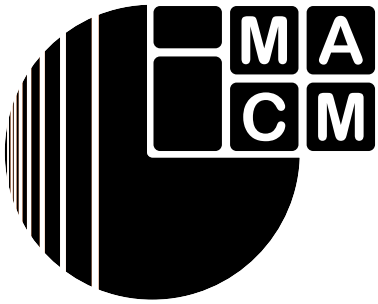

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Bilateral Counterparty Risk Valuation of CDS contracts with Simultaneous Defaults

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Abstract

We analyze the general risk-neutral valuation for counterparty risk embedded in a Credit Default Swap (CDS) contract by adapting the recent findings of Brigo and Capponi to allow for simultaneous defaults among the two parties and the underlying reference credit, while the counterparty risk is considered bilaterally. For the default intensities we employ a Markov copula model allowing for the possibility of a simultaneous default. The dependence between defaults of three names in a CDS contract and the wrong-way risk will thus be represented by the possibility of simultaneous defaults.

Using our numerical results we investigate the effect of considering simultaneous defaults on the counterparty risk valuation of a CDS contract. Finally, we study a CDS contract between Royal Dutch Shell and British Airways based on Lehman Brothers applying this methodology, illustrating the bilateral adjustments inclusive of the possibility of simultaneous defaults in concrete crisis situations.

Keywords *Credit Default Swaps, Counterparty Risk, Risk-neutral Credit Valuation Adjustment, Default Intensity, Default Correlation, Simultaneous Default, Markov Copula Model.*

1 Introduction

In the last few years the CDS market has grown rapidly, the corresponding notional exposures had a peak around 60 trillion dollars in 2007 and despite the financial crisis. Maybe this growth should be anticipated in advance, since the CDS index market(iTraxx) is deep;

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however, the single-name CDS market is illiquid and dwindling, so that the easiest way to adjust the exposure to credit risk has been to make new contracts rather than cancelling CDS agreements already in running. Besides, due to the fact that many financial contracts are traded over the counter (OTC) so that the credit quality of the counterparty may be important, the counterparty having worse credit rating will default with more possibility. This has become even more relevant in the recent years as financial institutions have witnessed increasing downfalls, the case of Citigroup and Lehman Brothers is an evident example. How is the default possibility treated? A risk premium will be required as a reward for assuming the default risk when investing in default risky assets, so that the reliable evaluation of counterparty risk becomes necessary. In last years, many authors investigated the counterparty risk in the CDS contracts, e.g. [4], [6], [8] and [10].

A general arbitrage-free valuation framework for bilateral counterparty default risk was introduced by Brigo and Capponi [7], see also [5,6]. They provided a general formula for the *bilateral risk credit valuation adjustment (BR-CVA)* for portfolios exchanged between risky counterparties. This bilateral adjustment has the *great symmetry property* which means that the adjustment seen from the point of view of each other is exactly the opposite, namely the parties will agree on the value of the counterparty risk adjustment. They showed that the value of the BR-CVA is the same as the sum of the value of a long position in a zero-strike put option and the value of a short position in a zero-strike call option, both on the residual net value of the contract at the relevant default times. Brigo and Capponi focused their attention on the application of this symmetric valuation to CDS contracts, namely not only the counterparty as protection buyer but also the investor as protection seller is subject to default risk. Besides, the default events among parties are correlated by a trivariate copula function, but the simultaneous defaults among the counterparties are not considered.

Indeed, a simultaneous default can happen in the real financial market. Mathematically we define simultaneous defaults among the counterparties as that the default times of them are exactly the same. But in the real world we already can say that they default simultaneously if they filed for bankruptcy protection on one day or within a few days, e.g. the collapses of Lehman Brothers and Merrill Lynch were just within two days (September 13-14, 2008). In reality, it is possible that the defaults among the counterparties do not occur simultaneously, but if one's default has triggered a jump in the default probability of the other one (e.g. if they are very highly correlated), which might end up defaulting only within a short time period. For example, the protection seller's default could trigger a jump in the default probability of the reference credit so that the protection buyer has to suffer a loss. In this work we refer to this case also as simultaneous default.

