing (2.15a), (2.14), (2.13) and (2.15b) we obtain the following sequence of vector bundle maps (we can drop the base space maps since they are all given by id_M)

$$M \times \{0\} \xrightarrow{\ \alpha\ } M \times \mathfrak{g} \xrightarrow{\ \iota\ } TP/G \xrightarrow{\ \pi_*\ } TM \xrightarrow{\ \beta\ } M \times \{0\}\ . \tag{2.16}$$

This is the **Atiyah sequence** [Ati57]. For completeness, we review the following

Proposition 2.18. *The Atiyah sequence (2.16) is a short exact sequence.*

Proof. First, we have to show that the composition of two subsequent maps is the trivial map, i.e. the vector bundle map which restricted to all fibres is 0. For $\iota \circ \alpha$ this property holds true due to linearity. Let now $(x,\xi) \in M \times \mathfrak{g}$, then $\pi_*(\iota(x,\xi)) = \pi_*(X_p^\xi) = 0$, since X_p^ξ is by construction a vertical vector. For $\beta \circ \pi_*$ this property holds trivially.

Next, we have to prove exactness at every step: Let $(x,\xi) \in M \times \mathfrak{g}$ be such that $\iota(x,\xi) = [X_p^\xi] = 0$. This implies that $X_p^\xi = 0$ and since $X_p^\bullet$ is a vector space isomorphism between $\mathfrak{g}$ and vertical vectors at $p \in P$ we find $\xi = 0$. Let now $[Y] \in TP/G|_x$ be such that $\pi_*([Y]) = 0$. This implies that any representative $Y \in T_pP$ (where $p \in \pi^{-1}[\{x\}]$) is vertical and due to the aforementioned isomorphism there exists a $\xi \in \mathfrak{g}$, such that $\iota(x,\xi) = [X_p^\xi] = [Y]$. For the last step let $X \in T_xM$ be such that $\beta(X) = (x,0)$. This condition is satisfied for all X. Using a local trivialization of P we can lift $X \in T_xM$ to a vector $\widehat{X} \in T_pP$ (where $p \in \pi^{-1}[\{x\}]$), such that $\pi_*(\widehat{X}) = X$. The equivalence class $[\widehat{X}] \in TP/G|_x$ is the element which proves exactness at this step. □

Similar to Lemma 2.8, one can show that all vector bundles appearing in the Atiyah sequence (2.16) are assigned by a covariant functor from PrBuGlobHyp to VeBuGlobHyp. We do not repeat all the steps in this proof and just give an explicit expression for the induced maps: Let $F = \big(f : P_1 \to P_2, \phi : G_1 \to G_2\big)$ be a morphism between two objects $\Xi_i = \big((M_i, \mathfrak{o}_i, g_i, \mathfrak{t}_i), (G_i, h_i), (P_i, r_i)\big)$, $i = 1,2$, in PrBuGlobHyp. Then the induced vector bundle maps (covering $\underline{f}$) are given by

$$F_{M\times\{0\}} : M_1 \times \{0\} \to M_2 \times \{0\}\ ,\quad (x,0) \mapsto \big(\underline{f}(x), 0\big)\ , \tag{2.17a}$$
$$F_{M\times\mathfrak{g}} : M_1 \times \mathfrak{g}_1 \to M_2 \times \mathfrak{g}_2\ ,\quad (x,\xi) \mapsto \big(\underline{f}(x), \phi_*(\xi)\big)\ , \tag{2.17b}$$
$$F_{TP/G} : TP_1/G_1 \to TP_2/G_2\ ,\quad [Y] \mapsto [f_*(Y)]\ , \tag{2.17c}$$
$$F_{TM} : TM_1 \to TM_2\ ,\quad X \mapsto \underline{f}_*(X)\ . \tag{2.17d}$$

Notice further that for α_i, ι_i, $\pi_{i\,*}$, β_i denoting the vector bundle maps in the Atiyah sequence (2.16) for the object Ξ_i in PrBuGlobHyp, $i = 1,2$, we obtain the commuting diagram:

$$\begin{array}{ccccccccc} M_1 \times \{0\} & \xrightarrow{\alpha_1} & M_1 \times \mathfrak{g}_1 & \xrightarrow{\iota_1} & TP_1/G_1 & \xrightarrow{\pi_{1\,*}} & TM_1 & \xrightarrow{\beta_1} & M_1 \times \{0\} \\ {\scriptstyle F_{M\times\{0\}}}\big\downarrow & & \big\downarrow{\scriptstyle F_{M\times\mathfrak{g}}} & & \big\downarrow{\scriptstyle F_{TP/G}} & & \big\downarrow{\scriptstyle F_{TM}} & & \big\downarrow{\scriptstyle F_{M\times\{0\}}} \\ M_2 \times \{0\} & \xrightarrow{\alpha_2} & M_2 \times \mathfrak{g}_2 & \xrightarrow{\iota_2} & TP_2/G_2 & \xrightarrow{\pi_{2\,*}} & TM_2 & \xrightarrow{\beta_2} & M_2 \times \{0\} \end{array} \tag{2.18}$$

2.5 The bundle of connections

We show that the affine space of connections $\big(\mathrm{Con}(P), \Omega^1(M,\mathfrak{g}), \Phi\big)$ constructed in Lemma 2.14 is isomorphic to the affine space of sections of an affine bundle over M.

Consider the bundle $\big(\mathrm{Hom}(TM,TP/G), M, \pi_{\mathrm{Hom}(TM,TP/G)}, \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)},\mathbb{R}^{\dim(P)})\big)$ of homomorphisms, that is a vector bundle. Sections of this bundle are in bijective correspondence with vector bundle maps $\big(\lambda : TM \to TP/G, \mathrm{id}_M : M \to M\big)$. We say that such a vector bundle map is a **splitting of the Atiyah sequence** (2.16), if $\pi_* \circ \lambda = \mathrm{id}_{TM}$. These splittings can be described equivalently by sections of a subbundle of $\mathrm{Hom}(TM,TP/G)$.

Definition 2.19. The **bundle of connections** $\big(\mathcal{C}(P), M, \pi_{\mathcal{C}(P)}, A\big)$ is the subbundle of the homomorphism bundle $\big(\mathrm{Hom}(TM,TP/G), M, \pi_{\mathrm{Hom}(TM,TP/G)}, \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)},\mathbb{R}^{\dim(P)})\big)$ specified by the submanifold $\mathcal{C}(P) := \big\{\lambda \in \mathrm{Hom}(TM,TP/G) : \pi_* \circ \lambda = \mathrm{id}_{TM}\big\}$.

Remark 2.20. The typical fibre A is the set of all linear maps $L \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)},\mathbb{R}^{\dim(P)})$ satisfying $\widetilde{\pi_*} \circ L = \mathrm{id}_{\mathbb{R}^{\dim(M)}}$, where $\widetilde{\pi_*} \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(P)},\mathbb{R}^{\dim(M)})$ is given in a basis $\{e_i \in \mathbb{R}^{\dim(M)} : i = 1,\dots,\dim(M)\}$ and $\{E_a \in \mathbb{R}^{\dim(P)} : a = 1,\dots,\dim(P)\}$ by

$$\widetilde{\pi_*}(E_a) = \begin{cases} e_i & , \text{ for } a = i \in \{1,\dots,\dim(M)\}\ , \\ 0 & , \text{ else }. \end{cases} \tag{2.19}$$

Notice that A is an affine space modeled on $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)},\mathbb{R}^{\dim(P)-\dim(M)})$, which is the typical fibre of the homomorphism bundle $\mathrm{Hom}(TM, M\times\mathfrak{g})$.

We define affine bundles following [KMS93, Chapter 6.22] and [BDS12].

Definition 2.21. An **affine bundle** is a triple $\big(M, (\mathsf{A}, M, \pi_{\mathsf{A}}, A), (\mathsf{V}, M, \pi_{\mathsf{V}}, V)\big)$, where M is a manifold, $(\mathsf{A}, M, \pi_{\mathsf{A}}, A)$ is a fibre bundle over M and $(\mathsf{V}, M, \pi_{\mathsf{V}}, V)$ is a vector bundle over M, such that

(i) for all $x \in M$, the fibre $\mathsf{A}|_x$ is an affine space modeled on $\mathsf{V}|_x$,

(ii) the typical fibre A is an affine space modeled on the typical fibre V,

(iii) for all $x \in M$, there exists a local bundle chart (U,ψ) of $(\mathsf{A}, M, \pi_{\mathsf{A}}, A)$ and a local vector bundle chart (U,ψ_{V}) of $(\mathsf{V}, M, \pi_{\mathsf{V}}, V)$, such that, for all $y \in U$, $\psi|_y : \mathsf{A}|_y \to A$ is an affine space isomorphism with linear part $\psi|_{yV} = \psi_{\mathsf{V}}|_y : \mathsf{V}|_y \to V$. We call the triple $(U,\psi,\psi_{\mathsf{V}})$ a local affine bundle chart.

Proposition 2.22. *Let M be a manifold, G an Abelian Lie group and (P,r) a principal G-bundle over M. The bundle of connections $\big(\mathcal{C}(P), M, \pi_{\mathcal{C}(P)}, A\big)$ is an affine bundle modeled on $\big(\mathrm{Hom}(TM, M\times\mathfrak{g}), M, \pi_{\mathrm{Hom}(TM,M\times\mathfrak{g})}, \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)},\mathbb{R}^{\dim(\mathfrak{g})})\big)$.*

Proof. Using the vector bundle embedding ι (see (2.16)) we can consider $\mathrm{Hom}(TM, M\times\mathfrak{g})$ as a vector subbundle of the homomorphism bundle $\mathrm{Hom}(TM,TP/G)$. By definition, the bundle of connections is also a subbundle of $\mathrm{Hom}(TM,TP/G)$. The vector space structure on the fibres of $\mathrm{Hom}(TM,TP/G)$ induces an affine space structure on the fibres of $\mathcal{C}(P)$ with underlying vector space given by the fibres of $\mathrm{Hom}(TM, M\times\mathfrak{g})$. By Remark 2.20, the typical fibre A is an affine space modeled on the typical fibre $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)},\mathbb{R}^{\dim(\mathfrak{g})})$ of $\mathrm{Hom}(TM, M\times\mathfrak{g})$. The local vector bundle charts of $\mathrm{Hom}(TM,TP/G)$ induce the required local affine bundle charts. □

By [BDS12, Lemma 2.20], the set of sections $\Gamma^\infty(M,\mathcal{C}(P))$ of the bundle of connections is an affine space modeled on the $C^\infty(M)$-module $\Gamma^\infty(M,\mathrm{Hom}(TM, M\times\mathfrak{g}))$. The latter is isomorphic (as a $C^\infty(M)$-module) to the $\mathfrak{g}$-valued one-forms on M, i.e. $\Omega^1(M,\mathfrak{g})$. Hence, $\big(\Gamma^\infty(M,\mathcal{C}(P)), \Omega^1(M,\mathfrak{g}), \widetilde{\Phi}\big)$ is an affine space, with action $\widetilde{\Phi} : \Gamma^\infty(M,\mathcal{C}(P)) \times \Omega^1(M,\mathfrak{g}) \to \Gamma^\infty(M,\mathcal{C}(P))$ given by, for all $\lambda \in \Gamma^\infty(M,\mathcal{C}(P))$, $\eta \in \Omega^1(M,\mathfrak{g})$ and $X \in T_xM$, $x \in M$,

$$\big(\widetilde{\Phi}(\lambda,\eta)\big)(X) := \lambda(X) + \iota\big(x, \eta(X)\big)\ . \tag{2.20}$$

Proposition 2.23. $\big(\Gamma^\infty(M,\mathcal{C}(P)),\Omega^1(M,\mathfrak{g}),\widetilde{\Phi}\big)$ *and* $\big(\mathrm{Con}(P),\Omega^1(M,\mathfrak{g}),\Phi\big)$ *(cf. Lemma 2.14) are isomorphic as affine spaces.*

Proof. Let $\omega\in\mathrm{Con}(P)$ be arbitrary. We define an element $\lambda_\omega\in\Gamma^\infty(M,\mathcal{C}(P))$ by, for all $X\in T_xM$, $x\in M$,

$$\lambda_\omega(X) := [X_p^{\uparrow\omega}]\;, \tag{2.21}$$

where $p\in\pi^{-1}[\{x\}]$ and $X_p^{\uparrow\omega}\in T_pP$ denotes the horizontal lift at p with respect to ω. By definition we have that $\pi_*\big(\lambda_\omega(X)\big)=X$. The equivalence class $[X_p^{\uparrow\omega}]$ is independent on the choice of p, since for any other $p'\in\pi^{-1}[\{x\}]$ there exists a $g\in G$, such that $p'=p\,g$ and hence $X_{p'}^{\uparrow\omega}=X_{p\,g}^{\uparrow\omega}=r_{g\,*}(X_p^{\uparrow\omega})$. The last equality follows from the G-equivariance of the horizontal subspaces, $\pi_*(r_{g\,*}(X_p^{\uparrow\omega}))=X$ and the uniqueness of the horizontal lift.

Let now $\lambda\in\Gamma^\infty(M,\mathcal{C}(P))$ and $Y\in T_pP$ with $p\in\pi^{-1}[\{x\}]$, $x\in M$, be arbitrary. By the splitting lemma, the element $[Y]\in TP/G|_x$ can be decomposed uniquely as $[Y]=\iota(x,\xi)+\lambda(X)=[X_p^\xi]+\lambda(X)$, where $\xi\in\mathfrak{g}$ and $X\in T_xM$. For the fixed element $p\in\pi^{-1}[\{x\}]$, there exist unique representatives $X_p^\xi\in T_pP$ of $[X_p^\xi]$ and $X_p^\uparrow\in T_pP$ of $\lambda(X)$, such that $Y=X_p^\xi+X_p^\uparrow$. We define $\omega_\lambda\in\Omega^1(P,\mathfrak{g})$ by setting

$$\omega_\lambda(Y)=\omega_\lambda(X_p^\xi+X_p^\uparrow)=\xi\;. \tag{2.22}$$

Condition (i) of Definition 2.10 is satisfied. Furthermore, ω_λ is G-equivariant (since G is Abelian this means G-invariant), for all $Y\in T_pP$,

$$\big(r_g^*(\omega_\lambda)\big)(Y)=\omega_\lambda\big(r_{g\,*}(Y)\big)=\omega_\lambda(X_{p\,g}^\xi+X_{p\,g}^\uparrow)=\xi=\omega_\lambda(Y)\;. \tag{2.23}$$

This shows that $\omega_\lambda\in\mathrm{Con}(P)$. The maps defined above provide a bijection between $\mathrm{Con}(P)$ and $\Gamma^\infty(M,\mathcal{C}(P))$.

We now show that they are also affine space isomorphisms. Let $\omega\in\mathrm{Con}(P)$, $\eta\in\Omega^1(M,\mathfrak{g})$ and consider $\omega':=\Phi(\omega,\eta)=\omega+\overline{\eta}\in\mathrm{Con}(P)$. The corresponding element $\lambda_{\omega'}\in\Gamma^\infty(M,\mathcal{C}(P))$ is defined by, for all $X\in T_xM$, $\lambda_{\omega'}(X)=[X_p^{\uparrow\omega'}]$, with $p\in\pi^{-1}[\{x\}]$ arbitrary. Using that $X_p^{\uparrow\omega'}=X_p^{\uparrow\omega}+X_p^\xi$ for some $\xi\in\mathfrak{g}$, we find

$$0=\omega'\big(X_p^{\uparrow\omega'}\big)=\omega(X_p^{\uparrow\omega'})+\overline{\eta}(X_p^{\uparrow\omega'})=\omega(X_p^{\uparrow\omega}+X_p^\xi)+\eta(X)=\xi+\eta(X)\;, \tag{2.24}$$

hence $\xi=-\eta(X)$. We obtain

$$\begin{aligned}\lambda_{\Phi(\omega,\eta)}(X)&=[X_p^{\uparrow\omega'}]=[X_p^{\uparrow\omega}+X_p^\xi]=\lambda_\omega(X)+[X_p^\xi]\\&=\lambda_\omega(X)+\iota(x,\xi)=\lambda_\omega(X)-\iota(x,\eta(X))\\&=\big(\widetilde{\Phi}(\lambda_\omega,-\eta)\big)(X)\;,\end{aligned} \tag{2.25}$$

which shows that the isomorphism $\lambda_\bullet:\mathrm{Con}(P)\to\Gamma^\infty(M,\mathcal{C}(P))\,,\;\omega\mapsto\lambda_\omega$ is an affine space isomorphism with linear part $\Omega^1(M,\mathfrak{g})\to\Omega^1(M,\mathfrak{g})\,,\;\eta\mapsto-\eta$. □

Corollary 2.24. *The map (denoted with a slight abuse of notation by the same symbol as the map in Lemma 2.17)*

$$\underline{\mathcal{F}}:\Gamma^\infty(M,\mathcal{C}(P))\to\Omega^2(M,\mathfrak{g})\;,\quad\lambda\mapsto\underline{\mathcal{F}}(\lambda):=\underline{\mathcal{F}(\omega_\lambda)} \tag{2.26}$$

is an affine differential operator in the sense of [BDS12, Section 3] with linear part $\underline{\mathcal{F}}_V:\Omega^1(M,\mathfrak{g})\to\Omega^2(M,\mathfrak{g})\,,\;\eta\mapsto-\mathrm{d}\eta$.

Proof. Combining Proposition 2.23 and Lemma 2.17 we obtain, for all $\lambda\in\Gamma^\infty(M,\mathcal{C}(P))$ and $\eta\in\Omega^1(M,\mathfrak{g})$,

$$\underline{\mathcal{F}}\big(\widetilde{\Phi}(\lambda,\eta)\big)=\underline{\mathcal{F}(\omega_{\widetilde{\Phi}(\lambda,\eta)})}=\underline{\mathcal{F}(\Phi(\omega_\lambda,-\eta))}=\underline{\mathcal{F}(\omega_\lambda)-\mathrm{d}\overline{\eta}}=\underline{\mathcal{F}}(\lambda)-\mathrm{d}\eta\;. \tag{2.27}$$

□

2.6 Morphisms and gauge transformations of connections

In this subsection we study in detail how morphisms in PrBuGlobHyp act on the bundle of connections. This will eventually lead to a functor from PrBuGlobHyp to a category AfBuGlobHyp of affine bundles. Finally, we focus on a special class of morphisms, namely that of gauge transformations.

We have seen in (2.17) that all vector bundles in the Atiyah sequence are obtained by covariant functors from PrBuGlobHyp to VeBuGlobHyp. Let us also consider the homomorphism bundles entering the bundle of connections, i.e. $\big(\mathrm{Hom}(TM, TP/G), M, \pi_{\mathrm{Hom}(TM,TP/G)}, \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)}, \mathbb{R}^{\dim(P)})\big)$ and $\big(\mathrm{Hom}(TM, M \times \mathfrak{g}), M, \pi_{\mathrm{Hom}(TM,M\times\mathfrak{g})}, \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{\dim(M)}, \mathbb{R}^{\dim(\mathfrak{g})})\big)$. Given two objects Ξ_i, $i = 1, 2$, and a morphism $F : \Xi_1 \to \Xi_2$ in PrBuGlobHyp we can induce from (2.17) the vector bundle maps (covering $\underline{f} : M_1 \to M_2$)

$$\begin{aligned} F_{\mathrm{Hom}(TM,TP/G)} : \mathrm{Hom}(TM_1, TP_1/G_1) &\to \mathrm{Hom}(TM_2, TP_2/G_2)\,, \\ \lambda &\mapsto F_{TP/G} \circ \lambda \circ F_{TM}^{-1}\,, \end{aligned} \tag{2.28a}$$

$$\begin{aligned} F_{\mathrm{Hom}(TM,M\times\mathfrak{g})} : \mathrm{Hom}(TM_1, M_1 \times \mathfrak{g}_1) &\to \mathrm{Hom}(TM_2, M_2 \times \mathfrak{g}_2)\,, \\ \eta &\mapsto F_{M\times\mathfrak{g}} \circ \eta \circ F_{TM}^{-1}\,. \end{aligned} \tag{2.28b}$$

Explicitly, (2.28a) maps $\lambda \in \mathrm{Hom}(TM_1, TP_1/G_1)|_x$ to $F_{\mathrm{Hom}(TM,TP/G)}(\lambda) = F_{TP/G}|_x \circ \lambda \circ F_{TM}|_x^{-1} \in \mathrm{Hom}(TM_2, TP_2/G_2)|_{\underline{f}(x)}$, which is well-defined since $F_{TM}|_x$ is a vector space isomorphism. Restricting the vector bundle maps (2.28a) and (2.28b) to fibres provides vector space isomorphisms, since by the hypotheses of Definition 2.4 also $F_{TP/G}|_x$ and $F_{M\times\mathfrak{g}}|_x$ are vector space isomorphisms, for all $x \in M_1$. As a consequence, these homomorphism bundles are obtained by covariant functors from PrBuGlobHyp to VeBuGlobHyp.

Also the bundle of connections of Proposition 2.22 is obtained functorially.

Definition 2.25. The category AfBuGlobHyp consists of the following objects and morphisms:

- An object in AfBuGlobHyp is a triple $\big((M, \mathfrak{o}, g, \mathfrak{t}), (\mathsf{A}, M, \pi_{\mathsf{A}}, A), (\mathsf{V}, M, \pi_{\mathsf{V}}, V)\big)$, where $(M, \mathfrak{o}, g, \mathfrak{t})$ is a globally hyperbolic spacetime and $(\mathsf{A}, M, \pi_{\mathsf{A}}, A)$ is an affine bundle over M modeled on the vector bundle $(\mathsf{V}, M, \pi_{\mathsf{V}}, V)$.
- A morphism between two objects $\big((M_i, \mathfrak{o}_i, g_i, \mathfrak{t}_i), (\mathsf{A}_i, M_i, \pi_{i\,\mathsf{A}}, A_i), (\mathsf{V}_i, M_i, \pi_{i\,\mathsf{V}}, V_i)\big)$, $i = 1, 2$, in AfBuGlobHyp is a fibre bundle map $\big(f : \mathsf{A}_1 \to \mathsf{A}_2, \underline{f} : M_1 \to M_2\big)$, such that $f|_x : \mathsf{A}_1|_x \to \mathsf{A}_2|_{\underline{f}(x)}$ is an affine space isomorphism, for all $x \in M_1$, and $\underline{f} : M_1 \to M_2$ is an orientation and time-orientation preserving isometric embedding with $\underline{f}[M_1] \subseteq M_2$ causally compatible and open.

Remark 2.26. Every morphism $(f, \underline{f})$ in AfBuGlobHyp determines a unique vector bundle map between the underlying vector bundles (that is a morphism in VeBuGlobHyp) by taking fibre-wise the linear part. We call this vector bundle map with a slight abuse of notation the linear part of $(f, \underline{f})$ and denote it by $(f_V, \underline{f})$.

Proposition 2.27. *There is a covariant functor $\mathcal{C}$: PrBuGlobHyp $\to$ AfBuGlobHyp. It associates to any object Ξ in PrBuGlobHyp the bundle of connections (cf. Proposition 2.22). To any morphism $F : \Xi_1 \to \Xi_2$ in PrBuGlobHyp the functor associates the restriction of the vector bundle map (2.28a) to the bundles of connections. The linear part is (2.28b)*

Proof. The nontrivial step is to show that (2.28a) restricts to a morphism between the bundles of connections. We define the induced fibre bundle map (covering $\underline{f}$)

$$\begin{aligned} F_{\mathcal{C}(P)} : \mathcal{C}(P_1) &\to \mathrm{Hom}(TM_2, TP_2/G_2)\,, \\ \lambda &\mapsto F_{TP/G} \circ \lambda \circ F_{TM}^{-1} \end{aligned} \tag{2.29}$$

and obtain, for all $\lambda \in \mathcal{C}(P_1)$,

$$\begin{aligned} \pi_{2*} \circ F_{\mathcal{C}(P)}(\lambda) &= \pi_{2*} \circ F_{TP/G} \circ \lambda \circ F_{TM}^{-1} = F_{TM} \circ \pi_{1*} \circ \lambda \circ F_{TM}^{-1} \\ &= F_{TM} \circ F_{TM}^{-1} = \mathrm{id}_{TM_2}\,, \end{aligned} \tag{2.30}$$

where we used in the second equality (2.18) and in the third equality that λ is a splitting of the Atiyah sequence. This implies that $F_{\mathcal{C}(P)} : \mathcal{C}(P_1) \to \mathcal{C}(P_2)$ is a fibre bundle map covering $\underline{f}$.

It remains to show that the restrictions $F_{\mathcal{C}(P)}|_x : \mathcal{C}(P_1)|_x \to \mathcal{C}(P_2)|_{\underline{f}(x)}$ are affine space isomorphisms, for all $x \in M_1$. We obtain, for all $\lambda \in \mathcal{C}(P_1)|_x$ and $\eta \in \mathrm{Hom}(TM_1, M_1 \times \mathfrak{g}_1)|_x$,

$$\begin{aligned} F_{\mathcal{C}(P)}\big(\lambda + \iota_1 \circ \eta\big) &= F_{TP/G} \circ (\lambda + \iota_1 \circ \eta) \circ F_{TM}^{-1} = F_{\mathcal{C}(P)}(\lambda) + \iota_2 \circ F_{M\times\mathfrak{g}} \circ \eta \circ F_{TM}^{-1} \\ &= F_{\mathcal{C}(P)}(\lambda) + \iota_2 \circ F_{\mathrm{Hom}(TM, M\times\mathfrak{g})}(\eta) \ , \end{aligned} \tag{2.31}$$

where in the second equality we have used again (2.18) and in the last one (2.28b). Fibre-wise invertibility follows from the fibre-wise invertibility of (2.28a) and (2.28b). □

Remark 2.28. A morphism $F : \Xi_1 \to \Xi_2$ in $\mathsf{PrBuGlobHyp}$ acts via pull-back on sections of the bundle of connections, $F^* : \Gamma^\infty(M_2, \mathcal{C}(P_2)) \to \Gamma^\infty(M_1, \mathcal{C}(P_1))\ ,\ \lambda \mapsto F^*(\lambda) = F_{\mathcal{C}(P)}^{-1} \circ \lambda \circ \underline{f}$. A short calculation shows compatibility with the affine space structure (2.20), for all $\lambda \in \Gamma^\infty(M_2, \mathcal{C}(P_2))$ and $\eta \in \Omega^1(M_2, \mathfrak{g}_2)$,

$$F^*\big(\widetilde{\Phi}_2(\lambda, \eta)\big) = \widetilde{\Phi}_1\big(F^*(\lambda), \underline{f}^*(\phi_*^{-1}(\eta))\big) \ , \tag{2.32}$$

where $\underline{f}^*$ is the pull-back of differential forms along $\underline{f} : M_1 \to M_2$ and $\phi_*^{-1}(\eta) \in \Omega^k(M_2, \mathfrak{g}_1)$ is defined by, for all $X_1, \dots, X_k \in T_x M_2$, $x \in M_2$, $\big(\phi_*^{-1}(\eta)\big)(X_1, \dots, X_k) := \phi_*^{-1}\big(\eta(X_1, \dots, X_k)\big)$.

We now study in detail a special, however very important, class of morphisms.

Definition 2.29. Let M be a manifold, G a Lie group and (P, r) a principal G-bundle over M. A **gauge transformation** is a G-equivariant diffeomorphism $f : P \to P$, such that $\underline{f} = \mathrm{id}_M$. We denote by $\mathrm{Gau}(P)$ the group of all gauge transformations of (P, r), where the group operation is given by the usual composition of morphisms.

Notice that whenever $\Xi = \big((M, \mathfrak{o}, g, \mathfrak{t}), (G, h), (P, r)\big)$ is an object in $\mathsf{PrBuGlobHyp}$, a gauge transformation $f \in \mathrm{Gau}(P)$ is an automorphism $F = (f, \mathrm{id}_G)$ in the same category.

Lemma 2.30. *Let M be a manifold, G an Abelian Lie group and (P, r) a principal G-bundle over M. Then there is a group isomorphism between $\mathrm{Gau}(P)$ and $C^\infty(M, G)$, where the latter group is equipped with the point-wise group operation.*

Proof. Let $f \in \mathrm{Gau}(P)$ be arbitrary. Then there exists a unique $\widetilde{f} \in C^\infty(P, G)$, such that $f(p) = p\,\widetilde{f}(p)$, for all $p \in P$. Since f is G-equivariant and G is Abelian we obtain that $\widetilde{f}$ is G-invariant, i.e. $\widetilde{f}(p\,g) = \widetilde{f}(p)$, for all $g \in G$ and $p \in P$. Hence, it canonically induces a unique $\widehat{f} \in C^\infty(M, G)$ on the quotient $M = P/G$. Vice versa, for any $\widehat{f} \in C^\infty(M, G)$ we define an element $f \in \mathrm{Gau}(P)$ by $f(p) = p\,\widehat{f}(\pi(p))$, for all $p \in P$. This bijection is a group isomorphism, since for all $f_1, f_2 \in \mathrm{Gau}(P)$ and $p \in P$,

$$\begin{aligned} \big(f_1 \circ f_2\big)(p) &= f_1\big(p\,\widehat{f}_2(\pi(p))\big) = f_1(p)\,\widehat{f}_2(\pi(p)) \\ &= p\,\widehat{f}_1(\pi(p))\,\widehat{f}_2(\pi(p)) = p\,(\widehat{f}_1\,\widehat{f}_2)(\pi(p)) \ . \end{aligned} \tag{2.33}$$

□

By Remark 2.28 we obtain that a gauge transformation $f \in \mathrm{Gau}(P)$ acts on $\lambda \in \Gamma^\infty(M, \mathcal{C}(P))$ via

$$f^*(\lambda) = f_{TP/G}^{-1} \circ \lambda \ , \tag{2.34}$$

where we have used that $f_{TM} = \mathrm{id}_{TM}$ and $\underline{f} = \mathrm{id}_M$ for $f \in \mathrm{Gau}(P)$. Notice that, for all $\lambda \in \Gamma^\infty(M, \mathcal{C}(P))$ and $\eta \in \Omega^1(M, \mathfrak{g})$, $f^*\big(\widetilde{\Phi}(\lambda, \eta)\big) = \widetilde{\Phi}\big(f^*(\lambda), \eta\big)$, i.e. gauge transformations have trivial linear parts.

The next proposition provides a characterization of the action of gauge transformations on $\Gamma^\infty(M, \mathcal{C}(P))$ in terms of the Abelian group action $\widetilde{\Phi}$ of elements in $\Omega^1(M, \mathfrak{g})$.

Proposition 2.31. *Let M be a manifold, G an Abelian Lie group and (P,r) a principal G-bundle over M. For any $f \in \mathrm{Gau}(P)$ and $\lambda \in \Gamma^\infty(M,\mathcal{C}(P))$ the following identity holds true*

$$f^*(\lambda) = \widetilde{\Phi}\big(\lambda, \widehat{f}^{-1\,*}(\mu_G)\big) \ , \tag{2.35}$$

where $\mu_G \in \Omega^1(G,\mathfrak{g})$ is the Maurer-Cartan form on G and $\widehat{f} \in C^\infty(M,G)$ is obtained from f via Lemma 2.30.

Proof. Let $X \in T_xM$, $x \in M$, be arbitrary. Let us fix any $p \in \pi^{-1}[\{x\}]$ and pick from the equivalence class $\lambda(X) \in TP/G|_x$ the unique element $Y \in T_pP$. We have by definition $\lambda(X) = [Y]$ and furthermore it holds true that $\pi_*(Y) = X$, since λ is a splitting of the Atiyah sequence. From (2.17) and (2.34) we obtain $\big(f^*(\lambda)\big)(X) = \big[f_*^{-1}(Y)\big]$. In order to compute $f_*^{-1}(Y) \in T_{f^{-1}(p)}P$ let us define $\widehat{f} \in C^\infty(M,G)$ according to Lemma 2.30 and introduce the map $\kappa_p : G \to P\,,\ g \mapsto p\,g$. We obtain by using $f^{-1}(p) = p\,\widehat{f}^{-1}(\pi(p)) = p\,\widehat{f}^{-1}(x)$,

$$\begin{aligned} f_*^{-1}(Y) &= r_{\widehat{f}^{-1}(x)\,*}(Y) + \big(\kappa_{p\,*} \circ \widehat{f}_*^{-1} \circ \pi_*\big)(Y) \\ &= r_{\widehat{f}^{-1}(x)\,*}(Y) + \kappa_{p\,*}\big(\widehat{f}_*^{-1}(X)\big) = r_{\widehat{f}^{-1}(x)\,*}(Y) + X_{f^{-1}(p)}^{\widehat{f}^{-1\,*}(\mu_G)(X)} \ , \end{aligned} \tag{2.36}$$

where the second term after the last equality denotes the fundamental vector at $f^{-1}(p)$ corresponding to $\widehat{f}^{-1\,*}(\mu_G)(X) \in \mathfrak{g}$. It follows that

$$\begin{aligned} \big(f^*(\lambda)\big)(X) &= [r_{\widehat{f}^{-1}(x)\,*}(Y) + X_{f^{-1}(p)}^{\widehat{f}^{-1\,*}(\mu_G)(X)}] \\ &= \lambda(X) + \iota\big(x, \widehat{f}^{-1\,*}(\mu_G)(X)\big) = \big(\widetilde{\Phi}(\lambda, \widehat{f}^{-1\,*}(\mu_G))\big)(X) \ , \end{aligned} \tag{2.37}$$

which concludes the proof since $X \in T_xM$ was arbitrary. □

3 The phase space for an object

Let $\Xi = \big((M,\mathfrak{o},g,\mathfrak{t}),(G,h),(P,r)\big)$ be an object in $\mathsf{PrBuGlobHyp}$, $\big(\mathcal{C}(P),M,\pi_{\mathcal{C}(P)},A\big)$ the associated bundle of connections and $\Gamma^\infty(M,\mathcal{C}(P))$ its sections. We denote the vector dual bundle (see [BDS12, Definition 2.15]) by $\big(\mathcal{C}(P)^\dagger, M, \pi^\dagger_{\mathcal{C}(P)}, A^\dagger\big)$ and its compactly supported sections by $\Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)$. The aim of this section is to construct a gauge invariant phase space for dynamical principal connections on Ξ.

Maxwell's equations are described in our setting by the affine differential operator

$$\mathsf{MW} := \delta \circ \underline{\mathcal{F}} : \Gamma^\infty(M,\mathcal{C}(P)) \to \Omega^1(M,\mathfrak{g}) \ , \ \ \lambda \mapsto \mathsf{MW}(\lambda) = \delta\underline{\mathcal{F}}(\lambda) \ , \tag{3.1a}$$

where δ is the codifferential and $\underline{\mathcal{F}}$ is the curvature affine differential operator, see Corollary 2.24. The linear part of MW is given by (cf. Corollary 2.24)

$$\mathsf{MW}_V : \Omega^1(M,\mathfrak{g}) \to \Omega^1(M,\mathfrak{g}) \ , \ \ \eta \mapsto \mathsf{MW}_V(\eta) = \delta\underline{\mathcal{F}}_V(\eta) = -\delta\mathrm{d}\eta \ . \tag{3.1b}$$

Due to [BDS12, Theorem 3.5], the affine differential operator MW is formally adjoinable to a differential operator $\mathsf{MW}^* : \Omega^1(M,\mathfrak{g}^*) \to \Gamma^\infty(M,\mathcal{C}(P)^\dagger)$, with $\mathfrak{g}^*$ denoting the dual of the Lie algebra $\mathfrak{g}$. Explicitly, MW^* is determined (up to the ambiguities to be discussed below) by the condition, for all $\lambda \in \Gamma^\infty(M,\mathcal{C}(P))$ and $\eta \in \Omega^1_0(M,\mathfrak{g}^*)$,

$$\langle \eta, \mathsf{MW}(\lambda)\rangle := \int_M \eta \wedge *\big(\mathsf{MW}(\lambda)\big) = \int_M \mathrm{vol}\,\big(\mathsf{MW}^*(\eta)\big)(\lambda) \ , \tag{3.2}$$

where $*$ denotes the Hodge operator and vol the volume form. We will always suppress the duality pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ in order to simplify the notation.

As it is proven in [BDS12, Theorem 3.5], the formal adjoint differential operator $\mathsf{MW}^* : \Omega^1_0(M,\mathfrak{g}^*) \to \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)$ is not unique. Uniqueness is restored if we quotient out the trivial elements[1]

$$\mathrm{Triv} := \left\{ a\,\mathbb{1} \in \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger) : a \in C^\infty_0(M) \text{ satisfies } \int_M \mathrm{vol}\, a = 0 \right\} , \tag{3.3}$$

i.e. if we consider the operator $\mathsf{MW}^* : \Omega^1_0(M,\mathfrak{g}^*) \to \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv}$. By $\mathbb{1} \in \Gamma^\infty(M,\mathcal{C}(P)^\dagger)$ we denote the canonical section which associates to every $x \in M$ the normalized constant affine map in the fibre $\mathcal{C}(P)^\dagger|_x$. The quotient by Triv does not influence the linear part of $\mathsf{MW}^*(\eta)$: Indeed, for all $\eta \in \Omega^1_0(M,\mathfrak{g}^*)$, $\lambda \in \Gamma^\infty(M,\mathcal{C}(P))$ and $\eta' \in \Omega^1(M,\mathfrak{g})$,

$$\begin{aligned} \int_M \mathrm{vol}\, \big(\mathsf{MW}^*(\eta)\big)\big(\widetilde{\Phi}(\lambda,\eta')\big) &= \Big\langle \eta, \mathsf{MW}\big(\widetilde{\Phi}(\lambda,\eta')\big) \Big\rangle \\ &= \big\langle \eta, \mathsf{MW}(\lambda) - \delta \mathrm{d}\eta' \big\rangle \\ &= \int_M \mathrm{vol}\, \big(\mathsf{MW}^*(\eta)\big)(\lambda) + \big\langle -\delta\mathrm{d}\eta, \eta' \big\rangle \end{aligned} \tag{3.4}$$

implies that the linear part is $\mathsf{MW}^*(\eta)_V = -\delta\mathrm{d}\eta$, for all $\eta \in \Omega^1_0(M,\mathfrak{g}^*)$.