Although the protection seller has to compensate the protection buyer when he has a default, but this claim is not big as long as the default risk of the reference entity does not jump substantially within the seller's default time. But in the extreme case, where the

default of the protection seller defaults simultaneously with the default of the reference entity, the payment would be the same as the full insurance payment. In this work, we adapt the valuation framework of Brigo and Capponi using the valuation methods in [3] which can take simultaneous defaults into account, so that simultaneous defaults between contract partners can be considered. We design, a new BR-CVA formula which includes the case of simultaneous defaults, keeping the two properties of the original formula explained above. One of them is the great symmetry property, the other one is to say the value of the BR-CVA can be represented as the sum of the value of a long position in a put option and the value of a short position in a call option.

In the CDS contracts, the CDS price contains information about the joint default risk of the contract partners, see [12]. Hence, CDS Spreads supposed contain a certain amount of the implicit information about simultaneous defaults. For this reason, we employ the Markov Copula Model with simultaneous defaults in [2] for default intensities, including of the simultaneous intensities. This model can be simulated and calibrated only based on CDS Spreads. Applying this model the dependence between defaults of three names in a CDS contract will be represented by the possibilities of simultaneous defaults, if we choose the correlations between intensity processes as zero. With numerical results we want to study the effect of considering simultaneous defaults on the valuation of counterparty risk.

The outline of this paper is as follows. In the next section we give the BR-CVA formula for a CDS which takes account of simultaneous defaults. Section 3 is devoted to the underlying model of stochastic default intensity with the specification and the simulation. Section 4 gives the numerical computation of that BR-CVA formula. In Section 5 numerical results are presented and discussed. In Section 6 we apply the methodology to compute the mark-to-market value of a concrete CDS contract between British Airways, Lehman Brothers and Royal Dutch Shell. Finally, Section 7 concludes this work. The proofs are provided in the appendix.

2 Pricing Counterparty Risk of a CDS contract

2.1 General Set-Up

We consider a standard running CDS contract on which the protection payment and recoveries are paid exactly at the times of default. Respectively, we label by C, I and R the counterparty, the investor and the underlying reference entity. Each of the three names may default before the maturity of the CDS contract, and we denote by τ_C, τ_I and τ_R their respective default times. These times are modeled as non-negative random variables given in a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$, where $\mathbb{Q}$ is the risk neutral measure. We define the enlarged filtration $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t \ \forall t \in \mathbb{R}^+$ to model the whole information in the market, where the right-continuous and complete sub-filtration $\mathcal{F}_t$ represents all the observable

market quantities and $\mathcal{H}_t = \sigma\left(\{\tau_R \leq u\} \vee \{\tau_C \leq u\} \vee \{\tau_I \leq u\} : u \leq t\right)$ denotes the right-continuous filtration generated by the default events of three names under contract. In particular, the random times $\tau_j, j = C, I, R$ are $\mathcal{G}_t$ stopping time for $t \in \mathbb{R}^+$. Next, we define the first default time as the minimum of τ_C, τ_I and $\tau_R : \tau = \tau_C \wedge \tau_I \wedge \tau_R$. In addition, we define the first default time of the two counterparties : $\widehat{\tau} = \tau_C \wedge \tau_I$.

2.2 Pricing Formula

In this paper, all cash flows and the prices are considered from the perspective of the investor as protection seller.