The next step is to restrict to those elements in $\Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv}$ that describe gauge invariant observables. It is enlightening to introduce the vector space of classical affine observables $\{\mathcal{O}_\varphi : \varphi \in \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv}\}$, where $\mathcal{O}_\varphi$ is the functional on the configuration space $\Gamma^\infty(M,\mathcal{C}(P))$ defined by

$$\mathcal{O}_\varphi : \Gamma^\infty(M,\mathcal{C}(P)) \to \mathbb{R} \; , \quad \lambda \mapsto \mathcal{O}_\varphi(\lambda) = \int_M \mathrm{vol}\, \varphi(\lambda) \; . \tag{3.5}$$

Let $\widehat{f}^{-1} \in C^\infty(M,G) \simeq \mathrm{Gau}(P)$ be an element in the gauge group (cf. Lemma 2.30). As we have shown in Proposition 2.31, the gauge transformations on $\Gamma^\infty(M,\mathcal{C}(P))$ are given by $\lambda \mapsto \widetilde{\Phi}\big(\lambda, \widehat{f}^*(\mu_G)\big)$. Demanding invariance of $\mathcal{O}_\varphi$ under gauge transformations, i.e. $\mathcal{O}_\varphi\big(\widetilde{\Phi}(\lambda,\widehat{f}^*(\mu_G))\big) = \mathcal{O}_\varphi(\lambda)$ for all $\lambda \in \Gamma^\infty(M,\mathcal{C}(P))$ and $\widehat{f} \in C^\infty(M,G)$, leads to the following condition for the linear part $\varphi_V \in \Omega^1_0(M,\mathfrak{g}^*)$ of $\varphi \in \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv}$, for all $\widehat{f} \in C^\infty(M,G)$,

$$\Big\langle \varphi_V, \widehat{f}^*(\mu_G) \Big\rangle = 0 \; . \tag{3.6}$$

This motivates us to define the following vector space

$$\mathcal{E}^{\mathrm{inv}} := \left\{ \varphi \in \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv} : \Big\langle \varphi_V, \widehat{f}^*(\mu_G) \Big\rangle = 0 \; , \; \forall \widehat{f} \in C^\infty(M,G) \right\} , \tag{3.7}$$

which serves as a starting point to construct the phase space.

Lemma 3.1. *a) For all $\varphi \in \mathcal{E}^{\mathrm{inv}}$ the linear part $\varphi_V \in \Omega^1_0(M,\mathfrak{g}^*)$ is coclosed, i.e. $\delta\varphi_V = 0$.*

b) All $\varphi \in \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv}$ satisfying $\varphi_V = \delta\eta$ for some $\eta \in \Omega^2_0(M,\mathfrak{g}^*)$ are elements in $\mathcal{E}^{\mathrm{inv}}$.

Proof. Proof of a): As G is by hypothesis a connected Abelian Lie group it is isomorphic to $\mathbb{T}^k \times \mathbb{R}^l$, see e.g. [Ada69, Theorem 2.19]. Denoting by x_i, $i = 1,\dots,l$, Cartesian coordinates on $\mathbb{R}^l$ and by ϕ_j, $j = 1,\dots,k$, angles on $\mathbb{T}^k$, the Maurer-Cartan form reads $\mu_G = \sum_{j=1}^k \mathrm{d}\phi_j \otimes_{\mathbb{R}} t^j + \sum_{i=1}^l \mathrm{d}x_i \otimes_{\mathbb{R}} t^{k+i}$, where $\mathrm{d}\phi_j$ denotes the dual 1-form of the vector field ∂_{ϕ_j} (we follow the usual abuse of notation and denote these forms by $\mathrm{d}\phi_j$, even though they are not exact!).

Let $\chi \in C^\infty(M,\mathfrak{g})$ and consider the element of the gauge group specified by $\widehat{f}_\chi := \exp \circ \chi \in C^\infty(M,G)$, where $\exp : \mathfrak{g} \to G$ denotes the exponential map. The pull-back of the Maurer-Cartan form

[1] By trivial we mean that the corresponding classical affine observables (3.5), i.e. functionals on the configuration space $\Gamma^\infty(M,\mathcal{C}(P))$, vanish.

then reads $\widehat{f}_\chi^*(\mu_G) = \mathrm{d}\chi$. Let $\varphi \in \mathcal{E}^{\mathrm{inv}}$ be arbitrary. Due to (3.7) the linear part φ_V of φ satisfies, for all $\chi \in C^\infty(M,\mathfrak{g})$,

$$0 = \Big\langle \varphi_V, \widehat{f}_\chi^*(\mu_G) \Big\rangle = \langle \varphi_V, \mathrm{d}\chi \rangle = \langle \delta\varphi_V, \chi \rangle \ , \tag{3.8}$$

which implies $\delta\varphi_V = 0$.

Proof of b): For all $\widehat{f} \in C^\infty(M,G)$,

$$\Big\langle \varphi_V, \widehat{f}^*(\mu_G) \Big\rangle = \Big\langle \delta\eta, \widehat{f}^*(\mu_G) \Big\rangle = \Big\langle \eta, \mathrm{d}\widehat{f}^*(\mu_G) \Big\rangle = \Big\langle \eta, \widehat{f}^*(\mathrm{d}\mu_G) \Big\rangle = 0 \ , \tag{3.9}$$

since the Maurer-Cartan form of Abelian Lie groups is closed. □

Corollary 3.2. *Let us define the vector spaces*

$$\mathcal{E}^{\mathrm{min}} := \big\{ \varphi \in \Gamma_0^\infty(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv} : \varphi_V \in \delta\Omega_0^2(M,\mathfrak{g}^*) \big\} \ , \tag{3.10a}$$

$$\mathcal{E}^{\mathrm{max}} := \big\{ \varphi \in \Gamma_0^\infty(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv} : \varphi_V \in \Omega_{0,\delta}^1(M,\mathfrak{g}^*) \big\} \ . \tag{3.10b}$$

Then the following inclusions of vector spaces hold true

$$\mathcal{E}^{\mathrm{min}} \subseteq \mathcal{E}^{\mathrm{inv}} \subseteq \mathcal{E}^{\mathrm{max}} \ . \tag{3.11}$$

Remark 3.3. This corollary provides us with a lower and upper bound on the vector space $\mathcal{E}^{\mathrm{inv}}$. Notice that in case M has a trivial first de Rham cohomology group $H^1_{\mathrm{dR}}(M,\mathfrak{g}) = \{0\}$ (which implies that the dual cohomology group is trivial $H^1_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*) := \Omega^1_{0,\delta}(M,\mathfrak{g}^*)/\delta\Omega_0^2(M,\mathfrak{g}^*) = \{0\}$), the lower and upper bounds coincide, i.e. $\mathcal{E}^{\mathrm{min}} = \mathcal{E}^{\mathrm{inv}} = \mathcal{E}^{\mathrm{max}}$. In general, the explicit characterization of $\mathcal{E}^{\mathrm{inv}}$ is rather complicated and will be postponed to Section 4.

The equation of motion $\mathsf{MW}(\lambda) = 0$ is implemented at a dual level on $\mathcal{E}^{\mathrm{inv}}$ by considering the quotient vector space $\mathcal{E}^{\mathrm{inv}}/\mathsf{MW}^*\big[\Omega_0^1(M,\mathfrak{g}^*)\big]$. To construct a presymplectic structure on this space let us consider the Hodge-d'Alembert operators $\square_{(k)} := \delta\circ\mathrm{d} + \mathrm{d}\circ\delta : \Omega^k(M,\mathfrak{g}^*) \to \Omega^k(M,\mathfrak{g}^*)$, that are normally hyperbolic operators. The corresponding unique retarded/advanced Green's operators are denoted by $G^\pm_{(k)} : \Omega_0^k(M,\mathfrak{g}^*) \to \Omega^k(M,\mathfrak{g}^*)$ and the causal propagators are defined by $G_{(k)} := G^+_{(k)} - G^-_{(k)} : \Omega_0^k(M,\mathfrak{g}^*) \to \Omega^k(M,\mathfrak{g}^*)$. We notice the relations

$$\square_{(k)}\circ\mathrm{d} = \mathrm{d}\circ\square_{(k-1)} \ , \quad \square_{(k)}\circ\delta = \delta\circ\square_{(k+1)} \ , \tag{3.12a}$$

which imply

$$G^\pm_{(k)}\circ\mathrm{d} = \mathrm{d}\circ G^\pm_{(k-1)} \ , \quad G^\pm_{(k)}\circ\delta = \delta\circ G^\pm_{(k+1)} \ . \tag{3.12b}$$

The G-invariant pseudo-Riemannian metric h on the Lie group G determines an ad-invariant inner product (possibly indefinite) on the Lie algebra $\mathfrak{g}$ and hence a vector space isomorphism (denoted with a slight abuse of notation by the same symbol) $h : \mathfrak{g} \to \mathfrak{g}^*$. We denote by $h^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ the inverse vector space isomorphism. Using also the pairing $\langle\ ,\ \rangle$ we define for all $\eta,\eta' \in \Omega^k(M,\mathfrak{g}^*)$ with compact overlapping support the non-degenerate (indefinite) inner product

$$\langle \eta,\eta'\rangle_h := \big\langle \eta, h^{-1}(\eta')\big\rangle \ . \tag{3.13}$$

We notice that $\square_{(k)}$ is formally self-adjoint with respect to $\langle\ ,\ \rangle_h$ and hence $G_{(k)}$ is formally skew-adjoint with respect to $\langle\ ,\ \rangle_h$ for all elements in $\Omega_0^k(M,\mathfrak{g}^*)$ (that is the domain of $G_{(k)}$).

Proposition 3.4. *Let* $\Xi = \big((M,\mathfrak{o},g,\mathfrak{t}),(G,h),(P,r)\big)$ *be an object in* $\mathsf{PrBuGlobHyp}$. *Then the vector space* $\mathcal{E} := \mathcal{E}^{\mathrm{inv}}/\mathsf{MW}^*\big[\Omega_0^1(M,\mathfrak{g}^*)\big]$ *can be equipped with the presymplectic structure*

$$\tau : \mathcal{E}\times\mathcal{E} \to \mathbb{R} \ , \quad ([\varphi],[\psi]) \mapsto \tau([\varphi],[\psi]) = \big\langle \varphi_V, G_{(1)}(\psi_V)\big\rangle_h \ . \tag{3.14}$$

In other words, $\big(\mathcal{E},\tau\big)$ *is a presymplectic vector space.*

Proof. We have to proof that τ is well-defined, i.e. that for every $\varphi = \mathsf{MW}^*(\eta)$, $\eta \in \Omega^1_0(M, \mathfrak{g}^*)$, we have $\big\langle \varphi_V, G_{(1)}(\psi_V)\big\rangle_h = 0$ and $\big\langle \psi_V, G_{(1)}(\varphi_V)\big\rangle_h = 0$ for the linear parts ψ_V of all elements $\psi \in \mathcal{E}^{\mathrm{inv}}$. Lemma 3.1 implies that $\delta\psi_V = 0$. The first property holds true:

$$\begin{aligned}\big\langle \varphi_V, G_{(1)}(\psi_V)\big\rangle_h &= \big\langle \mathsf{MW}^*(\eta)_V, G_{(1)}(\psi_V)\big\rangle_h = -\big\langle \delta\mathrm{d}\eta, G_{(1)}(\psi_V)\big\rangle_h \\ &= -\big\langle \eta, \delta\mathrm{d}G_{(1)}(\psi_V)\big\rangle_h = -\big\langle \eta, (\square_{(1)} - \mathrm{d}\delta)\big(G_{(1)}(\psi_V)\big)\big\rangle_h \\ &= \big\langle \eta, \mathrm{d}G_{(0)}(\delta\psi_V)\big\rangle_h = 0\,. \end{aligned} \tag{3.15}$$

The second property follows analogously, since $G_{(1)}$ is formally skew-adjoint with respect to $\langle\ ,\ \rangle_h$. From the latter property it also follows that τ is antisymmetric. □

Remark 3.5. The presymplectic structure (3.14) can be derived from a Lagrangian form by generalizing the method of Peierls [Pei52] to gauge theories. This generalization has already been studied in [Mar93] and it was put on mathematically solid grounds recently in [SDH12] for the vector potential of $U(1)$-connections. Since in our approach the configuration space $\Gamma^\infty(M, \mathcal{C}(P))$ is different, we have to adapt the relevant arguments to our setting: Let us consider the Lagrangian form $\mathcal{L}[\lambda] := -\frac{1}{2}h\big(\underline{\mathcal{F}}(\lambda)\big) \wedge *\big(\underline{\mathcal{F}}(\lambda)\big)$ and its perturbation by an element $\varphi \in \mathcal{E}^{\mathrm{inv}}$, i.e. $\mathcal{L}_\varphi[\lambda] := \mathcal{L}[\lambda] + \mathrm{vol}\,\varphi(\lambda)$. The Euler-Lagrange equation corresponding to $\mathcal{L}_\varphi$ is given by $\mathsf{MW}(\lambda) + h^{-1}(\varphi_V) = 0$, where $\varphi_V \in \Omega^1_0(M, \mathfrak{g}^*)$ is the linear part of φ. Let us take any $\lambda \in \Gamma^\infty(M, \mathcal{C}(P))$ satisfying $\mathsf{MW}(\lambda) = 0$. The goal is to construct the retarded/advanced effect of φ on this solution. Let $\Sigma^\pm \subset M$ be two Cauchy surfaces (with Σ^+ being in the future of Σ^-) such that $\mathrm{supp}(\varphi_V) \subseteq J^-_M(\Sigma^+) \cap J^+_M(\Sigma^-)$ (this means that φ_V has support in the spacetime region between Σ^+ and Σ^-). We are looking for a $\lambda^\pm_\varphi \in \Gamma^\infty(M, \mathcal{C}(P))$ satisfying the equation of motion $\mathsf{MW}(\lambda^\pm_\varphi) + h^{-1}(\varphi_V) = 0$ and $\lambda^\pm_\varphi|_{J^\mp_M(\Sigma^\mp)} = \widetilde{\Phi}\big(\lambda, \widehat{f}^*_\pm(\mu_G)\big)|_{J^\mp_M(\Sigma^\mp)}$ for some $\widehat{f}_\pm \in C^\infty(M, G)$. The latter condition states that $\lambda^\pm_\varphi$ agrees up to a gauge transformation with λ in the past/future of $\Sigma^\mp$. Since $\Gamma^\infty(M, \mathcal{C}(P))$ is an affine space over $\Omega^1(M, \mathfrak{g})$ we find a unique $\eta^\pm_\varphi \in \Omega^1(M, \mathfrak{g})$ such that $\lambda^\pm_\varphi = \widetilde{\Phi}(\lambda, \eta^\pm_\varphi)$. The equations of motion for λ and $\lambda^\pm_\varphi$ then imply $-\delta\mathrm{d}\eta^\pm_\varphi + h^{-1}(\varphi_V) = 0$ and the asymptotic condition reads $\big(\eta^\pm_\varphi - \widehat{f}^*_\pm(\mu_G)\big)|_{J^\mp_M(\Sigma^\mp)} = 0$ for some $\widehat{f}_\pm \in C^\infty(M, G)$. Since any $\eta^\pm_\varphi \in \Omega^1(M, \mathfrak{g})$ is gauge equivalent to a coclosed one-form, we can assume without loss of generality that $\eta^\pm_\varphi$ satisfies $\delta\eta^\pm_\varphi = 0$, and hence the equation of motion reads $\square_{(1)}\eta^\pm_\varphi = h^{-1}(\varphi_V)$. For the support condition $\eta^\pm_\varphi|_{J^\mp_M(\Sigma^\mp)} = 0$ (that is contained in the asymptotic condition above) the unique solution of this equation is $\eta^\pm_\varphi = G^\pm_{(1)}\big(h^{-1}(\varphi_V)\big) = h^{-1}\big(G^\pm_{(1)}(\varphi_V)\big)$. All solutions of the equation $-\delta\mathrm{d}\eta^\pm_\varphi + h^{-1}(\varphi_V) = 0$ subject to the asymptotic condition $\big(\eta^\pm_\varphi - \widehat{f}^*_\pm(\mu_G)\big)|_{J^\mp_M(\Sigma^\mp)} = 0$, for some $\widehat{f}_\pm \in C^\infty(M, G)$, are obtained by gauge transformations of $\eta^\pm_\varphi = h^{-1}\big(G^\pm_{(1)}(\varphi_V)\big)$. Let now $\psi \in \mathcal{E}^{\mathrm{inv}}$ and consider the gauge invariant functional $\mathcal{O}_\psi$ as in (3.5). The retarded/advanced effect of $\varphi \in \mathcal{E}^{\mathrm{inv}}$ on $\mathcal{O}_\psi$ is defined by $E^\pm_\varphi(\mathcal{O}_\psi)(\lambda) := \mathcal{O}_\psi(\lambda^\pm_\varphi) - \mathcal{O}_\psi(\lambda) = \langle \psi_V, \eta^\pm_\varphi\rangle = \Big\langle \psi_V, h^{-1}\big(G^\pm_{(1)}(\varphi_V)\big)\Big\rangle = \Big\langle \psi_V, G^\pm_{(1)}(\varphi_V)\Big\rangle_h$. Notice that this expression is well-defined since $\mathcal{O}_\psi$ is gauge invariant. The presymplectic structure (3.14) is given by the difference of the retarded and advanced effect, i.e. $\tau([\psi], [\varphi]) = E^+_\varphi(\mathcal{O}_\psi)(\lambda) - E^-_\varphi(\mathcal{O}_\psi)(\lambda)$, which agrees with the idea of Peierls [Pei52].

We come to the characterization of the radical $\mathcal{N} \subseteq \mathcal{E}$ of the presymplectic structure τ. An element $[\psi] \in \mathcal{E}$ is in $\mathcal{N}$ if and only if, for all $[\varphi] \in \mathcal{E}$, $\tau\big([\varphi], [\psi]\big) = 0$. In this section we will only provide a lower and upper estimate for the vector space $\mathcal{N}$. The explicit characterization will be content of Section 4.

Lemma 3.6. *a) Let $[\psi] \in \mathcal{N}$ be arbitrary. Then any representative $\psi \in \mathcal{E}^{\mathrm{inv}}$ is such that $\psi_V = \delta\alpha$ for some $\alpha \in \Omega^2_{0,\mathrm{d}}(M, \mathfrak{g}^*)$.*

b) Let $\psi \in \mathcal{E}^{\mathrm{inv}}$ be such that $\psi_V = \delta\mathrm{d}\gamma$ with $\gamma \in \Omega^1_{\mathrm{tc}}(M, \mathfrak{g}^*)$ and $\mathrm{d}\gamma \in \Omega^2_0(M, \mathfrak{g}^*)$. Then $[\psi] \in \mathcal{N}$. The subscript $_{\mathrm{tc}}$ denotes forms of timelike compact support.

Proof. Proof of a): By hypothesis $[\psi]$ satisfies, for all $[\varphi] \in \mathcal{E}$,

$$\tau\big([\varphi], [\psi]\big) = \big\langle \varphi_V, G_{(1)}(\psi_V)\big\rangle_h = 0\,. \tag{3.16}$$

By Corollary 3.2 we have that $\mathcal{E}^{\min} \subseteq \mathcal{E}^{\mathrm{inv}}$ and thus it is necessary for $[\psi]$ to fulfill, for all $\eta \in \Omega^2_0(M, \mathfrak{g}^*)$,

$$0 = \big\langle \delta\eta, G_{(1)}(\psi_V)\big\rangle_h = \big\langle \eta, G_{(2)}(\mathrm{d}\psi_V)\big\rangle_h \ . \tag{3.17}$$

This implies that $G_{(2)}(\mathrm{d}\psi_V) = 0$ and hence due to the fact that $G_{(2)}$ is the causal propagator of a normally hyperbolic operator we obtain $\mathrm{d}\psi_V = \square_{(2)}(\alpha)$ for some $\alpha \in \Omega^2_0(M, \mathfrak{g}^*)$. Applying d to this equation shows that $\mathrm{d}\alpha = 0$, i.e. $\alpha \in \Omega^2_{0,\mathrm{d}}(M, \mathfrak{g}^*)$. Applying δ and using that $\delta\psi_V = 0$ (cf. Lemma 3.1) we find $\square_{(1)}(\psi_V) = \square_{(1)}(\delta\alpha)$. This implies $\psi_V = \delta\alpha$ and completes the proof.

Proof of b): Let now $\psi \in \mathcal{E}^{\mathrm{inv}}$ be as specified above. Then we obtain, for all $[\varphi] \in \mathcal{E}$,

$$\begin{aligned}\tau\big([\varphi], [\psi]\big) &= \big\langle \varphi_V, G_{(1)}(\delta \mathrm{d}\gamma)\big\rangle_h = \big\langle \varphi_V, \delta\mathrm{d}G_{(1)}(\gamma)\big\rangle_h \\ &= \big\langle \varphi_V, (\square_{(1)} - \mathrm{d}\delta)\big(G_{(1)}(\gamma)\big)\big\rangle_h = -\big\langle \varphi_V, \mathrm{d}\delta G_{(1)}(\gamma)\big\rangle_h \\ &= -\big\langle \delta\varphi_V, \delta G_{(1)}(\gamma)\big\rangle_h = 0\ ,\end{aligned} \tag{3.18}$$

where in the second equality we have used that the domain of $G_{(1)}$ can be extended to $\Omega^1_{\mathrm{tc}}(M, \mathfrak{g}^*)$ [SDH12] and in the last equality that $\delta\varphi_V = 0$. □

Corollary 3.7. *Let us define the vector spaces*

$$\mathcal{N}_{\min} := \big\{\psi \in \mathcal{E}^{\mathrm{inv}} : \psi_V \in \delta\big(\Omega^2_0(M, \mathfrak{g}^*) \cap \mathrm{d}\Omega^1_{\mathrm{tc}}(M, \mathfrak{g}^*)\big)\big\}/\mathsf{MW}^*\big[\Omega^1_0(M, \mathfrak{g}^*)\big]\ , \tag{3.19a}$$

$$\mathcal{N}_{\max} := \big\{\psi \in \mathcal{E}^{\mathrm{inv}} : \psi_V \in \delta\Omega^2_{0,\mathrm{d}}(M, \mathfrak{g}^*)\big\}/\mathsf{MW}^*\big[\Omega^1_0(M, \mathfrak{g}^*)\big]\ . \tag{3.19b}$$

Then the following inclusions of vector spaces hold true

$$\mathcal{N}_{\min} \subseteq \mathcal{N} \subseteq \mathcal{N}_{\max} \subseteq \mathcal{E}^{\min} \subseteq \mathcal{E} \subseteq \mathcal{E}^{\max}\ . \tag{3.20}$$

Remark 3.8. The radical $\mathcal{N}$ of the theory under consideration is in general different from that of affine matter field theories, see [BDS12, Proposition 4.4]. Even though the constant affine observables $[a\,\mathbb{1}]$, with $a \in C^\infty_0(M)$, are contained in $\mathcal{N}$, in general they do not exhaust all elements. The lower bound on $\mathcal{N}$ given in Corollary 3.7 coincides with the radical obtained in [SDH12] (up to the constant affine observables which are not present in this paper since it does not exploit the complete geometric structure of the bundle of connections).

Remark 3.9. If M has compact Cauchy surfaces the vector space $\mathcal{N}_{\min}$ is trivial. That this is not generically the case is shown by the following explicit example: Let us consider the case in which $G = \mathbb{R}$ (implying $\mathfrak{g}^* = \mathbb{R}$) and M is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^{m-2}$, where $m > 2$ and $\mathbb{S}^{m-2}$ denotes the $m-2$-sphere (we suppress this diffeomorphism in the following). Any Cauchy surface $\Sigma \subseteq M$ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{m-2}$. Since $H^1_{0\,\mathrm{dR}}(\mathbb{R}) = \mathbb{R}$ is nontrivial, we can find an $\alpha \in \Omega^1_{0,\mathrm{d}}(\mathbb{R})$ that is not exact. Let us introduce Cartesian coordinates (t, x) on the $\mathbb{R}^2$ factor of M. We denote by $\alpha_t \in \Omega^1_{0,\mathrm{d}}(M)$ the pull-back of α along the projection to the time direction t and by $\alpha_x \in \Omega^1_{0,\mathrm{d}}(M)$ the pull-back of α along the projection to the space direction x. We define $\eta := \alpha_t \wedge \alpha_x$. The support property of α and the compatibility between d and the pull-backs entail that $\eta \in \Omega^2_{0,\mathrm{d}}(M)$. Furthermore, since $H^1_{\mathrm{dR}}(M) = \{0\}$, there exists a $\beta \in C^\infty(M)$ such that $\alpha_x = -\mathrm{d}\beta$, which implies $\eta = \mathrm{d}(\beta\,\alpha_t)$, where $\beta\,\alpha_t \in \Omega^1_{\mathrm{tc}}(M)$. We now show that $\eta \notin \mathrm{d}\Omega^1_0(M)$: Let $\nu_{\mathbb{S}^{m-2}}$ be the normalized volume form on $\mathbb{S}^{m-2}$ and let $\mathrm{pr} : M \to \mathbb{S}^{m-2}$ be the projection from M to $\mathbb{S}^{m-2}$. Notice that the integral $\int_M \eta \wedge \mathrm{pr}^*(\nu_{\mathbb{S}^{m-2}}) = \big(\int_{\mathbb{R}} \alpha\big)^2 \neq 0$, since α is not exact. If there would exist a $\gamma \in \Omega^1_0(M)$, such that $\eta = \mathrm{d}\gamma$, then by Stokes' theorem the integral vanishes, which is a contradiction. Hence, $\eta = \mathrm{d}(\beta\,\alpha_t)$, with $\beta\,\alpha_t \in \Omega^1_{\mathrm{tc}}(M)$, defines a nontrivial element in $H^2_{0\,\mathrm{dR}}(M)$. Furthermore, $\delta\eta$ is a representative of a nontrivial class in $\mathcal{N}_{\min}$: Indeed, suppose that there exists $\gamma \in \Omega^1_0(M)$ such that $\delta\eta = \delta\mathrm{d}\gamma$. Using that η is closed and of compact support, this equation entails $\square_{(2)}(\eta) = \square_{(2)}(\mathrm{d}\gamma)$ which yields the contradiction $\eta = \mathrm{d}\gamma$, since $\square_{(2)}$ is a normally hyperbolic operator.

4 Explicit characterization of $\mathcal{E}^{\mathrm{inv}}$ and $\mathcal{N}$

So far we have obtained for the vector spaces $\mathcal{E}^{\mathrm{inv}}$ and $\mathcal{N}$ only upper and lower bounds, see Corollary 3.2 and Corollary 3.7. The goal of this section is to provide an explicit characterization of $\mathcal{E}^{\mathrm{inv}}$ and $\mathcal{N}$. For this we have to understand more explicitly how the gauge group $\mathrm{Gau}(P) \simeq C^\infty(M,G)$ acts on $\Gamma^\infty(M,\mathcal{C}(P))$. Due to Proposition 2.31 this amounts to characterizing the Abelian subgroup

$$\big\{\widehat{f}^*(\mu_G) : \widehat{f} \in C^\infty(M,G)\big\} \subseteq \Omega^1(M,\mathfrak{g}) \ . \tag{4.1}$$

In the proof of Lemma 3.1 we have shown that, for every $\chi \in C^\infty(M,\mathfrak{g})$, the map $\widehat{f}_\chi := \exp\circ\chi \in C^\infty(M,G)$ leads to $\widehat{f}_\chi^*(\mu_G) = \mathrm{d}\chi$. Furthermore, since the exterior differential commutes with the pull-back $\widehat{f}^*$ and μ_G is closed, we have that $\widehat{f}^*(\mu_G) \in \Omega^1_{\mathrm{d}}(M,\mathfrak{g})$. This implies the inclusions of Abelian groups

$$\mathrm{d}C^\infty(M,\mathfrak{g}) \subseteq \big\{\widehat{f}^*(\mu_G) : \widehat{f} \in C^\infty(M,G)\big\} \subseteq \Omega^1_{\mathrm{d}}(M,\mathfrak{g}) \tag{4.2}$$

and taking the quotient by $\mathrm{d}C^\infty(M,\mathfrak{g})$ we are led to consider the Abelian subgroup

$$A_G := \big\{\widehat{f}^*(\mu_G) : \widehat{f} \in C^\infty(M,G)\big\}/\mathrm{d}C^\infty(M,\mathfrak{g}) \subseteq H^1_{\mathrm{dR}}(M,\mathfrak{g}) \ . \tag{4.3}$$

Lemma 4.1. *Let us consider the following equivalence relation on the gauge group $C^\infty(M,G)$*

$$\widehat{g} \sim \widehat{h} \quad :\Leftrightarrow \quad \exists \chi \in C^\infty(M,\mathfrak{g}) \ \text{ such that } \ \widehat{g} = \widehat{h}\,\widehat{f}_\chi \ , \tag{4.4}$$

where $\widehat{f}_\chi = \exp\circ\chi \in C^\infty(M,G)$. Then $C^\infty(M,G)/{\sim}$ is an Abelian group and the following map is an isomorphism of Abelian groups

$$C^\infty(M,G)/{\sim} \ \to A_G \ , \ \ [\widehat{f}] \mapsto [\widehat{f}^*(\mu_G)] \ . \tag{4.5}$$

Proof. $C^\infty(M,G)/{\sim}$ is an Abelian group with group operation given by $[\widehat{f}]\,[\widehat{g}] := [\widehat{f}\,\widehat{g}]$. The map (4.5) is obviously a map of Abelian groups and it is well-defined, since for $\widehat{f}\,\widehat{f}_\chi$ we have $(\widehat{f}\,\widehat{f}_\chi)^*(\mu_G) = \widehat{f}^*(\mu_G) + \mathrm{d}\chi$. Surjectivity holds by definition of A_G and injectivity is shown as follows: Let $[\widehat{f}] \in C^\infty(M,G)/{\sim}$ be such that $[\widehat{f}^*(\mu_G)] = 0$. This implies that for any representative $\widehat{f}$ the pull-back is exact, $\widehat{f}^*(\mu_G) = \mathrm{d}\chi$ for some $\chi \in C^\infty(M,\mathfrak{g})$. Considering the representative $\widehat{f}\,\widehat{f}_{-\chi}$ of the same class, we can set without loss of generality $\chi = 0$, i.e. $\widehat{f}^*(\mu_G) = 0$. This implies, for all $X \in TM$, $0 = \big(\widehat{f}^*(\mu_G)\big)(X) = \mu_G\big(\widehat{f}_*(X)\big)$ and since the Maurer-Cartan form is non-degenerate we obtain, for all $X \in TM$, $\widehat{f}_*(X) = 0$. It follows that $\widehat{f} : M \to G$ is the constant map and hence $[\widehat{f}]$ is the identity of the group $C^\infty(M,G)/{\sim}$. □

Remark 4.2. Due to this lemma the Abelian group A_G characterizes exactly the gauge transformations which are not of exponential form $\exp\circ\chi$, for some $\chi \in C^\infty(M,\mathfrak{g})$.

Since any connected Abelian Lie group is isomorphic to $\mathbb{T}^k \times \mathbb{R}^l$, the map $\widehat{f} \in C^\infty(M,G)$ is given by a $k+l$-tuple of maps $\big(\widehat{f}_1,\dots,\widehat{f}_{k+l}\big)$, where $\widehat{f}_i \in C^\infty(M,\mathbb{T})$, for $i = 1,\dots,k$, and $\widehat{f}_i \in C^\infty(M,\mathbb{R})$, for $i = k{+}1,\dots,k{+}l$. The Abelian group $C^\infty(M,G)/{\sim}$ factorizes into the direct product $\big(C^\infty(M,\mathbb{T})/{\sim}\big)^k \times \big(C^\infty(M,\mathbb{R})/{\sim}\big)^l$, where $\sim$ denotes respectively the equivalence relation of Lemma 4.1 for $G = \mathbb{T}$ and $G = \mathbb{R}$. Furthermore, the Lie algebra $\mathfrak{g}$ of G is given by the direct sum of k copies of the Lie algebra $i\mathbb{R}$ of $\mathbb{T}$ and l copies of the Lie algebra $\mathbb{R}$ of $\mathbb{R}$, i.e. $\mathfrak{g} = (i\mathbb{R})^{\oplus_k} \oplus \mathbb{R}^{\oplus_l}$. This allow for a splitting of the cohomology group into a direct sum $H^1_{\mathrm{dR}}(M,\mathfrak{g}) = H^1_{\mathrm{dR}}(M,i\mathbb{R})^{\oplus_k} \oplus H^1_{\mathrm{dR}}(M,\mathbb{R})^{\oplus_l}$. The Abelian group A_G is thus given by a direct sum of Abelian groups $A_G = A_{\mathbb{T}}^{\oplus_k} \oplus A_{\mathbb{R}}^{\oplus_l}$ (remember that the direct product and direct sum of groups over a finite index set yield the same group). In this way the problem of characterizing A_G is reduced to the problem of characterizing $A_{\mathbb{T}}$ and $A_{\mathbb{R}}$.

Proposition 4.3. $A_{\mathbb{R}} = \{0\}$.

Proof. Since the Maurer-Cartan form $\mu_{\mathbb{R}} = \mathrm{d}x$ is exact (x is a Cartesian coordinate function on $G = \mathbb{R}$), for any $\widehat{f} \in C^\infty(M,G)$ the one-form $\widehat{f}^*(\mu_{\mathbb{R}}) = \mathrm{d}\widehat{f}^*(x)$ is also exact. This implies $A_{\mathbb{R}} = \{0\}$. □

To characterize $A_{\mathbb{T}}$ we require techniques from Čech cohomology which we are going to review now, see [BT82, §10] for more details. Let $\mathcal{A}$ be a presheaf of Abelian groups on M and $\mathcal{U} := \{U_\alpha\}_{\alpha\in\mathcal{I}}$ a finite good cover, i.e. $\mathcal{I}$ is finite and all non-empty intersections $U_{\alpha_1\dots\alpha_n} := U_{\alpha_1}\cap\cdots\cap U_{\alpha_n}$ are diffeomorphic to $\mathbb{R}^{\dim(M)}$. The existence of a finite good cover is part of our assumptions on M. A p-cochain $\{\eta_{\alpha_0\dots\alpha_p}\}_{\alpha_0<\dots<\alpha_p} \in C^p(\mathcal{U},\mathcal{A})$ is a collection of elements $\eta_{\alpha_0\dots\alpha_p} \in \mathcal{A}(U_{\alpha_0\dots\alpha_p})$, for all $\alpha_0 < \alpha_1 < \cdots < \alpha_p$. For not having to keep track of the index orderings we follow the usual antisymmetry convention to define $\eta_{\alpha_0\dots\alpha_p}$ for all $\alpha_0,\alpha_1,\dots,\alpha_p$. The Čech differential $\check{\delta} : C^p(\mathcal{U},\mathcal{A}) \to C^{p+1}(\mathcal{U},\mathcal{A})$ is given by, for all $\{\eta_{\alpha_0\dots\alpha_p}\} \in C^p(\mathcal{U},\mathcal{A})$,

$$(\check{\delta}\eta)_{\alpha_0\dots\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i\,\eta_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_{p+1}}~, \tag{4.6}$$

where on the right hand side the restriction of $\eta_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_{p+1}}$ to $U_{\alpha_0\dots\alpha_{p+1}}$ is suppressed. The cohomology of the complex

$$C^0(\mathcal{U},\mathcal{A}) \xrightarrow{\ \check{\delta}\ } C^1(\mathcal{U},\mathcal{A}) \xrightarrow{\ \check{\delta}\ } C^2(\mathcal{U},\mathcal{A}) \xrightarrow{\ \check{\delta}\ } \cdots \tag{4.7}$$

is denoted by $\check{H}^\ast(\mathcal{U},\mathcal{A})$ and called the **Čech cohomology** of the cover $\mathcal{U}$ with values in $\mathcal{A}$.

For our purposes we shall require only the first Čech cohomology group $\check{H}^1(\mathcal{U},\mathcal{A})$ for the constant presheaves $\mathcal{A} = i\mathbb{R}$ and $\mathcal{A} = 2\pi i\,\mathbb{Z}$. In these cases, on account of [BT82, Theorem 8.9 and Theorem 15.8], $\check{H}^1(\mathcal{U},\mathcal{A})$ does not depend on the choice of the good cover $\mathcal{U}$. Furthermore, due to $\mathbb{Z}\hookrightarrow\mathbb{R}$ there exists a canonical injection of Abelian groups

$$\check{H}^1(\mathcal{U},2\pi i\,\mathbb{Z}) \to \check{H}^1(\mathcal{U},i\mathbb{R})~,~~[\{\eta_{\alpha\beta}\}] \mapsto [\{\eta_{\alpha\beta}\}]~, \tag{4.8}$$

which we are going to suppress in the following. By [BT82, Theorem 8.9] there exists for any good cover $\mathcal{U}$ an isomorphism $H^1_{\mathrm{dR}}(M,i\mathbb{R}) \simeq \check{H}^1(\mathcal{U},i\mathbb{R})$. We also require an explicit expression for this isomorphism: Let $[\eta]\in H^1_{\mathrm{dR}}(M,i\mathbb{R})$ be arbitrary and take any representative $\eta\in\Omega^1_{\mathrm{d}}(M,i\mathbb{R})$. Restricting η to the open subsets U_α of the good cover, $\eta|_{U_\alpha}\in\Omega^1_{\mathrm{d}}(U_\alpha,i\mathbb{R})$, there exist $\chi_\alpha\in C^\infty(U_\alpha,i\mathbb{R})$, such that $\eta|_{U_\alpha} = \mathrm{d}\chi_\alpha$. Notice that χ_α is not unique, since we can add arbitrary constant functions $c_\alpha\in i\mathbb{R}$ on U_α, i.e. $\eta|_{U_\alpha} = \mathrm{d}(\chi_\alpha + c_\alpha) = \mathrm{d}\chi_\alpha$. On double intersections $U_{\alpha\beta}$ we have to satisfy the condition $\mathrm{d}\chi_\alpha|_{U_{\alpha\beta}} = \mathrm{d}\chi_\beta|_{U_{\alpha\beta}}$, which implies that $\eta_{\alpha\beta} := \chi_\alpha - \chi_\beta = \mathrm{const}\in i\mathbb{R}$ on $U_{\alpha\beta}$. It is easy to see that $(\check{\delta}\eta)_{\alpha\beta\gamma} = 0$ and hence $[\{\eta_{\alpha\beta}\}]$ defines an element in $\check{H}^1(\mathcal{U},i\mathbb{R})$. This element does not depend on the choice of χ_α, since for $\chi^\prime_\alpha = \chi_\alpha + c_\alpha$ with $c_\alpha = \mathrm{const}\in i\mathbb{R}$, we find that $\eta^\prime_{\alpha\beta} = \chi^\prime_\alpha - \chi^\prime_\beta = \eta_{\alpha\beta} + c_\alpha - c_\beta = \eta_{\alpha\beta} + (\check{\delta}c)_{\alpha\beta}$. Furthermore, this element does not depend on the choice of representative in the class $[\eta]\in H^1_{\mathrm{dR}}(M,i\mathbb{R})$, since for $\eta^\prime = \eta + \mathrm{d}\zeta$, with $\zeta\in C^\infty(M,i\mathbb{R})$, $\chi^\prime_\alpha = \chi_\alpha + \zeta|_{U_\alpha}$ and hence on $U_{\alpha\beta}$, $\eta^\prime_{\alpha\beta} = \chi^\prime_\alpha - \chi^\prime_\beta = \chi_\alpha - \chi_\beta + \zeta|_{U_{\alpha\beta}} - \zeta|_{U_{\alpha\beta}} = \eta_{\alpha\beta}$. For constructing the inverse of this map let us take a partition of unity $\{\psi_\alpha\}_{\alpha\in\mathcal{I}}$ subordinated to the good cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$. Let $[\{\eta_{\alpha\beta}\}]\in\check{H}^1(\mathcal{U},i\mathbb{R})$ be arbitrary and take some representative $\{\eta_{\alpha\beta}\}$. Let us define $\chi_\alpha := \sum_{\beta\in\mathcal{I}}\eta_{\alpha\beta}\psi_\beta \in C^\infty(U_\alpha,i\mathbb{R})$ and consider the local one-forms $\mathrm{d}\chi_\alpha\in\Omega^1_{\mathrm{d}}(U_\alpha,i\mathbb{R})$. On the double intersections $U_{\alpha\beta}$ we find $\chi_\alpha - \chi_\beta = \sum_{\gamma\in\mathcal{I}}(\eta_{\alpha\gamma} - \eta_{\beta\gamma})\psi_\gamma = \eta_{\alpha\beta}\sum_{\gamma\in\mathcal{I}}\psi_\gamma = \eta_{\alpha\beta}$, where in the second equality we have used that $(\check{\delta}\eta)_{\alpha\beta\gamma} = \eta_{\beta\gamma} - \eta_{\alpha\gamma} + \eta_{\alpha\beta} = 0$. It follows that $\mathrm{d}\chi_\alpha|_{U_{\alpha\beta}} = \mathrm{d}\chi_\beta|_{U_{\alpha\beta}}$ and hence the collection of local forms $\mathrm{d}\chi_\alpha$ defines a global closed one-form $\eta\in\Omega^1_{\mathrm{d}}(M,i\mathbb{R})$ and an element $[\eta]\in H^1_{\mathrm{dR}}(M,i\mathbb{R})$. The latter element does not depend on the choice of representative in $[\{\eta_{\alpha\beta}\}]$, since for $\{\eta^\prime_{\alpha\beta}\} = \{\eta_{\alpha\beta} + c_\alpha - c_\beta\}$ we obtain $\chi^\prime_\alpha = \sum_{\beta\in\mathcal{I}}\eta^\prime_{\alpha\beta}\psi_\beta = \chi_\alpha + c_\alpha - \sum_{\beta\in\mathcal{I}}c_\beta\gamma_\beta = \chi_\alpha + c_\alpha + \zeta|_{U_\alpha}$, where $\zeta\in C^\infty(M,i\mathbb{R})$. This implies that $\mathrm{d}\chi^\prime_\alpha = \mathrm{d}\chi_\alpha + \mathrm{d}\zeta|_{U_\alpha}$ and hence $\eta^\prime = \eta + \mathrm{d}\zeta$. The two maps presented above are one the inverse of the other and thus they provide the desired isomorphism.