Definition 2.1. *We define the discounted payoff of a CDS with a default-free counterparty at time t as:*

$$\begin{aligned}\Pi(t,T) :=& D(t,\tau_R)(\tau_R - T_{\gamma(\tau_R)-1})\mathscr{P}\mathbb{I}_{\{T_a<\tau_R<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i\mathscr{P}\mathbb{I}_{\{\tau_R\geq T_i\}} \\ &- D(t,\tau_R)\mathscr{L}_R\mathbb{I}_{\{T_a<\tau_R\leq T_b\}},\end{aligned} \tag{1}$$

where $t \in [T_{\gamma(t)-1}, T_{\gamma(t)})$, i.e. $T_{\gamma(t)}$ is the first date among the $T_i's$ that follows t, and where α_i is the year fraction between T_{i-1} and T_i. $D(t,T)$ is the stochastic discount factor at time t for maturity T. We assume the Loss Given Default $\mathscr{L}_j$ to be deterministic and $\mathscr{L}_j = 1-\mathscr{R}_j$, where the recovery Rate $\mathscr{R}_j$, for $j = C, I, R$ is also assumed to be deterministic and the notional is set to one. The periodic premium rate is denoted by $\mathscr{P}$.

Definition 2.2. *We denote by S_t the price of a counterparty risk-free CDS contract maturing at time T and*

$$S_t = \mathbb{E}\left\{\Pi(t,T)|\mathcal{G}_t\right\},\ t \in [0,T], \tag{2}$$

where $\mathbb{E}$ denotes the expected value under the risk neutral measure.

Now, we are in position to define the discounted payoff for a CDS contract with any credit event, i.e. that is the CDS contract that also accounts for the counterparty risk associated with the two counterparties of the contract. For this goal we adapt the conventions of so called *close-out cash flows* in [3] as follows.

We define the following events ordering the default times of three names in the CDS contract between valuation t and maturity T:

$$\begin{aligned}&\mathfrak{A} = \{t < \tau = \tau_R \leq T\}, && \mathfrak{D} = \{t < \tau = \tau_C = \tau_I \leq T\},\\ &\mathfrak{B} = \{t < \tau = \tau_C \leq T\}, && \mathfrak{E} = \{t < \widehat{\tau} = \tau_R \leq T\},\\ &\mathfrak{C} = \{t < \tau = \tau_I \leq T\}.\end{aligned} \tag{3}$$

Definition 2.3. *The discounted payoff of a counterparty-risky CDS contract at time t can be written as:*

$$\begin{aligned}\Pi^A(t,T) :=& \mathbb{I}_{\mathfrak{A}} D(t,\tau)(-\mathscr{L}_R) \\ &+ \mathbb{I}_{\mathfrak{B}}\left[D(t,\tau)\left(\mathscr{R}_C(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^-\right)\right] \\ &+ \mathbb{I}_{\mathfrak{C}}\left[D(t,\tau)\left((S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - \mathscr{R}_I(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^-\right)\right] \\ &+ \mathbb{I}_{\mathfrak{D}}\left[D(t,\tau)\left(-(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)\right)\right] \\ &+ \mathbb{I}_{\mathfrak{E}}\left[D(t,\tau)\mathscr{L}_R\right] \\ &+ D(t,\tau)(\tau - T_{\gamma(\tau)-1})\mathscr{P}\mathbb{I}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i\mathscr{P}\mathbb{I}_{\{\tau\geq T_i\}},\end{aligned} \tag{4}$$

where the term $\mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R$ represents the exposure in case when the reference entity simultaneously default with any other default times.

- $\mathfrak{B}$: When the counterparty defaults, at default time τ, the value of the CDS until maturity $S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R$ is computed. If that is negative, the investor closes out the position by paying the defaulting counterparty this price. If the value is positive, the investor closes out the position and only receives a fraction $\mathscr{R}_C$ of this value from his counterparty. Therefore, in this case, we can define the close-out payment as

$$\mathscr{R}_C(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^-.$$

- $\mathfrak{C}$: In case of an investor default, if the value of CDS until maturity $S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R$ is positive, the counterparty closes out the position by paying this price in full. If this value is negative, the counterparty only receives a fraction $\mathscr{R}_I$ of this value to close out the position. Hence, the close-out payment is defined as

$$(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - \mathscr{R}_I(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^-.$$