Proposition 4.4. $A_{\mathbb{T}} \simeq \check{H}^1(\mathcal{U},2\pi i\,\mathbb{Z})$.

Proof. Consider an arbitrary element $[\widehat{f}^\ast(\mu_{\mathbb{T}})]\in A_{\mathbb{T}}\subseteq H^1_{\mathrm{dR}}(M,i\mathbb{R})$ and a representative $\widehat{f}\in C^\infty(M,\mathbb{T})$. Let us restrict $\widehat{f}$ to the open subsets U_α of the good cover, $\widehat{f}|_{U_\alpha}\in C^\infty(U_\alpha,\mathbb{T})$. Then $\widehat{f}^\ast(\mu_{\mathbb{T}})|_{U_\alpha} =$

$\widehat{f}|^*_{U_\alpha}(\mu_{\mathbb{T}}) = \mathrm{d}\chi_\alpha$ are exact local one-forms, with $\chi_\alpha \in C^\infty(U_\alpha, i\mathbb{R})$. In the proof of Lemma 4.1 we have shown that this implies $\widehat{f}|_{U_\alpha} = \exp\circ(\chi_\alpha + c_\alpha)$, for some $c_\alpha \in i\mathbb{R}$. Redefining χ_α by $\chi_\alpha + c_\alpha$ we can set without loss of generality $c_\alpha = 0$. Since $\widehat{f}$ is a global function we have to satisfy the consistency conditions in the double intersections $\widehat{f}|_{U_{\alpha\beta}} = \exp\circ\chi_\alpha|_{U_{\alpha\beta}} = \exp\circ\chi_\beta|_{U_{\alpha\beta}}$. This implies that on $U_{\alpha\beta}$, $\eta_{\alpha\beta} := \chi_\alpha - \chi_\beta = \mathrm{const} \in 2\pi i\,\mathbb{Z}$. Hence, $[\widehat{f}^*(\mu_{\mathbb{T}})] \in A_{\mathbb{T}}$ defines an element $[\{\eta_{\alpha\beta}\}] \in \check{H}^1(\mathcal{U}, 2\pi i\,\mathbb{Z}) \subseteq \check{H}^1(\mathcal{U}, i\mathbb{R})$. This element is independent on the representative $\widehat{f}$ we choose.

Let us now take an arbitrary element $[\{\eta_{\alpha\beta}\}] \in \check{H}^1(\mathcal{U}, 2\pi i\,\mathbb{Z}) \subseteq \check{H}^1(\mathcal{U}, i\mathbb{R})$ and a representative $\{\eta_{\alpha\beta}\}$. The Čech-de Rham isomorphism provides us with local functions $\chi_\alpha := \sum_{\beta\in\mathcal{I}} \eta_{\alpha\beta}\psi_\beta \in C^\infty(U_\alpha, i\mathbb{R})$. Let us define also the local functions $\widehat{f}_\alpha := \exp\circ\chi_\alpha \in C^\infty(U_\alpha, \mathbb{T})$. On double intersections we have $\widehat{f}_\alpha|_{U_{\alpha\beta}} = \exp\circ\chi_\alpha|_{U_{\alpha\beta}} = \exp\circ(\chi_\beta + \eta_{\alpha\beta})|_{U_{\alpha\beta}} = \exp\circ\chi_\beta|_{U_{\alpha\beta}} = \widehat{f}_\beta|_{U_{\alpha\beta}}$, since $\eta_{\alpha\beta} \in 2\pi i\,\mathbb{Z}$. Thus, we can construct a global function $\widehat{f} \in C^\infty(M, \mathbb{T})$ and define an element $[\widehat{f}^*(\mu_{\mathbb{T}})] \in A_{\mathbb{T}} \subseteq H^1_{\mathrm{dR}}(M, i\mathbb{R})$. This element does not depend on the choice of representative $\{\eta_{\alpha\beta}\}$. The two maps are one the inverse of the other and provide the desired isomorphism. □

Corollary 4.5. $A_G \simeq \check{H}^1(\mathcal{U}, 2\pi i\,\mathbb{Z})^{\oplus_k}$.

For providing an explicit characterization of $\mathcal{E}^{\mathrm{inv}}$ we use that by assumption M is of finite type with $\mathcal{U}$ denoting a finite good cover. Following the arguments of [Voi07, Chapter 7.1.1] we obtain an isomorphism

$$\check{H}^1(\mathcal{U}, 2\pi i\,\mathbb{Z}) \otimes \mathbb{R} \simeq \check{H}^1(\mathcal{U}, i\mathbb{R})\ . \tag{4.9}$$

In the generic case when M is not of finite type, this isomorphism receives corrections from the $\mathfrak{Ext}$ and $\mathfrak{Tor}$ functors, see the universal coefficient theorems [BT82, §15]. Since most (if not all) physically relevant globally hyperbolic spacetimes are of finite type (in particular $M = \mathbb{R}^m \times K$ with K compact is of finite type), we are restricting ourselves to this case and thereby avoid the characterization of the $\mathfrak{Ext}$ and $\mathfrak{Tor}$ parts.

Theorem 4.6. *Let $\Xi = \big((M, \mathfrak{o}, g, \mathfrak{t}), (G, h), (P, r)\big)$ be any object in* PrBuGlobHyp *($G \simeq \mathbb{T}^k \times \mathbb{R}^l$). Then the gauge invariant subspace $\mathcal{E}^{\mathrm{inv}}$ (3.7) is given by*

$$\mathcal{E}^{\mathrm{inv}} = \big\{\varphi \in \Gamma^\infty_0(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv} : \varphi_V \in \delta\Omega^2_0(M, i\mathbb{R})^{\oplus_k} \oplus \Omega^1_{0,\delta}(M, \mathbb{R})^{\oplus_l}\big\}\ . \tag{4.10}$$

Proof. By definition, $\mathcal{E}^{\mathrm{inv}}$ is the vector subspace of $\Gamma^\infty_0(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv}$, such that the linear parts annihilate $\{\widehat{f}^*(\mu_G) : \widehat{f} \in C^\infty(M, G)\}$. Due to Corollary 3.2 we have that $\mathcal{E}^{\mathrm{inv}} \subseteq \mathcal{E}^{\mathrm{max}} = \{\varphi \in \Gamma^\infty_0(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv} : \varphi_V \in \Omega^1_{0,\delta}(M, \mathfrak{g}^*)\}$ and hence we can pair the linear parts of elements $\varphi \in \mathcal{E}^{\mathrm{inv}}$ with cohomology classes $[\eta] \in H^1_{\mathrm{dR}}(M, \mathfrak{g})$, $\langle\varphi_V, [\eta]\rangle = \int_M \varphi_V \wedge *(\eta)$. The gauge invariance condition amounts to $\langle\varphi_V, A_G\rangle = \{0\}$, for all $\varphi \in \mathcal{E}^{\mathrm{inv}}$, and by Corollary 4.5 this is equivalent to

$$\big\langle\varphi_V, \check{H}^1(\mathcal{U}, 2\pi i\,\mathbb{Z})^{\oplus_k}\big\rangle = \{0\}\ . \tag{4.11}$$

Since $H^1_{\mathrm{dR}}(M, i\mathbb{R}) \simeq \check{H}^1(\mathcal{U}, i\mathbb{R}) \simeq \check{H}^1(\mathcal{U}, 2\pi i\,\mathbb{Z}) \otimes \mathbb{R}$ and since the map $\langle\varphi_V, \ \rangle : H^1_{\mathrm{dR}}(M, \mathfrak{g}) \to \mathbb{R}$ is linear, (4.11) implies that, for all $\varphi \in \mathcal{E}^{\mathrm{inv}}$,

$$\big\langle\varphi_V, H^1_{\mathrm{dR}}(M, i\mathbb{R})^{\oplus_k}\big\rangle = \{0\}\ . \tag{4.12}$$

As a consequence of Poincaré duality, $\varphi_V \in \delta\Omega^2_0(M, i\mathbb{R})^{\oplus_k} \oplus \Omega^1_{0,\delta}(M, \mathbb{R})^{\oplus_l}$ which completes the proof. □

Remark 4.7. Notice that if $G \simeq \mathbb{T}^k \times \mathbb{R}^l$ contains a nontrivial compact factor (i.e. $k > 0$), the vector space of gauge invariant classical affine functionals $\{\mathcal{O}_\varphi : \varphi \in \mathcal{E}^{\mathrm{inv}}\}$ (cf. (3.5)) does not separate all gauge equivalence classes of connections: Given two connections $\lambda_1, \lambda_2 \in \Gamma^\infty(M, \mathcal{C}(P))$ with the same curvature, then there exists $\eta \in \Omega^1_{\mathrm{d}}(M, \mathfrak{g})$ such that $\lambda_2 = \widetilde{\Phi}(\lambda_1, \eta)$. Let us assume that $[\eta] \in H^1_{\mathrm{dR}}(M, i\mathbb{R})^{\oplus_k} \subseteq H^1_{\mathrm{dR}}(M, \mathfrak{g})$, but $[\eta] \notin A_G$ such that λ_1 and λ_2 are not gauge equivalent (this exists e.g. for $M \simeq \mathbb{R}^{m-1} \times \mathbb{T}$). Then by (4.12) we obtain, for all $\varphi \in \mathcal{E}^{\mathrm{inv}}$, $\mathcal{O}_\varphi(\lambda_2) = \mathcal{O}_\varphi(\lambda_1) + \langle\varphi_V, \eta\rangle = \mathcal{O}_\varphi(\lambda_1)$. The origin of this pathology is the fact that A_G is only an Abelian group and not a vector space (cf. Corollary 4.5). Performing the quotient of the configuration space $\Gamma^\infty(M,\mathcal{C}(P))$ by the gauge transformations that are of exponential form (that are all for $k=0$) we obtain again an affine space. However, performing the quotient of the resulting affine space by the Abelian group A_G we obtain no affine space anymore (compare this with the quotient $\mathbb{R}/\mathbb{Z}\simeq\mathbb{T}$). The gauge invariant classical affine functionals $\{\mathcal{O}_\varphi:\varphi\in\mathcal{E}^{\mathrm{inv}}\}$ do not take into account the nontrivial topology of the full gauge invariant configuration space. For this reason one should enlarge the algebra of gauge invariant observables constructed in this paper to include additional elements which can separate all gauge equivalence classes of connections. A natural candidate are Wilson loops, but, being too singular objects localized on curves, they cannot be added easily to the present formalism used in algebraic quantum field theory. We hope to come back to this issue in our future investigations.

To conclude this section we characterize the radical $\mathcal{N}$ of the presymplectic vector space $\big(\mathcal{E},\tau\big)$ of Proposition 3.4.

Theorem 4.8. *Let $\Xi=\big((M,\mathfrak{o},g,\mathfrak{t}),(G,h),(P,r)\big)$ be any object in* PrBuGlobHyp *($G\simeq\mathbb{T}^k\times\mathbb{R}^l$). Then the radical $\mathcal{N}$ of $\big(\mathcal{E},\tau\big)$ is given by*

$$\mathcal{N}=\big\{\psi\in\mathcal{E}^{\mathrm{inv}}:h^{-1}(\psi_V)\in\delta\Omega^2_{0,\mathrm{d}}(M,i\mathbb{R})^{\oplus_k}\oplus\delta\big(\Omega^2_0(M,\mathbb{R})\cap\mathrm{d}\Omega^1_{\mathrm{tc}}(M,\mathbb{R})\big)^{\oplus_l}\big\}/\mathsf{MW}^*\big[\Omega^1_0(M,\mathfrak{g}^*)\big]\;. \tag{4.13}$$

Proof. Let $[\psi]$ be an element of the vector space on the right hand side of (4.13). Any representative ψ is such that $h^{-1}(\psi_V)=\delta\eta+\delta\mathrm{d}\zeta$ for some $\eta\in\Omega^2_{0,\mathrm{d}}(M,i\mathbb{R})^{\oplus_k}$ and $\zeta\in\Omega^1_{\mathrm{tc}}(M,\mathbb{R})^{\oplus_l}$. By Theorem 4.6 any $\varphi\in\mathcal{E}^{\mathrm{inv}}$ is such that $\varphi_V=\delta\alpha+\beta$ for some $\alpha\in\Omega^2_0(M,i\mathbb{R})^{\oplus_k}$ and $\beta\in\Omega^1_{0,\delta}(M,\mathbb{R})^{\oplus_l}$. As a consequence,

$$\begin{aligned}\tau([\varphi],[\psi])&=\big\langle\varphi_V,G_{(1)}\big(h^{-1}(\psi_V)\big)\big\rangle=\big\langle\delta\alpha,G_{(1)}(\delta\eta)\big\rangle+\big\langle\beta,G_{(1)}(\delta\mathrm{d}\zeta)\big\rangle\\&=\big\langle\alpha,\mathrm{d}\delta G_{(2)}(\eta)\big\rangle+\big\langle\beta,\delta\mathrm{d}G_{(1)}(\zeta)\big\rangle=-\big\langle\alpha,\delta\mathrm{d}G_{(2)}(\eta)\big\rangle-\big\langle\beta,\mathrm{d}\delta G_{(1)}(\zeta)\big\rangle=0\;,\end{aligned}\tag{4.14}$$

hence the vector space on the right hand side of (4.13) is contained in the radical $\mathcal{N}$. To show that it is equal to the radical let $\psi\in\mathcal{E}^{\mathrm{inv}}$ be any element satisfying, for all $\varphi\in\mathcal{E}^{\mathrm{inv}}$, $\tau([\varphi],[\psi])=0$. Using again the decomposition $\varphi_V=\delta\alpha+\beta$ for some $\alpha\in\Omega^2_0(M,i\mathbb{R})^{\oplus_k}$ and $\beta\in\Omega^1_{0,\delta}(M,\mathbb{R})^{\oplus_l}$, as well as the decomposition $h^{-1}(\psi_V)=\delta\eta+\delta\epsilon$, where $\eta\in\Omega^2_{0,\mathrm{d}}(M,i\mathbb{R})^{\oplus_k}$ and $\epsilon\in\Omega^2_{0,\mathrm{d}}(M,\mathbb{R})^{\oplus_l}$ (which is possible due to Corollary 3.7), this condition yields

$$0=\tau([\varphi],[\psi])=\big\langle\delta\alpha,G_{(1)}(\delta\eta)\big\rangle+\big\langle\beta,G_{(1)}(\delta\epsilon)\big\rangle=\big\langle\beta,G_{(1)}(\delta\epsilon)\big\rangle\;. \tag{4.15}$$

By (4.15) and Poincaré duality there exists a $\gamma\in C^\infty(M,\mathbb{R})^{\oplus_l}$, such that $G_{(1)}(\delta\epsilon)=\mathrm{d}\gamma$. Applying the codifferential to this equation we find that γ satisfies the wave equation $\delta\mathrm{d}\gamma=\Box_{(0)}(\gamma)=0$, hence by [SDH12] there exists a $\theta\in C^\infty_{\mathrm{tc}}(M,\mathbb{R})^{\oplus_l}$ such that $\gamma=G_{(0)}(\theta)$. Plugging this into the equation above yields $G_{(1)}(\delta\epsilon)=\mathrm{d}\gamma=G_{(1)}(\mathrm{d}\theta)$, which implies $\delta\epsilon=\mathrm{d}\theta+\Box_{(1)}(\zeta)$ for some $\zeta\in\Omega^1_{\mathrm{tc}}(M,\mathbb{R})^{\oplus_l}$. Applying d and using that ϵ is closed we obtain $\epsilon=\mathrm{d}\zeta$, which shows that any element in the radical is contained in the vector space on the right hand side of (4.13). □

5 The phase space functor and $\mathfrak{CCR}$-quantization

In this section we show that the association of the presymplectic vector space $\big(\mathcal{E},\tau\big)$ in Proposition 3.4 to objects $\Xi=\big((M,\mathfrak{o},g,\mathfrak{t}),(G,h),(P,r)\big)$ in PrBuGlobHyp is functorial. We are going to construct a covariant functor $\mathfrak{PhSp}$: PrBuGlobHyp $\to$ PreSymp, where the latter category is that of presymplectic vector spaces with compatible morphisms, that are however not assumed to be injective (see the definition below). We will then derive some important properties of the functor.

Definition 5.1. The category PreSymp consists of the following objects and morphisms:

- An object in PreSymp is a tuple $\big(\mathcal{E},\tau\big)$, where $\mathcal{E}$ is a (possibly infinite dimensional) vector space over $\mathbb{R}$ and $\tau:\mathcal{E}\times\mathcal{E}\to\mathbb{R}$ is an antisymmetric bilinear map (a presymplectic structure).