- $\mathfrak{D}$: If the investor and the counterparty default simultaneously, then compute the value of CDS like in case $\mathfrak{B}$ und $\mathfrak{C}$, that is $S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R$, and if it is negative, the counterparty receives a fraction $\mathscr{R}_I$ of this value; however, if it is positive, the investor receives a fraction $\mathscr{R}_C$ of this value. Together, we set the close-out payment for this case as

$$-(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R).$$

- $\mathfrak{E}$: If the investor or the counterparty default simultaneously with the reference entity, investor receives a fraction $\mathscr{R}_C$ of the remaining recovery amount, $(-\mathscr{L}_R)^+$, when the counterparty defaults. Similarly, if the investor defaults, the counterparty receives a

portion $\mathscr{R}_I$ of the remaining recovery amount, $(-\mathscr{L}_R)^-$. The close-out payment in joint default including the reference entity has the form,

$$\mathscr{L}_R.$$

Definition 2.4. *We denote by S_t^A the price of a counterparty-risky CDS contract maturing at time T, i.e.*

$$S_t^A = \mathbb{E}\{\Pi^A(t,T)|\mathcal{G}_t\},\ t \in [t,T]. \tag{5}$$

Definition 2.5. *The bilateral credit valuation adjustment (CVA) on a CDS contract maturing at time T is defined as*

$$BVA_t = S_t - S_t^A, \tag{6}$$

for every $t \in [0,T]$.

Proposition 2.1. *At valuation time t, the bilateral CVA on a CDS contract maturing at time T can be written as*

$$\begin{aligned} BVA_t =& \mathbb{E}\left\{\mathbb{I}_{\mathfrak{B}} \cdot \mathscr{L}_C \cdot D(t,\tau) \cdot \left(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R\right)^+ |\mathcal{G}_t\right\} \\ & - \mathbb{E}\left\{\mathbb{I}_{\mathfrak{C}} \cdot \mathscr{L}_I \cdot D(t,\tau) \cdot \left(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R\right)^- |\mathcal{G}_t\right\}, \end{aligned} \tag{7}$$

for every $t \in [0,T]$.

The proof can be found in the Appendix.

Remark 2.1. *The above discussion shows that the BVA equals the sum of the value of a long position in a zero-strike call option on the residual price of CDS and the value of a short position in a zero-strike put option on the residual price of CDS. The option only gives contribution, if the corresponding party default earlier.*

Remark 2.2. *Similar to the counterparty pricing formula in [7] the formula (7) has the great advantage of being symmetric. This property means that the BVA from the view of the counterparty is exactly the opposite of the investor $(-BVA_t)$, this is to say that the parties agree on the value of the BVA. Besides, from (7) we can conclude that the value of this BVA can be negative and positive, the sign depends on which party is more risky to default.*

The following remark is directed towards the numerical evaluation.

Remark 2.3. *From (1) and (2) we can compute* S_τ *as*

$$S_\tau = \mathbb{I}_{\{\tau_R > \tau\}} \left\{ \mathscr{P}\left[-\int_{max\{T_a,\tau\}}^{T_b} D(\tau,t)(t - T_{\gamma(t)-1}) d\mathbb{Q}(\tau_R > t|\mathcal{G}_\tau) + \sum_{i=max\{a,j\}+1}^{b} \alpha_i D(\tau,T_i)\mathbb{Q}(\tau_R > T_i|\mathcal{G}_\tau) \right] + \mathscr{L}_R\left[\int_{max\{T_a,\tau\}}^{T_b} D(\tau,t) d\mathbb{Q}(\tau_R > t|\mathcal{G}_\tau) \right] \right\}. \tag{8}$$

For the corresponding computation we refer to [11].

3 The Multivariate Markov Default Model

In this section we propose an underlying stochastic model following [2] and [1]. To this end, we define a Markov Copula model of multivariate default times with factor processes $y = (y_C, y_I, y_R)$ and the corresponding default indicator processes $H = (H_C, H_I, H_R)$ for a CDS contract which will have the following key features

(i) The pair (y, H) is Markov in its natural filtration,

(ii) Each pair (y_j, H_j) is a Markov process,

(iii) At every instant, either each name on CDS contracts defaults individually or simultaneously with other names.