- A morphism between two objects $(\mathcal{E}_1, \tau_1)$ and $(\mathcal{E}_2, \tau_2)$ in $\mathsf{PreSymp}$ is a linear map $L : \mathcal{E}_1 \to \mathcal{E}_2$ (not necessarily injective) , which preserves the presymplectic structures, i.e. $\tau_2(L(v), L(w)) = \tau_1(v, w)$, for all $v, w \in \mathcal{E}_1$.

Before constructing the phase space functor $\mathfrak{PhSp}$ we require two lemmas characterizing the compatibility of Maxwell's affine differential operator MW, the Hodge-d'Alembert operators $\square_{(k)}$ and their Green's operators $G^{\pm}_{(k)}$ with morphisms in $\mathsf{PrBuGlobHyp}$.

Lemma 5.2. *Let Ξ_i, $i = 1, 2$, be two objects and $F : \Xi_1 \to \Xi_2$ a morphism in $\mathsf{PrBuGlobHyp}$. Then the following diagram commutes:*

$$\begin{array}{ccc} \Gamma^\infty(M_2, \mathcal{C}(P_2)) & \xrightarrow{\mathsf{MW}_2} & \Omega^1(M_2, \mathfrak{g}_2) \\ {\scriptstyle F^*}\big\downarrow & & \big\downarrow{\scriptstyle \underline{f}^* \circ \phi_*^{-1}} \\ \Gamma^\infty(M_1, \mathcal{C}(P_1)) & \xrightarrow{\mathsf{MW}_1} & \Omega^1(M_1, \mathfrak{g}_1) \end{array} \tag{5.1}$$

F^* is defined in Remark 2.28, $\underline{f}^*$ is the usual pull-back along the induced map $\underline{f} : M_1 \to M_2$ and $\phi_*^{-1} : \mathfrak{g}_2 \to \mathfrak{g}_1$ is the inverse of the push-forward of $\phi : G_1 \to G_2$.

Proof. Let $\lambda \in \Gamma^\infty(M_2, \mathcal{C}(P_2))$ be arbitrary and let $\omega_\lambda \in \mathrm{Con}(P_2)$ be the associated connection form (cf. Proposition 2.23). Then $\omega_{F^*(\lambda)} = f^*\big(\phi_*^{-1}(\omega_\lambda)\big) \in \mathrm{Con}(P_1)$, where on the right hand side f^* denotes the usual pull-back of forms along $f : P_1 \to P_2$. For the curvatures $\mathcal{F}_i : \mathrm{Con}(P_i) \to \Omega^2_{\text{hor}}(P_i, \mathfrak{g}_i)^{\text{eqv}}$ we obtain, for all $\lambda \in \Gamma^\infty(M_2, \mathcal{C}(P_2))$,

$$\mathcal{F}_1\big(\omega_{F^*(\lambda)}\big) = \mathrm{d}_1 \omega_{F^*(\lambda)} = \mathrm{d}_1 f^*\big(\phi_*^{-1}(\omega_\lambda)\big) = f^*\big(\phi_*^{-1}(\mathrm{d}_2 \omega_\lambda)\big) = f^*\big(\phi_*^{-1}(\mathcal{F}_2(\omega_\lambda))\big)\ . \tag{5.2}$$

This implies for the associated curvature affine differential operators $\underline{\mathcal{F}}_i : \Gamma^\infty(M_i, \mathcal{C}(P_i)) \to \Omega^2(M_i, \mathfrak{g}_i)$,

$$\underline{\mathcal{F}}_1 \circ F^* = \underline{f}^* \circ \phi_*^{-1} \circ \underline{\mathcal{F}}_2\ . \tag{5.3}$$

Using that by hypothesis $\underline{f} : M_1 \to M_2$ is an isometric and orientation preserving embedding, we obtain for the codifferentials $\delta_1 \circ \underline{f}^* = \underline{f}^* \circ \delta_2$ and, hence, for the Maxwell operators $\mathsf{MW}_1 \circ F^* = \underline{f}^* \circ \phi_*^{-1} \circ \mathsf{MW}_2$, which shows the commutativity of the diagram (5.1). □

Lemma 5.3. *Let Ξ_i, $i = 1, 2$, be two objects and $F : \Xi_1 \to \Xi_2$ a morphism in $\mathsf{PrBuGlobHyp}$.*

a) The following diagram commutes for all k:

$$\begin{array}{ccc} \Omega^k(M_2, \mathfrak{g}_2^*) & \xrightarrow{\square_{2\,(k)}} & \Omega^k(M_2, \mathfrak{g}_2^*) \\ {\scriptstyle \underline{f}^* \circ \phi^*}\big\downarrow & & \big\downarrow{\scriptstyle \underline{f}^* \circ \phi^*} \\ \Omega^k(M_1, \mathfrak{g}_1^*) & \xrightarrow{\square_{1\,(k)}} & \Omega^k(M_1, \mathfrak{g}_1^*) \end{array} \tag{5.4}$$

$\phi^* : \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ is the pull-back of $\phi : G_1 \to G_2$.

b) The Green's operators satisfy $G^{\pm}_{1\,(k)} = \underline{f}^* \circ \phi^* \circ G^{\pm}_{2\,(k)} \circ \underline{f}_* \circ \phi^{-1\,*}$, where $\underline{f}_*$ denotes the push-forward of compactly supported forms along $\underline{f} : M_1 \to M_2$ and $\phi^{-1\,*} : \mathfrak{g}_1^* \to \mathfrak{g}_2^*$ is the pull-back of $\phi^{-1} : G_2 \to G_1$.

Proof. Notice that the operators $\square_{i\,(k)}$ act as the identity on $\mathfrak{g}_i^*$. The commutative diagram (5.4) is then a consequence of $\underline{f}^* \circ \mathrm{d}_2 = \mathrm{d}_1 \circ \underline{f}^*$, which holds for any smooth map $\underline{f} : M_1 \to M_2$, and of $\underline{f}^* \circ \delta_2 = \delta_1 \circ \underline{f}^*$, which holds since $\underline{f}$ is an isometric and orientation preserving embedding.

To prove b) first notice that $\underline{f}_*\big(\underline{f}^*(\eta)\big) = \eta$, for all $\eta \in \Omega^k_0(\underline{f}[M_1], \mathfrak{g}_2^*) \subseteq \Omega^k_0(M_2, \mathfrak{g}_2^*)$, and that $\underline{f}^*\big(\underline{f}_*(\eta)\big) = \eta$, for all $\eta \in \Omega^k_0(M_1, \mathfrak{g}_1^*)$. Let us define $\widetilde{G}^{\pm}_{1\,(k)} := \underline{f}^* \circ \phi^* \circ G^{\pm}_{2\,(k)} \circ \underline{f}_* \circ \phi^{-1\,*}$. We show that $\widetilde{G}^{\pm}_{1\,(k)}$ are retarded/advanced Green's operators for $\Box_{1\,(k)}$ and thus by uniqueness it follows the claim $\widetilde{G}^{\pm}_{1\,(k)} = G^{\pm}_{1\,(k)}$. Due to the diagram (5.4) and the above properties of $\underline{f}_*$ and $\underline{f}^*$ we obtain

$$\begin{aligned}\Box_{1\,(k)} \circ \widetilde{G}^{\pm}_{1\,(k)} &= \Box_{1\,(k)} \circ \underline{f}^* \circ \phi^* \circ G^{\pm}_{2\,(k)} \circ \underline{f}_* \circ \phi^{-1\,*} \\ &= \underline{f}^* \circ \phi^* \circ \Box_{2\,(k)} \circ G^{\pm}_{2\,(k)} \circ \underline{f}_* \circ \phi^{-1\,*} = \mathrm{id}_{\Omega^k_0(M_1,\mathfrak{g}_1^*)} \end{aligned} \tag{5.5a}$$

and on $\Omega^k_0(M_1,\mathfrak{g}_1^*)$

$$\begin{aligned}\widetilde{G}^{\pm}_{1\,(k)} \circ \Box_{1\,(k)} &= \underline{f}^* \circ \phi^* \circ G^{\pm}_{2\,(k)} \circ \underline{f}_* \circ \phi^{-1*} \circ \Box_{1\,(k)} \\ &= \underline{f}^* \circ \phi^* \circ G^{\pm}_{2\,(k)} \circ \Box_{2\,(k)} \circ \underline{f}_* \circ \phi^{-1\,*} = \mathrm{id}_{\Omega^k_0(M_1,\mathfrak{g}_1^*)} \ . \end{aligned} \tag{5.5b}$$

Thus, $\widetilde{G}^{\pm}_{1\,(k)}$ are Green's operators for $\Box_{1\,(k)}$. They are retarded/advanced Green's operators, since for all $\eta \in \Omega^k_0(M_1,\mathfrak{g}_1^*)$,

$$\mathrm{supp}\big(\widetilde{G}^{\pm}_{1\,(k)}(\eta)\big) \subseteq \underline{f}^{-1}\big[J^{\pm}_{M_2}\big(\underline{f}[\mathrm{supp}(\eta)]\big)\big] = J^{\pm}_{M_1}\big(\mathrm{supp}(\eta)\big) \ , \tag{5.6}$$

where in the second step we have used that $\underline{f}[M_1] \subseteq M_2$ is by hypothesis causally compatible. $\square$

Definition 5.4. Let Ξ_i, $i=1,2$, be two objects and $F : \Xi_1 \to \Xi_2$ a morphism in $\mathsf{PrBuGlobHyp}$. Furthermore, let $F^* : \Gamma^\infty(M_2,\mathcal{C}(P_2)) \to \Gamma^\infty(M_1,\mathcal{C}(P_1))$ be the affine map constructed in Remark 2.28. We define the linear map $F_* : \Gamma^\infty_0(M_1,\mathcal{C}(P_1)^\dagger)/\mathrm{Triv}_1 \to \Gamma^\infty_0(M_2,\mathcal{C}(P_2)^\dagger)/\mathrm{Triv}_2$ by duality, for all $\varphi \in \Gamma^\infty_0(M_1,\mathcal{C}(P_1)^\dagger)/\mathrm{Triv}_1$ and $\lambda \in \Gamma^\infty(M_2,\mathcal{C}(P_2))$,

$$\int_{M_2} \mathrm{vol}_2\,\big(F_*(\varphi)\big)(\lambda) = \int_{M_1} \mathrm{vol}_1\,\varphi\big(F^*(\lambda)\big) \ . \tag{5.7}$$

Theorem 5.5. *There is a covariant functor $\mathfrak{PhSp} : \mathsf{PrBuGlobHyp} \to \mathsf{PreSymp}$. It associates to any object Ξ in $\mathsf{PrBuGlobHyp}$ the object $\mathfrak{PhSp}(\Xi) = \big(\mathcal{E},\tau\big)$ in $\mathsf{PreSymp}$ which has been constructed in Proposition 3.4. Given a morphism $F : \Xi_1 \to \Xi_2$ between two objects Ξ_i, $i=1,2$, in $\mathsf{PrBuGlobHyp}$ the functor associates a morphism in $\mathsf{PreSymp}$ as follows*

$$\mathfrak{PhSp}(F) : \mathfrak{PhSp}(\Xi_1) \to \mathfrak{PhSp}(\Xi_2) \ , \quad [\varphi] \mapsto [F_*(\varphi)] \ , \tag{5.8}$$

where the linear map F_* is given in Definition 5.4.

Proof. First, we show that F_* maps $\mathcal{E}^{\mathrm{inv}}_1$ to $\mathcal{E}^{\mathrm{inv}}_2$. Let $\varphi \in \mathcal{E}^{\mathrm{inv}}_1$ be arbitrary, i.e. for all $\widehat{f} \in C^\infty(M_1,G_1)$, $\big\langle \varphi_V, \widehat{f}^*(\mu_{G_1})\big\rangle_1 = 0$. By Remark 2.28 and Definition 5.4 we obtain $F_*(\varphi)_V = \underline{f}_*\big(\phi^{-1\,*}(\varphi_V)\big)$ and hence, for all $\widehat{f} \in C^\infty(M_2,G_2)$,

$$\Big\langle F_*(\varphi)_V, \widehat{f}^*(\mu_{G_2})\Big\rangle_2 = \Big\langle \varphi_V, \underline{f}^*\big(\widehat{f}^*(\phi_*^{-1}(\mu_{G_2}))\big)\Big\rangle_1 = \Big\langle \varphi_V, (\phi^{-1}\circ\widehat{f}\circ\underline{f})^*(\mu_{G_1})\big)\Big\rangle_1 = 0 \ . \tag{5.9}$$

In the second equality we have used that $\phi^*\big(\phi_*^{-1}(\mu_{G_2})\big) = \mu_{G_1}$, where ϕ^* is the pull-back of forms along $\phi : G_1 \to G_2$.

Next, we prove that (5.8) is well-defined, that is, for all $\eta \in \Omega^1_0(M_1,\mathfrak{g}_1^*)$ we have $F_*\big(\mathsf{MW}_1^*(\eta)\big) \in \mathsf{MW}_2^*\big[\Omega^1_0(M_2,\mathfrak{g}_2^*)\big]$. This property is a consequence of the following short calculation, for all $\lambda \in \Gamma^\infty(M_2,\mathcal{C}(P_2))$,

$$\begin{aligned}\int_{M_2} \mathrm{vol}_2\,\Big(F_*\big(\mathsf{MW}_1^*(\eta)\big)\Big)(\lambda) &= \big\langle \eta, \mathsf{MW}_1\big(F^*(\lambda)\big)\big\rangle_1 = \big\langle \eta, \underline{f}^*\big(\phi_*^{-1}(\mathsf{MW}_2(\lambda))\big)\big\rangle_1 \\ &= \Big\langle \underline{f}_*(\phi^{-1\,*}(\eta)), \mathsf{MW}_2(\lambda)\Big\rangle_2 = \int_{M_2} \mathrm{vol}_2\,\Big(\mathsf{MW}_2^*\big(\underline{f}_*(\phi^{-1\,*}(\eta))\big)\Big)(\lambda) \ , \end{aligned} \tag{5.10}$$

where in the second equality we have used Lemma 5.2.

It remains to be shown that the linear map $\mathfrak{PhSp}(F)$ in (5.8) preserves the presymplectic structures. Let us take two arbitrary $[\varphi], [\psi] \in \mathcal{E}_1$. Then

$$\tau_2\big([F_*(\varphi)], [F_*(\psi)]\big) = \big\langle F_*(\varphi)_V, G_{2\,(1)}\big(F_*(\psi)_V\big)\big\rangle_{h_2} \ . \tag{5.11}$$

Using again that $F_*(\varphi)_V = \underline{f}_*(\phi^{-1\,*}(\varphi_V))$ (and similar for ψ) yields

$$\begin{aligned}\tau_2\big([F_*(\varphi)], [F_*(\psi)]\big) &= \Big\langle \underline{f}_*(\phi^{-1\,*}(\varphi_V)), G_{2\,(1)}\big(\underline{f}_*(\phi^{-1\,*}(\psi_V))\big)\Big\rangle_{h_2} \\ &= \Big\langle \varphi_V, \big(\underline{f}^* \circ \phi^* \circ G_{2\,(1)} \circ \underline{f}_* \circ \phi^{-1\,*}\big)(\psi_V)\Big\rangle_{h_1} \\ &= \big\langle \varphi_V, G_{1\,(1)}(\psi_V)\big\rangle_{h_1} = \tau_1([\varphi], [\psi]) \ .\end{aligned} \tag{5.12}$$

In the second equality we used that ϕ is an isometry and in the third equality Lemma 5.3 b). □

Remark 5.6. The covariant functor $\mathfrak{PhSp} : \mathsf{PrBuGlobHyp} \to \mathsf{PreSymp}$ does not satisfy the locality property stating that for any morphism $F : \Xi_1 \to \Xi_2$ in $\mathsf{PrBuGlobHyp}$ the morphism $\mathfrak{PhSp}(F)$ is injective. We will show this failure by giving a simple example in the full subcategory $\mathsf{PrBuGlobHyp}^{U(1)}$ where $G = U(1) \simeq \mathbb{T}$ is fixed and we refer to Section 7 for a possible solution of this problem. Let Ξ_2 be an object in $\mathsf{PrBuGlobHyp}^{U(1)}$ such that $(M_2, \mathfrak{o}_2, g_2, \mathfrak{t}_2)$ is the m-dimensional Minkowski spacetime ($m > 2$). Let us denote by Ξ_1 the object in $\mathsf{PrBuGlobHyp}^{U(1)}$ that is obtained by restricting all data of Ξ_2 to the causally compatible and globally hyperbolic open subset $M_1 := M_2 \setminus J_{M_2}(\{0\})$, where $\{0\}$ is the set of a single point in Minkowski spacetime (cf. [BGP07, Lemma A.5.11]). Notice that M_1 is diffeomorphic to $\mathbb{R}^2 \times S^{m-2}$, where S^{m-2} is the $m-2$-sphere. The canonical embedding (via the identity) $F : \Xi_1 \to \Xi_2$ is a morphism in $\mathsf{PrBuGlobHyp}^{U(1)}$. Let us take any nonexact element in $\eta \in \Omega^2_{0,\mathrm{d}}(M_1, \mathfrak{g}^*)$, that exists since by Poincaré duality $H^{m-2}_{\mathrm{dR}}(M_1, \mathfrak{g}) \simeq H^2_{0\,\mathrm{dR}}(M_1, \mathfrak{g}^*)$ and $H^{m-2}_{\mathrm{dR}}(M_1, \mathfrak{g}) \simeq \mathfrak{g} \simeq i\,\mathbb{R}$ since M_1 is homotopy equivalent to S^{m-2}. Applying the formal adjoint of the curvature affine differential operator we obtain a nontrivial element $\big[\underline{\mathcal{F}_1}^*(\eta)\big] \in \mathfrak{PhSp}(\Xi_1)$ (this element is contained in the radical $\mathcal{N}_1$, cf. Theorem 4.8). Under the morphism $\mathfrak{PhSp}(F)$ we obtain by using (5.3)

$$\begin{aligned}\mathfrak{PhSp}(F)\big(\big[\underline{\mathcal{F}_1}^*(\eta)\big]\big) &= \big[F_*\big(\underline{\mathcal{F}_1}^*(\eta)\big)\big] = \big[\underline{\mathcal{F}_2}^*\big(\underline{f}_*(\phi^{-1\,*}(\eta)\big)\big] \\ &= \big[\underline{\mathcal{F}_2}^*(\mathrm{d}\xi)\big] = \big[\mathsf{MW}_2^*(\xi)\big] = 0 \ .\end{aligned} \tag{5.13}$$

In the third equality we have used that $\underline{f}_*\big(\phi^{-1\,*}(\eta)\big) \in \Omega^2_{0,\mathrm{d}}(M_2, \mathfrak{g}^*)$ is exact since M_2 is the Minkowski spacetime. By Remark 3.9 the same conclusion holds true for $G = \mathbb{R}$ and hence for generic $G \simeq \mathbb{T}^k \times \mathbb{R}^l$.