Remark 3.1. *The Property* (i) *allows us to address in a dynamic and theoretically consistent way the issues of pricing and hedging credit derivates. Property* (ii) *grants a quick valuation of single-name CDS contracts and independent calibration of each pair* (y_j, H_j)*, whereas* (iii) *will allow us to account for a dependence between defaults of each name.*

Towards these properties, we define henceforth the default intensities

$$\lambda_j(t) = y_j(t) + a_j, \quad t \geq 0, \quad j = C, I, R, \tag{9}$$

where a_j is a constant, and where each y_j is a Cox-Ingersoll-Ross (CIR) process given by

$$dy_j(t) = \kappa_j(\mu_j - y_j(t))dt + \sigma_j\sqrt{y_j(t)}dW_j(t), \quad j = C, I, R. \tag{10}$$

The parameters of each collection $(\kappa_j, \mu_j, \nu_j, y_j(0))$ are positive deterministic constants and we assume that each W_j is a standard Brownian motion process under the risk neutral measure.

Remark 3.2. *We assume that the processes W_j are independent with each other, under this assumption the specification as defined in (9) has Markov consistency. This is to say the intensities of surviving names would not be affected by past defaults and the model dependence between defaults is only represented by the possibility of common jumps.*

3.1 The Model Specification

We define a certain number of groups $M_l \subseteq \{C, I, R\} := M_{CDS}$ of contract parties who could like to default simultaneously, for $l \in \{\{C, I\}, \{C, R\}, \{I, R\}, \{C, I, R\}\} := \mathcal{L}$ and we simply set $M_l = l$. However, λ_{M_l} can not only be interpreted as intensity of all parties in l defaulting simultaneously. For example, the reference credit R will default with $M_l = \{C, R\}$ as long as he is still alive, if the investor I is already defaulted. Then for the default intensity we have the following:

- the counterparty C defaults with intensity $\lambda_{\{C\}} + \lambda_{\{C,I\}} + \lambda_{\{C,R\}} + \lambda_{\{C,I,R\}}$ as long as he is still alive,
- the investor I defaults with intensity $\lambda_{\{I\}} + \lambda_{\{C,I\}} + \lambda_{\{I,R\}} + \lambda_{\{C,I,R\}}$ as long as he is still alive,
- the reference credit R defaults with intensity $\lambda_{\{R\}} + \lambda_{\{C,R\}} + \lambda_{\{I,R\}} + \lambda_{\{C,I,R\}}$ as long as he is still alive,
- the counterparty C and the reference credit R default together with intensity $\lambda_{\{C,R\}} + \lambda_{\{C,I,R\}}$ as long as they are still alive,
- the counterparty C and the investor I default together with intensity $\lambda_{\{C,I\}} + \lambda_{\{C,I,R\}}$ as long as they are still alive,
- the investor I and the reference credit R default together with intensity $\lambda_{\{I,R\}} + \lambda_{\{C,I,R\}}$ as long as they are still alive,
- the counterparty C , the investor I and the reference credit R default together with intensity $\lambda_{\{C,I,R\}}$ as long as they are still alive.

Regarding with this specification we first set the non-negative bounded intensity functions $\tilde{a}_j(t)$ as

$$\tilde{a}_j(t) = \sum_{\{l \in \mathcal{L}; j \in l\}} \lambda_l(t), \tag{11}$$

but the calibration scheme will be computationally costly. It is thus useful to devise parsimonious model parameterizations. For instance, we use constant joint default intensities,

setting $\lambda_l(t) = \lambda_l$ and thus for a_j in (9) we have

$$a_j = \sum_{\{l \in \mathcal{L}; j \in l\}} \lambda_l, \qquad j \in M_{CDS}. \tag{12}$$

The default intensities for every $j \in M_{CDS}$ as defined in (9) can thus be written as

$$\lambda_j(t) = y_j(t) + \sum_{\{l \in \mathcal{L}; j \in l\}} \lambda_l, \quad t \geq 0. \tag{13}$$

We define the following integrated quantities which will be used in the remainder,

$$\Lambda_j(t_1, t_2) := \int_{t_1}^{t_2} \lambda_j(s) ds, \quad Y_j(t_1, t_2) := \int_{t_1}^{t_2} y_j(s) ds, \quad \Lambda_l(t_1, t_2) := \int_{t_1}^{t_2} \lambda_l(s) ds, \tag{14}$$

and

$$\Lambda_j(t) := \int_0^t \lambda_j(s) ds, \quad Y_j(t) := \int_0^t y_j(s) ds, \quad \Lambda_l(t) := \int_0^t \lambda_l(s) ds, \tag{15}$$

where $j \in M_{CDS}$ and $l \in \mathcal{L}$.

It is obvious from (6) and (7) that we need the following conditional survival probabilities to compute the counterparty risk adjustment as defined in (8)

$$\mathbb{Q}(\tau_R > t | \mathcal{G}_{\tau_C}), \tag{16}$$

and

$$\mathbb{Q}(\tau_R > t | \mathcal{G}_{\tau_I}). \tag{17}$$

Based on the above constructed model this two survival probabilities can be calculated by the following propositions.

Proposition 3.1.

$$\mathbb{Q}(\tau_R > t | \mathcal{G}_{\tau_C}) = \mathbb{E}\left\{\exp\left(-\Lambda_R(\tau_R, t)\right) | \mathcal{G}_{\tau_C}\right\} \tag{18}$$

$$= \mathbb{E}\left\{\exp\left(-Y_R(\tau_R, t) - \sum_{\{l \in \mathcal{L}; R \in l\}} \Lambda_l(\tau_R, t)\right) | \mathcal{G}_{\tau_C}\right\}. \tag{19}$$

Proposition 3.2.

$$\mathbb{Q}(\tau_R > t | \mathcal{G}_{\tau_I}) = \mathbb{E}\left\{\exp\left(-\Lambda_R(\tau_R, t)\right) | \mathcal{G}_{\tau_I}\right\} \tag{20}$$

$$= \mathbb{E}\left\{\exp\left(-Y_R(\tau_R, t) - \sum_{\{l \in \mathcal{L}; R \in l\}} \Lambda_l(\tau_R, t)\right) | \mathcal{G}_{\tau_I}\right\}. \tag{21}$$

The two propositions follow directly from the Markov probabilities, cf. [2].

3.2 Model Simulation

As described in [2] this model above allows a common shock model such that the simulation of a random time τ is quite fast. Given the previously simulated trajectories of the CIR processes y_j for $j \in M_{CDS}$, one essentiallly needs to simulate IID (Independent and identically) exponential random variables $\xi_{\hat{j}}$, for $\hat{j} \in \mathcal{L} \cup M_{CDS}$. Then one computes, for every $l \in \mathcal{L}$,

$$\hat{\tau}_l := \inf\{t > 0;\ \Lambda_l(t) \geq \xi_l\} \tag{22}$$

and for every $j \in M_{CDS}$,

$$\hat{\tau}_j := \inf\{t > 0;\ Y_j(t) \geq \xi_j\}. \tag{23}$$

Next we set for every $j \in M_{CDS}$

$$\tau_j = \hat{\tau}_j \wedge (\bigwedge_{\{l \in \mathcal{L}; j \in l\}} \hat{\tau}_l). \tag{24}$$

4 Numerical Evaluation of the BVA Adjustment

In this section we perform a numerical evaluation of the BVA as defined in (7) based on Monte-Carlo simulations.