Theorem 5.7. *The covariant functor $\mathfrak{PhSp} : \mathsf{PrBuGlobHyp} \to \mathsf{PreSymp}$ satisfies the classical causality property:*

Let Ξ_j, $j = 1, 2, 3$, be three objects and let $F_i : \Xi_i \to \Xi_3$, $i = 1, 2$, be two morphisms in $\mathsf{PrBuGlobHyp}$, such that $\underline{f_1}[M_1]$ and $\underline{f_2}[M_2]$ are causally disjoint in M_3. Then τ_3 acts trivially among the vector subspaces $\mathfrak{PhSp}(F_1)\big[\mathfrak{PhSp}(\Xi_1)\big]$ and $\mathfrak{PhSp}(F_2)\big[\mathfrak{PhSp}(\Xi_2)\big]$ of $\mathfrak{PhSp}(\Xi_3)$. That is, for all $[\varphi] \in \mathfrak{PhSp}(\Xi_1)$ and $[\psi] \in \mathfrak{PhSp}(\Xi_2)$,

$$\tau_3\big(\mathfrak{PhSp}(F_1)([\varphi]), \mathfrak{PhSp}(F_2)([\psi])\big) = 0 \ . \tag{5.14}$$

Proof. From (5.8) and (3.14) it follows that

$$\tau_3\big(\mathfrak{PhSp}(F_1)([\varphi]), \mathfrak{PhSp}(F_2)([\psi])\big) = \Big\langle \underline{f_1}_*\big(\phi_1^{-1\,*}(\varphi_V)\big), G_{3\,(1)}\big(\underline{f_2}_*\big(\phi_2^{-1\,*}(\psi_V)\big)\big)\Big\rangle_{h_3} = 0 \ , \tag{5.15}$$

since the supports $\mathrm{supp}\big(\underline{f_1}_*\big(\phi_1^{-1\,*}(\varphi_V)\big)\big) \subseteq \underline{f_1}[M_1]$ and $\mathrm{supp}\big(G_{3\,(1)}\big(\underline{f_2}_*\big(\phi_2^{-1\,*}(\psi_V)\big)\big)\big) \subseteq J_{M_3}\big(\underline{f_2}[M_2]\big)$ are by hypothesis disjoint. □

Theorem 5.8. *The covariant functor* $\mathfrak{PhSp} : \mathsf{PrBuGlobHyp} \to \mathsf{PreSymp}$ *satisfies the classical time-slice axiom:*

Let Ξ_i*,* $i = 1, 2$*, be two objects and* $F : \Xi_1 \to \Xi_2$ *a morphism in* $\mathsf{PrBuGlobHyp}$*, such that* $\underline{f}[M_1] \subseteq M_2$ *contains a Cauchy surface of* M_2*. Then*

$$\mathfrak{PhSp}(F) : \mathfrak{PhSp}(\Xi_1) \to \mathfrak{PhSp}(\Xi_2) \tag{5.16}$$

is an isomorphism.

Proof. Let us define $\Xi_2|_{\underline{f}[M_1]} := \big((\underline{f}[M_1], \mathfrak{o}_2|_{\underline{f}[M_1]}, g_2|_{\underline{f}[M_1]}, \mathfrak{t}_2|_{\underline{f}[M_1]}), (G_2, h_2), (P_2|_{\underline{f}[M_1]}, r_2)\big)$, where $P_2|_{\underline{f}[M_1]}$ denotes the restriction of the principal G_2-bundle (P_2, r_2) over M_2 to $\underline{f}[M_1] \subseteq M_2$. Notice that $\Xi_2|_{\underline{f}[M_1]}$ is an object in $\mathsf{PrBuGlobHyp}$ and by definition of the morphisms in this category, $F : \Xi_1 \to \Xi_2|_{\underline{f}[M_1]}$ is an isomorphism. As a consequence of functoriality, we obtain an isomorphism in $\mathsf{PreSymp}$

$$\mathfrak{PhSp}(F) : \mathfrak{PhSp}(\Xi_1) \to \mathfrak{PhSp}(\Xi_2|_{\underline{f}[M_1]}) \,. \tag{5.17}$$

Hence, the proof would follow if we could show that in the hypotheses of this theorem the canonical map $\mathfrak{PhSp}(\Xi_2|_{\underline{f}[M_1]}) \to \mathfrak{PhSp}(\Xi_2)\,,\ [\varphi] \mapsto [\varphi]$ is an isomorphism.

Let us first prove injectivity of the canonical map: Let $[\varphi] \in \mathfrak{PhSp}(\Xi_2|_{\underline{f}[M_1]})$ be such that when interpreted via the canonical map as an element in $\mathfrak{PhSp}(\Xi_2)$ we have $[\varphi] = 0$. As a consequence, $[\varphi] \in \mathfrak{PhSp}(\Xi_2|_{\underline{f}[M_1]})$ has to be in the radical $\mathcal{N}_2|_{\underline{f}[M_1]}$ and by Corollary 3.7 there exists for any representative φ an $\eta \in \Omega^2_{0,\mathrm{d}}(\underline{f}[M_1], \mathfrak{g}_2^\ast)$ such that $\varphi_V = \delta_2 \eta$. Notice that due to the quotient in Corollary 3.7 the equivalence class $[\varphi]$ only depends on the cohomology class $[\eta] \in H^2_{0\,\mathrm{dR}}(\underline{f}[M_1], \mathfrak{g}_2^\ast)$. By a theorem of Bernal and Sánchez [BS05] and the hypothesis that $\underline{f}[M_1]$ contains a Cauchy surface of M_2 we have that $\underline{f}[M_1]$ and M_2 are homotopy equivalent (notice also that $\dim(\underline{f}[M_1]) = \dim(M_2)$). By Poincaré duality $[\eta]$ specifies a unique element in $H^{\dim(M_2)-2}_{\mathrm{dR}}(\underline{f}[M_1], \mathfrak{g}_2)$, which by homotopy invariance of the de Rham cohomology groups and a further instance of Poincaré duality specifies a unique element in $H^2_{0\,\mathrm{dR}}(M_2, \mathfrak{g}_2^\ast)$. Using the fact that $[\varphi] = 0$ when regarded in $\mathfrak{PhSp}(\Xi_2)$ then implies that $[\eta]$ is the trivial element, i.e. $\eta = \mathrm{d}_2 \zeta$ for some $\zeta \in \Omega^1_0(\underline{f}[M_1], \mathfrak{g}_2^\ast)$. Thus, we can find a representative φ of the class $[\varphi] \in \mathfrak{PhSp}(\Xi_2|_{\underline{f}[M_1]})$ such that $\varphi_V = 0$, i.e. $\varphi = a\,\mathbb{1}_2$ with $a \in C^\infty_0(\underline{f}[M_1])$. Since $[\varphi]$ lies in the kernel of the canonical map we obtain $0 = \int_{M_2} \mathrm{vol}_2\, a = \int_{\underline{f}[M_1]} \mathrm{vol}_2\, a$ and thus $[\varphi] = 0$ in $\mathfrak{PhSp}(\Xi_2|_{\underline{f}[M_1]})$.

We now prove surjectivity of the canonical map: Let $[\varphi] \in \mathfrak{PhSp}(\Xi_2)$ be arbitrary and let φ be any representative. By hypothesis, there is a Cauchy surface Σ_2 in M_2 that is contained in $\underline{f}[M_1]$. Then $\Sigma_1 := \underline{f}^{-1}[\Sigma_2]$ is a Cauchy surface in M_1, since $\underline{f} : M_1 \to \underline{f}[M_1]$ is an isometry. Let us choose two other Cauchy surfaces $\Sigma_1^\pm$ with $\Sigma_1^\pm \cap \Sigma_1 = \emptyset$ in the future/past of Σ_1 and let us denote by $\Sigma_2^\pm := \underline{f}[\Sigma_1^\pm]$ their images, which are Cauchy surfaces in M_2 since $\underline{f}[M_1]$ is causally compatible. Let $\chi^+ \in C^\infty(M_2)$ be any function such that $\chi^+ \equiv 1$ on $J^+_{M_2}(\Sigma_2^+)$ and $\chi^+ \equiv 0$ on $J^-_{M_2}(\Sigma_2^-)$. We define $\chi^- \in C^\infty(M_2)$ by $\chi^+ + \chi^- \equiv 1$ on M_2. Then $\eta := \chi^+\, G^-_{(1)}(\varphi_V) + \chi^-\, G^+_{(1)}(\varphi_V) \in \Omega^1_0(M_2, \mathfrak{g}_2^\ast)$ is of compact support and the linear part of $\varphi' := \varphi + \mathsf{MW}_2^\ast(\eta)$, given by $\varphi'_V = \varphi_V - \delta_2 \mathrm{d}_2 \eta$, vanishes outside of $\underline{f}[M_1]$ (remember that by Lemma 3.1 $\delta_2 \varphi_V = 0$). The constant affine part of φ' can be treated as in [BDS12, Theorem 5.6] by adding a suitable element of Triv_2 to φ', which leads to a representative φ'' of the same class $[\varphi]$ that has compact support in $\underline{f}[M_1]$. The class $[\varphi''] \in \mathfrak{PhSp}(\Xi_2|_{\underline{f}[M_1]})$ proves surjectivity of the canonical map. □

We quantize our theory by using the $\mathfrak{CCR}$-functor, which we are going to briefly review to be self-contained.

Definition 5.9. The category ${}^\ast\mathsf{Alg}$ consists of the following objects and morphisms:

- An object in ${}^\ast\mathsf{Alg}$ is a unital $\ast$-algebra $\mathcal{A}$ over $\mathbb{C}$.
- A morphism between two objects $\mathcal{A}_i$, $i = 1, 2$, in ${}^\ast\mathsf{Alg}$ is a unital $\ast$-algebra homomorphism $\kappa : \mathcal{A}_1 \to \mathcal{A}_2$ (not necessarily injective).

The $\mathfrak{CCR}$-functor is the covariant functor $\mathfrak{CCR} : \mathsf{PreSymp} \to {}^*\mathsf{Alg}$ which associates to any object $(\mathcal{E}, \tau)$ the unital $*$-algebra $\mathfrak{CCR}(\mathcal{E}, \tau) = \mathcal{T}(\mathcal{E})/\mathcal{I}(\mathcal{E}, \tau)$. $\mathcal{T}(\mathcal{E})$ is the complex tensor algebra over $\mathcal{E}$ and $\mathcal{I}(\mathcal{E}, \tau)$ is the two-sided ideal generated by the elements $v \otimes_{\mathbb{C}} w - w \otimes_{\mathbb{C}} v - i\, \tau(v, w)\, \mathbf{1}$, for all $v, w \in \mathcal{E}$. To any morphism $L : (\mathcal{E}_1, \tau_1) \to (\mathcal{E}_2, \tau_2)$ in $\mathsf{PreSymp}$ the functor associates the morphism $\mathfrak{CCR}(L)$ in ${}^*\mathsf{Alg}$ which is defined on the tensor algebra by $\mathfrak{CCR}(L)\big(v_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} v_k\big) = L(v_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} L(v_k)$, for all $k \geq 1$ and $v_1, \dots, v_k \in \mathcal{E}_1$. Since L preserves the presymplectic structures, this unital $*$-algebra homomorphism canonically induces to the quotients.

Using the same arguments as in [BDS12, Theorem 6.3] it follows immediately from Theorem 5.7 and Theorem 5.8 the following

Theorem 5.10. *The covariant functor* $\mathfrak{A} := \mathfrak{CCR} \circ \mathfrak{PhSp} : \mathsf{PrBuGlobHyp} \to {}^*\mathsf{Alg}$ *satisfies:*

(i) The quantum causality property:

Let Ξ_j, $j = 1, 2, 3$, *be three objects and let* $F_i : \Xi_i \to \Xi_3$, $i = 1, 2$, *be two morphisms in* $\mathsf{PrBuGlobHyp}$, *such that* $\underline{f_1}[M_1]$ *and* $\underline{f_2}[M_2]$ *are causally disjoint in* M_3. *Then* $\mathfrak{A}(F_1)\big[\mathfrak{A}(\Xi_1)\big]$ *and* $\mathfrak{A}(F_2)\big[\mathfrak{A}(\Xi_2)\big]$ *commute as subalgebras of* $\mathfrak{A}(\Xi_3)$.

(ii) The quantum time-slice axiom:

Let Ξ_i, $i = 1, 2$, *be two objects and* $F : \Xi_1 \to \Xi_2$ *a morphism in* $\mathsf{PrBuGlobHyp}$, *such that* $\underline{f}[M_1] \subseteq M_2$ *contains a Cauchy surface of* M_2. *Then*

$$\mathfrak{A}(F) : \mathfrak{A}(\Xi_1) \to \mathfrak{A}(\Xi_2) \tag{5.18}$$

is an isomorphism.

6 Generally covariant topological quantum fields

According to [BFV03], a locally covariant quantum field is a natural transformation from a covariant functor describing test sections to the covariant functor $\mathfrak{A}$. In this section we introduce the concept of generally covariant topological quantum fields, that are natural transformations from a covariant functor describing topological information to the functor $\mathfrak{A}$, and construct two examples which can be interpreted as magnetic and electric charge. We have added the attribute 'generally covariant' in 'generally covariant topological quantum field' in order to distinguish it from the usual notion of topological quantum field theory [Ati89]. For simplifying the discussion we restrict ourselves in this section to the full subcategory $\mathsf{PrBuGlobHyp}^{U(1)}$, where the structure group is fixed to $G = U(1) \simeq \mathbb{T}$. The covariant functor $\mathfrak{A}$ of Theorem 5.10 is also restricted, i.e. $\mathfrak{A} : \mathsf{PrBuGlobHyp}^{U(1)} \to {}^*\mathsf{Alg}$.

Definition 6.1. The category Vec consists of the following objects and morphisms:

- An object in Vec is a (possibly infinite dimensional) vector space V over $\mathbb{R}$.
- A morphism between two objects V_i, $i = 1, 2$, in Vec is a linear map $L : V_1 \to V_2$ (not necessarily injective).

Composing $\mathfrak{A} : \mathsf{PrBuGlobHyp}^{U(1)} \to {}^*\mathsf{Alg}$ with the forgetful functor from from ${}^*\mathsf{Alg}$ to Vec we can consider $\mathfrak{A}$ as a covariant functor from $\mathsf{PrBuGlobHyp}^{U(1)}$ to Vec (with a slight abuse of notation we denote this covariant functor again by $\mathfrak{A}$). The other covariant functors from $\mathsf{PrBuGlobHyp}^{U(1)}$ to Vec which enter our construction of generally covariant topological quantum fields are those of smooth singular homology with coefficients in the real vector space $\mathfrak{g}^* = i\,\mathbb{R}$ (since the smooth and continuous singular homology are isomorphic, the smooth singular homology only contains topological information). For being self-contained we review briefly the relevant concepts: Let M be a manifold of finite type. A smooth singular p-simplex, $p \in \mathbb{N}^0$, is a smooth map $\sigma : \Delta^p \to M$, where Δ^p is the standard p-simplex in $\mathbb{R}^p$. The real vector space generated by finite linear combinations of smooth singular p-simplices is denoted by $S_p(M, \mathfrak{g}^*)$ and its elements $\sum_{\text{finite}} a_j\,\sigma_j$, $a_j \in \mathfrak{g}^*$, are called smooth singular p-chains with coefficients in $\mathfrak{g}^* = i\,\mathbb{R}$. We will suppress the subscript finite in the following for a better readability. For all $p > 0$ there is a boundary operator $\partial_p : S_p(M,\mathfrak{g}^*) \to S_{p-1}(M,\mathfrak{g}^*)$ satisfying $\partial_p \circ \partial_{p+1} = 0$. The homology of the complex

$$\cdots \xrightarrow{\partial_{p+2}} S_{p+1}(M,\mathfrak{g}^*) \xrightarrow{\partial_{p+1}} S_p(M,\mathfrak{g}^*) \xrightarrow{\partial_p} S_{p-1}(M,\mathfrak{g}^*) \xrightarrow{\partial_{p-1}} \cdots \tag{6.1}$$

is denoted by $H_*(M,\mathfrak{g}^*)$ and called the **singular homology** with coefficients in $\mathfrak{g}^* = i\,\mathbb{R}$. Explicitly, the p-th singular homology group is the real vector space $H_p(M,\mathfrak{g}^*) = \mathrm{Ker}(\partial_p)/\mathrm{Im}(\partial_{p+1})$.

Let now $\Xi = \big((M,\mathfrak{o},g,\mathfrak{t}),(G,h),(P,r)\big)$ be an object in $\mathsf{PrBuGlobHyp}^{U(1)}$. The association of the p-th singular homology group of M is a covariant functor $\mathfrak{H}_p : \mathsf{PrBuGlobHyp}^{U(1)} \to \mathsf{Vec}$: To any object Ξ in $\mathsf{PrBuGlobHyp}^{U(1)}$ the functor associates $\mathfrak{H}_p(\Xi) = H_p(M,\mathfrak{g}^*)$. To any morphism $F = \big(f : P_1 \to P_2, \phi : G \to G\big) : \Xi_1 \to \Xi_2$ in $\mathsf{PrBuGlobHyp}^{U(1)}$ the functor associates

$$\mathfrak{H}_p(F) : \mathfrak{H}_p(\Xi_1) \to \mathfrak{H}_p(\Xi_2)~,~~\Big[\sum a_j\,\sigma_j\Big] \mapsto \Big[\sum \phi^{-1\,*}(a_j)\,(\underline{f}\circ\sigma_j)\Big]~. \tag{6.2}$$

The singular cohomology is defined by duality, $H^*(M,\mathfrak{g}) := \mathrm{Hom}_{\mathbb{R}}(H_*(M,\mathfrak{g}^*),\mathbb{R})$. Furthermore, by de Rham's theorem there exists a vector space isomorphism $\mathcal{J} : H^p_{\mathrm{dR}}(M,\mathfrak{g}) \to H^p(M,\mathfrak{g})~,~[\eta] \mapsto \mathcal{J}([\eta])$, where $\mathcal{J}([\eta])$ is the linear functional on $H_p(M,\mathfrak{g}^*)$ defined by, for all $\sum a_j\,\sigma_j$,

$$\mathcal{J}([\eta])\Big(\Big[\sum a_j\,\sigma_j\Big]\Big) = \sum a_j \int_{\Delta^p} \sigma_j^*(\eta)~, \tag{6.3}$$

where σ_j^* is the pull-back of $\sigma_j : \Delta^p \to M$ and the duality pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ is suppressed. By Poincaré duality there also exists a vector space isomorphism $\mathcal{K} : H_p(M,\mathfrak{g}^*) \to H^p_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)$ (by the subscript dR^* we denote the cohomology groups of the codifferential δ) specified by, for all $\big[\sum a_j\,\sigma_j\big] \in H_p(M,\mathfrak{g}^*)$ and $[\eta] \in H^p_{\mathrm{dR}}(M,\mathfrak{g})$,

$$\Big\langle \mathcal{K}\Big(\Big[\sum a_j\,\sigma_j\Big]\Big),[\eta]\Big\rangle = \mathcal{J}([\eta])\Big(\Big[\sum a_j\,\sigma_j\Big]\Big)~. \tag{6.4}$$

The pairing $\langle~,~\rangle : H^p_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*) \times H^p_{\mathrm{dR}}(M,\mathfrak{g}) \to \mathbb{R}$ on the left hand side is that induced by the pairing $\langle\zeta,\eta\rangle = \int_M \zeta\wedge *(\eta)$ of p-forms $\zeta \in \Omega^p_0(M,\mathfrak{g}^*)$ and $\eta \in \Omega^p(M,\mathfrak{g})$.

We now can construct our first example of a generally covariant topological quantum field, which by Remark 6.3 below should be interpreted as magnetic charge (Euler class).

Theorem 6.2. *Consider the two covariant functors $\mathfrak{H}_2, \mathfrak{A} : \mathsf{PrBuGlobHyp}^{U(1)} \to \mathsf{Vec}$. We associate to any object Ξ in $\mathsf{PrBuGlobHyp}^{U(1)}$ the morphism in Vec*

$$\Psi^{\mathrm{mag}}_{\Xi} : \mathfrak{H}_2(\Xi) \to \mathfrak{A}(\Xi)~,~~\Big[\sum a_j\,\sigma_j\Big] \mapsto \Big[\underline{\mathcal{F}}^*\Big(\mathcal{K}\Big(\Big[\sum a_j\,\sigma_j\Big]\Big)\Big)\Big]~, \tag{6.5}$$

where $\underline{\mathcal{F}}^* : \Omega^2_0(M,\mathfrak{g}^*) \to \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv}$ is the formal adjoint of the curvature affine differential operator (cf. Corollary 2.24). The collection $\Psi^{\mathrm{mag}} = \{\Psi^{\mathrm{mag}}_{\Xi}\}$ is a natural transformation from $\mathfrak{H}_2$ to $\mathfrak{A}$.

Proof. The map (6.5) is well-defined due the dual of the (Abelian) Bianchi identity $\mathrm{d}\circ\underline{\mathcal{F}} = 0$. Furthermore, since any representative of the class $\mathcal{K}\big(\big[\sum a_j\,\sigma_j\big]\big)$ is coclosed, the linear part of $\underline{\mathcal{F}}^*\big(\mathcal{K}\big(\big[\sum a_j\,\sigma_j\big]\big)\big)$ vanishes. Hence, $\underline{\mathcal{F}}^*\big(\mathcal{K}\big(\big[\sum a_j\,\sigma_j\big]\big)\big) \in \mathcal{E}^{\mathrm{inv}}$ is a representative of an element in $\mathcal{N}$ and the image of (6.5) is contained in $\mathcal{E} \subseteq \mathfrak{A}(\Xi)$.

Let $F : \Xi_1 \to \Xi_2$ be a morphism in $\mathsf{PrBuGlobHyp}^{U(1)}$. As a consequence of the dual of (5.3) and $\underline{f}_* \circ \phi^{-1\,*} \circ \mathcal{K}_1 = \mathcal{K}_2 \circ \mathfrak{H}_2(F)$, which descends from (6.4), we obtain that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{H}_2(\Xi_1) & \xrightarrow{\;\Psi^{\mathrm{mag}}_{\Xi_1}\;} & \mathfrak{A}(\Xi_1) \\ {\scriptstyle \mathfrak{H}_2(F)}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{A}(F)} \\ \mathfrak{H}_2(\Xi_2) & \xrightarrow{\;\Psi^{\mathrm{mag}}_{\Xi_2}\;} & \mathfrak{A}(\Xi_2) \end{array} \tag{6.6}$$

This proves that $\Psi^{\mathrm{mag}} = \{\Psi^{\mathrm{mag}}_{\Xi}\}$ is a natural transformation. □

Remark 6.3. The interpretation of the natural transformation Ψ^{mag} is as follows: The classical affine functional (3.5) corresponding to $\underline{\mathcal{F}}^*\big(\mathcal{K}\big(\big[\sum a_j\,\sigma_j\big]\big)\big)$ yields when evaluating on any $\lambda\in\Gamma^\infty(M,\mathcal{C}(P))$

$$\mathcal{O}_{\underline{\mathcal{F}}^*(\mathcal{K}([\sum a_j\,\sigma_j]))}(\lambda)=\Big\langle\mathcal{K}\Big(\Big[\sum a_j\,\sigma_j\Big]\Big),\underline{\mathcal{F}}(\lambda)\Big\rangle=\sum a_j\int_{\Delta^2}\sigma_j^*\big(\underline{\mathcal{F}}(\lambda)\big)\ . \tag{6.7}$$

Via this identification the elements in the image of the map $\Psi^{\mathrm{mag}}_{\Xi}$ determine the cohomology class $[\underline{\mathcal{F}}(\lambda)]\in H^2_{\mathrm{dR}}(M,\mathfrak{g})$ and hence the Euler class of the principal $U(1)$-bundle. In physics $[\underline{\mathcal{F}}(\lambda)]$ is called the magnetic charge. This is a purely topological information, which explains our notation generally covariant topological quantum field. After $\mathfrak{CCR}$-quantization, we should interpret the image of the map (6.5) as magnetic charge observables, which can be assigned coherently to all objects in $\mathsf{PrBuGlobHyp}^{U(1)}$ since Ψ^{mag} is a natural transformation. We note that the image of the map (6.5) lies in the center of the algebra $\mathfrak{A}(\Xi)$, hence magnetic charge observables are not subject to Heisenberg's uncertainty relation and can be measured without quantum fluctuations.

Motivated by [SDH12] we will now construct a generally covariant topological quantum field, which by Remark 6.5 below should be interpreted as electric charge. For this we require a covariant functor which associates to any object Ξ in $\mathsf{PrBuGlobHyp}^{U(1)}$ the singular homology group $H_{\dim(M)-2}(M,\mathfrak{g}^*)\simeq H^{\dim(M)-2}_{0\,\mathrm{dR}^*}(M,\mathfrak{g}^*)$. This functor exists since the set of morphisms $\{F:\Xi_1\to\Xi_2\}$ is only nonempty between objects Ξ_1 and Ξ_2 where M_1 and M_2 have the same dimension (cf. Definition 2.4). We shall denote this covariant functor by $\mathfrak{H}_{-2}:\mathsf{PrBuGlobHyp}^{U(1)}\to\mathsf{Vec}$.

Theorem 6.4. *Consider the two covariant functors $\mathfrak{H}_{-2},\mathfrak{A}:\mathsf{PrBuGlobHyp}^{U(1)}\to\mathsf{Vec}$. We associate to any object Ξ in $\mathsf{PrBuGlobHyp}^{U(1)}$ the morphism in Vec*

$$\Psi^{\mathrm{el}}_{\Xi}:\mathfrak{H}_{-2}(\Xi)\to\mathfrak{A}(\Xi)\ ,\quad\Big[\sum a_j\,\sigma_j\Big]\mapsto\Big[\underline{\mathcal{F}}^*\Big(*\Big(\mathcal{K}\Big(\Big[\sum a_j\,\sigma_j\Big]\Big)\Big)\Big)\Big]\ . \tag{6.8}$$

The collection $\Psi^{\mathrm{el}}=\{\Psi^{\mathrm{el}}_{\Xi}\}$ is a natural transformation from $\mathfrak{H}_{-2}$ to $\mathfrak{A}$.

Proof. The map (6.8) is well-defined, since for all $\chi\in\Omega^{\dim(M)-1}_0(M,\mathfrak{g}^*)$, $\underline{\mathcal{F}}^*\big(*(\delta\chi)\big)=\mathsf{MW}^*(*(\chi))$ yields the trivial class in $\mathcal{E}\subseteq\mathfrak{A}(\Xi)$. For any $\eta\in\Omega^{\dim(M)-2}_{0,\delta}(M,\mathfrak{g}^*)$ the linear part of $\underline{\mathcal{F}}^*\big(*(\eta)\big)$ is $\underline{\mathcal{F}}^*\big(*(\eta)\big)_V=\delta*(\eta)$, with $*(\eta)\in\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)$. Hence, $\underline{\mathcal{F}}^*\big(*\big(\mathcal{K}\big(\big[\sum a_j\,\sigma_j\big]\big)\big)\big)\in\mathcal{E}^{\mathrm{inv}}$ is a representative of an element in $\mathcal{N}$ and the image of (6.8) is contained in $\mathcal{E}\subseteq\mathfrak{A}(\Xi)$.

Let $F:\Xi_1\to\Xi_2$ be a morphism in $\mathsf{PrBuGlobHyp}^{U(1)}$. Using that $\underline{f}_*\circ\phi^{-1*}\circ*_1=*_2\circ\underline{f}_*\circ\phi^{-1*}$ and the same arguments as in the proof of Theorem 6.2 we obtain that the following diagram commutes:

$$\begin{array}{ccc}\mathfrak{H}_{-2}(\Xi_1) & \xrightarrow{\ \Psi^{\mathrm{el}}_{\Xi_1}\ } & \mathfrak{A}(\Xi_1)\\ {\scriptstyle\mathfrak{H}_{-2}(F)}\big\downarrow & & \big\downarrow{\scriptstyle\mathfrak{A}(F)}\\ \mathfrak{H}_{-2}(\Xi_2) & \xrightarrow{\ \Psi^{\mathrm{el}}_{\Xi_2}\ } & \mathfrak{A}(\Xi_2)\end{array} \tag{6.9}$$

This proves that $\Psi^{\mathrm{el}}=\{\Psi^{\mathrm{el}}_{\Xi}\}$ is a natural transformation. □

Remark 6.5. Following Remark 6.3 we can interpret Ψ^{el} as a coherent assignment of electric charge observables: The classical affine functional (3.5) corresponding to $\underline{\mathcal{F}}^*\big(*\big(\mathcal{K}\big(\big[\sum a_j\,\sigma_j\big]\big)\big)\big)$ yields when evaluating on any solution $\lambda\in\Gamma^\infty(M,\mathcal{C}(P))$ of the equation of motion $\mathsf{MW}(\lambda)=0$,

$$\mathcal{O}_{\underline{\mathcal{F}}^*(*(\mathcal{K}([\sum a_j\,\sigma_j])))}(\lambda)=\Big\langle\mathcal{K}\Big(\Big[\sum a_j\,\sigma_j\Big]\Big),*\big(\underline{\mathcal{F}}(\lambda)\big)\Big\rangle=\sum a_j\int_{\Delta^{\dim(M)-2}}\sigma_j^*\big(*\big(\underline{\mathcal{F}}(\lambda)\big)\big)\ . \tag{6.10}$$

Via this identification the elements in the image of the map Ψ^{el}_{Ξ} determine the cohomology class $\big[*(\underline{\mathcal{F}}(\lambda))\big]\in H^{\dim(M)-2}_{\mathrm{dR}}(M,\mathfrak{g})$ that, via Gauss' law, is the electric charge. Also in this case the image of the map (6.8) lies in the center of the algebra $\mathfrak{A}(\Xi)$, meaning that electric charge observables in the quantum theory are not subject to Heisenberg's uncertainty relation and can be measured without quantum fluctuations.

7 The charge-zero functor and the locality property

In the previous section we have identified electric and magnetic charge observables in the algebra $\mathfrak{A}(\Xi) = \mathfrak{CCR}\big(\mathfrak{PhSp}(\Xi)\big)$ for any object Ξ in $\mathsf{PrBuGlobHyp}^{U(1)}$. While magnetic charge observables are certainly very welcome in our framework since they can measure the topology of the principal bundle, electric charges play a different role. By construction, the covariant functor $\mathfrak{A} : \mathsf{PrBuGlobHyp}^{U(1)} \to {}^*\mathsf{Alg}$ models quantized principal $U(1)$-connections without the presence of any charged fields. As a consequence, all electric charge measurements should yield zero.[2] We are going to implement this physical feature into our framework by performing a different quotient in the presymplectic vector spaces $\big(\mathcal{E},\tau\big)$ of Proposition 3.4. It is then rather straightforward to show that there is a covariant functor $\mathfrak{PhSp}^0 : \mathsf{PrBuGlobHyp}^{U(1)} \to \mathsf{PreSymp}$, the charge-zero phase space functor, which associates these presymplectic vector spaces to objects in $\mathsf{PrBuGlobHyp}^{U(1)}$. Interestingly, the functor $\mathfrak{PhSp}^0$ satisfies, in addition to the classical causality property and the classical time-slice axiom, the locality property stating that for any morphism F in $\mathsf{PrBuGlobHyp}^{U(1)}$ the morphism $\mathfrak{PhSp}^0(F)$ in $\mathsf{PreSymp}$ is injective. Due to Remark 5.6 this is not the case for the functor $\mathfrak{PhSp}$ constructed in Section 5. Composing the charge-zero phase space functor with the $\mathfrak{CCR}$-functor we obtain a covariant functor $\mathfrak{A}^0$ that satisfies all axioms of locally covariant quantum field theory, i.e. the quantum causality property, the quantum time-slice axiom and injectivity of $\mathfrak{A}^0(F)$ for any morphism F in $\mathsf{PrBuGlobHyp}^{U(1)}$.

Let $\Xi = \big((M,\mathfrak{o},g,\mathfrak{t}),(G,h),(P,r)\big)$ be an object in $\mathsf{PrBuGlobHyp}^{U(1)}$ and $\mathcal{E}^{\mathrm{inv}}$ the gauge invariant vector space characterized in Theorem 4.6. Notice that the vector subspace $\underline{\mathcal{F}}^*\big[\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)\big] \subseteq \mathcal{E}^{\mathrm{inv}}$ contains $\mathsf{MW}^*\big[\Omega^1_0(M,\mathfrak{g}^*)\big]$ as a vector subspace as well as the electric charge observables of Theorem 6.4. Hence, by considering the quotient $\mathcal{E}^0 := \mathcal{E}^{\mathrm{inv}}/\underline{\mathcal{F}}^*\big[\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)\big]$ we implement the equation of motion and identify all electric charges with zero.

Lemma 7.1. *Let Ξ be an object in* $\mathsf{PrBuGlobHyp}$.

a) Then $\mathcal{E}^0 := \mathcal{E}^{\mathrm{inv}}/\underline{\mathcal{F}}^*\big[\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)\big]$ can be equipped with the presymplectic structure

$$\tau^0 : \mathcal{E}^0 \times \mathcal{E}^0 \to \mathbb{R}~,~~\big([\varphi],[\psi]\big) \mapsto \tau^0\big([\varphi],[\psi]\big) = \big\langle \varphi_V, G_{(1)}(\psi_V)\big\rangle_h ~. \tag{7.1}$$

In other words, $\big(\mathcal{E}^0,\tau^0\big)$ is a presymplectic vector space.

b) The radical $\mathcal{N}^0$ of $\big(\mathcal{E}^0,\tau^0\big)$ is

$$\mathcal{N}^0 = \Big[\big\{\varphi \in \mathcal{E}^{\mathrm{inv}} : \varphi_V = 0\big\}\Big]~. \tag{7.2}$$

Proof. This is a direct consequence of Theorem 4.8. □

Similar to Theorem 5.5 we obtain that the association of these presymplectic vector spaces is functorial.

Theorem 7.2. *There is a covariant functor $\mathfrak{PhSp}^0 : \mathsf{PrBuGlobHyp}^{U(1)} \to \mathsf{PreSymp}$. It associates to any object Ξ in $\mathsf{PrBuGlobHyp}^{U(1)}$ the object $\mathfrak{PhSp}^0(\Xi) = \big(\mathcal{E}^0,\tau^0\big)$ in $\mathsf{PreSymp}$ which has been constructed in Lemma 7.1. Given a morphism $F : \Xi_1 \to \Xi_2$ between two objects Ξ_i, $i = 1,2$, in $\mathsf{PrBuGlobHyp}^{U(1)}$ the functor associates a morphism in $\mathsf{PreSymp}$ as follows*

$$\mathfrak{PhSp}^0(F) : \mathfrak{PhSp}^0(\Xi_1) \to \mathfrak{PhSp}^0(\Xi_2)~,~~[\varphi] \mapsto [F_*(\varphi)]~, \tag{7.3}$$

where the linear map F_* is given in Definition 5.4.

Proof. The proof follows by similar arguments as in the proof of Theorem 5.5. □

By slightly modifying the proofs of Theorem 5.7 and Theorem 5.8 it is easy to show that the covariant functor $\mathfrak{PhSp}^0 : \mathsf{PrBuGlobHyp}^{U(1)} \to \mathsf{PreSymp}$ satisfies the classical causality property and the classical time-slice axiom. In addition, we have have the following

[2] We are very grateful to Jochen Zahn and Thomas-Paul Hack for comments which have led to this insight.

Theorem 7.3. *The covariant functor* $\mathfrak{PhSp}^0 : \mathsf{PrBuGlobHyp}^{U(1)} \to \mathsf{PreSymp}$ *satisfies the locality property:*

Let $F : \Xi_1 \to \Xi_2$ *be any morphism in* $\mathsf{PrBuGlobHyp}^{U(1)}$*, then* $\mathfrak{PhSp}^0(F)$ *is injective.*

Proof. Notice that any element $[\varphi] \in \mathfrak{PhSp}^0(\Xi_1)$ that satisfies $[F_*(\varphi)] = 0$ is necessarily contained in the radical $\mathcal{N}_1^0 \subseteq \mathfrak{PhSp}^0(\Xi_1)$. Let us now assume that $[\varphi] \in \mathcal{N}_1^0$ is such that $[F_*(\varphi)] = 0$. By Lemma 7.1 b) there exists a representative $\varphi \in \Gamma_0^\infty(M_1, \mathcal{C}(P_1)^\dagger)$ of $[\varphi]$ that is of the form $\varphi = a\, \mathbb{1}_1$ with $a \in C_0^\infty(M_1)$. The push-forward along F of this representative is then $F_*(a\, \mathbb{1}_1) = \underline{f}_*(a)\, \mathbb{1}_2$, where $\underline{f}_*(a) \in C_0^\infty(M_2)$ is the push-forward along $\underline{f} : M_1 \to M_2$. Since by hypothesis $[F_*(\varphi)] = 0$, the representative $\underline{f}_*(a)\, \mathbb{1}_2$ is equivalent to an element in Triv_2, i.e. for some $\eta \in \Omega^2_{0,\mathrm{d}}(M_2, \mathfrak{g}^*)$ and $b \in C_0^\infty(M_2)$ satisfying $\int_{M_2} \mathrm{vol}_2\, b = 0$, we have $\underline{f}_*(a)\, \mathbb{1}_2 = b\, \mathbb{1}_2 + \underline{\mathcal{F}_2}^*(\eta)$. Comparing the linear parts of both sides of the equality we obtain $\delta_2 \eta = 0$, i.e. $\eta \in \Omega^2_{0,\mathrm{d}}(M_2, \mathfrak{g}^*)$ is both closed and coclosed. As a consequence, $\square_{2\,(2)}(\eta) = 0$, which due to normal hyperbolicity implies that $\eta = 0$. We find $\underline{f}_*(a) = b$ and in particular $0 = \int_{M_2} \mathrm{vol}_2\, \underline{f}_*(a) = \int_{M_1} \mathrm{vol}_1\, a$. Thus, $[\varphi] = [a\, \mathbb{1}_1] = 0$ since $a\, \mathbb{1}_1 \in \mathrm{Triv}_1$. □

Let us denote by $\mathsf{PreSymp}^{\mathrm{inj}}$ the subcategory of $\mathsf{PreSymp}$ where all morphisms are injective. We have shown above the existence of the covariant functor $\mathfrak{PhSp}^0 : \mathsf{PrBuGlobHyp}^{U(1)} \to \mathsf{PreSymp}^{\mathrm{inj}}$. Since the $\mathfrak{CCR}$-functor restricts to a covariant functor $\mathfrak{CCR} : \mathsf{PreSymp}^{\mathrm{inj}} \to {}^*\mathsf{Alg}^{\mathrm{inj}}$, where we have used the obvious notation for the subcategory of ${}^*\mathsf{Alg}$ with injective morphisms, we obtain by composition a covariant functor $\mathfrak{A}^0 : \mathsf{PrBuGlobHyp}^{U(1)} \to {}^*\mathsf{Alg}^{\mathrm{inj}}$. The classical causality property and the classical time-slice axiom extend via the $\mathfrak{CCR}$-functor to the quantum case, see e.g. [BDS12, Theorem 6.3]. The main result of this section can be summarized as follows:

Theorem 7.4. *The covariant functor* $\mathfrak{A}^0 := \mathfrak{CCR} \circ \mathfrak{PhSp}^0 : \mathsf{PrBuGlobHyp}^{U(1)} \to {}^*\mathsf{Alg}^{\mathrm{inj}}$ *is a locally covariant quantum field theory, i.e.* $\mathfrak{A}^0$ *satisfies the quantum causality property, the quantum time-slice axiom and the locality property.*

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