First we need to generate the sample paths of the CIR processes, the simulation can be terminated using the fact that the distribution of $y(t)$ given $y(s)$ as defined (10), for some $s < t$ is, up to a scale factor, a noncentral chi-square distribution, see [11]. Alternatively, we can model this sample paths using the paths of the standard Brownian motion.

4.1 The generation of Break-even Spreads

Furthermore, we assume deterministic interest rates such that the default time τ and the discount factor $D(0,t)$ are independent. We know that the survival probabilities associated with a CIR intensity process are given by

$$\begin{aligned} \mathbb{Q}(\tau_R > t) :&= \mathbb{E}\left[\mathrm{e}^{-Y_R(t)}\right] \\ &= P^{CIR}(0,t), \end{aligned} \tag{25}$$

where $P^{CIR}(0,t)$ denotes the price at time 0 of a zero coupon bond maturing at time t under a stochastic interest rate dynamics given by the CIR process [11] and given by

$$P(t,T) := A(t,T)\mathrm{e}^{-B(t,T)y(t)}, \tag{26}$$

where

$$A(t,T) := \left[\frac{2h\exp\{(\kappa+h)(T-t)/2\}}{2h+(\kappa+h)(\exp\{(T-h)h\}-1)}\right]^{\frac{2\kappa\mu}{\sigma^2}}, \tag{27}$$

$$B(t,T) := \frac{2(\exp\{(T-t)h\}-1)}{2h+(\kappa+h)(\exp\{(T-h)h\}-1)}, \tag{28}$$

$$h := \sqrt{\kappa^2+2\sigma^2}. \tag{29}$$

We return now to defining CDS premium rate $\mathscr{P}$ which can be computed by zeroing (8) in $\mathscr{P}$ as break-even spreads.

We assume deterministic recovery rates and as well loss given defaults, so we set the loss given defaults of the low, medium and high risk entity are respectively to $LGD_{low} = 0.6, LGD_{medium} = 0.65$ and $LGD_{high} = 0.7$.

We show in Table 2 the premium rate $\mathscr{P}$ in basis points using the assumed deterministic Loss Given Defaults and collections of the parameters in Table 1, each collection of the parameters may take extreme values corresponding to a low, a medium or a high credit risk, where the Loss Given Defaults of the extremely entity is also set to be 0.6.

Credit Risk Level	κ	μ	σ	y_0
Extremely low	0.9	0.001	0.005	0.00001
Low	0.9	0.001	0.01	0.001
Medium	0.8	0.02	0.1	0.01
High	0.5	0.05	0.3	0.04

Table 1: Collection of parameters for initializing the CIR processes.

Maturity	Extremely low	Low Risk	Medium Risk	High Risk
1y	2	6	85	293
2y	3	6	97	298
3y	4	6	105	301
4y	4	6	110	302
5y	5	6	113	302
6y	5	6	115	303

Table 2: Break-even spreads in basis points generated using the collections of the parameters of the CIR processes in Table 1.

4.2 Market Intensities

Before producing the default times using the intensities we first have to determine the constant a_j for $j = C, I, R$ in the model defined in (12). For modeling interest rates, the condition

$$2\kappa\mu > \sigma^2 \tag{30}$$

has to be imposed to ensure that the origin is inaccessible to the process (10), such that we can assure that the interest rates remain positive. To model the default intensities with CIR model we relax the condition (30) such that we will not limit the parameters generated by the CIR model.

If we assume that the processes $y_j, j \in M_{CDS}$ are always non-negative. Then due to the definition (9) the constant a_j defined in (12) must be chosen as

$$a_j = \sum_{\{l \in \mathcal{L}; j \in l\}} \lambda_l \leq \lambda_j, \quad \forall j \in M_{CDS} \tag{31}$$

For instance, we can set as in [1], for every $l \in \mathcal{L}$,

$$\lambda_l = \alpha_l \inf_{j \in l} \lambda_j \tag{32}$$

for some non-negative model dependence parameters α_l such that $\sum_{l \in \mathcal{L}} \alpha_l \leq 1$. The value of α_l determines possibility of the simultaneous defaults between the parties in the group l, a larger value refers to the higher possibility of simultaneous defaults.

We denote the market implied intensity (hazard rate) for name j with λ_j^* which can be bootstrapped from the individual CDS quotes reported in Table 2. The bootstrapping procedure is model independent and performed assuming a piecewise linear hazard rate function, cf. [11]. Now we can calibrate the constant a_j for every $j \in M_{CDS}$ by choosing appropriate model dependence parameters α_l and setting

$$a_j = \sum_{\{l \in \mathcal{L}; j \in l\}} \lambda_l \leq \lambda_j^*. \tag{33}$$

From the CDS quotes of the higher risk the bootstrapped intensity λ_j^* is larger, thus $\lambda_l, j \in l$ is larger due to (32). Besides, for the same the intensity λ_j^*, a larger dependence parameter α_l constructs the larger $\lambda_l, j \in l$. Hence, with the same exponentially distributed trigger variable the simultaneous default time of the group l is smaller (earlier) through (22), if α_l is larger and consequently the possibility of the simultaneous defaults between parties in the group l is higher.

4.3 Monte-Carlo Evaluation

To compute the BVA on a CDS contract we perform the following steps based on Monte-Carlo simulations:

1. Produce default times τ_C, τ_I and τ_R using (22),(23) and (24).
2. In case of $\mathfrak{B}$ (see (3)), this is to say the counterparty defaults first. We need to compute the term inside the first expectation value which has positive sign. First we check, at the default time of counterparty whether the reference credit also defaults, for the case of a simultaneous default we just need the loss given default $\mathcal{L}_R$. Otherwise we compute S_{τ_C} given in (8), the survival possibility in S_{τ_C} can be computed by (19).
3. In the event of $\mathfrak{C}$ (see (3)) we need the term inside the second expectation value which with negative sign, the computation is similar to the last step.
4. Finally, we produce the BVA by discounting and averaging.

5 Numerical Results

We study a five years CDS contract on a reference entity traded by an investor and a counterparty, where both the investor and the counterparty are defaultable. We assume the payment dates to be every three months and the loss given default of the three names are taken from a market provider and are fixed to 60%. Besides, we set the three names having different levels of credit risk which are specified by the collections of the parameters in Table 1. The parameters $\alpha_l, l \neq \{C, R\}$ are assumed to be the same and equal with 0.01. We compute the BVAs for each following scenario by varying the parameter $\alpha_{\{C,R\}}$. We denote the BVA as a purchaser and a seller of the protection respectively by BVA_p and BVA_s, the results are described in basis points. In the results below the number between parentheses represents the Monte-Carlo standard error.

We assume the following scenarios:

- Scenario 1. The investor has low credit risk, the reference entity has high credit risk and the counterparty has medium credit risk. This situation is the most common in the real market.
- Scenario 2. The investor has low credit risk, the reference entity has medium credit risk and the counterparty has high credit risk. We are facing a risky counterparty in this case.
- Scenario 3. The investor has high credit risk, the reference entity has medium credit risk and the counterparty has low credit risk. The investor is most risky itself.

- Scenario 4. Both investor and counterparty have medium credit risk, while the reference entity has high credit risk (Risky Reference I).
- Scenario 5. Both investor and counterparty have low credit risk, while the reference entity has high credit risk (Risky Reference II).

Table 3: The values of the BVA for the different scenarios.

Scenario	Base Scenario	Risky Counterparty	Risky Investor	Risky Ref I	Risky Ref II
$(\alpha_{\{C,R\}})$	BVA_p BVA_s	BVA_p BVA_s	BVA_p BVA_s	BVA_p BVA_s	BVA_p BVA_s
0.01	7.0 (0.1) 4.1 (0.0)	6.5 (0.1) 0.7 (0.0)	-0.6 (0.0) -6.3 (0.1)	2.9 (0.1) -2.9 (0.1)	0.2 (0.0) -0.1 (0.0)
0.03	12.5 (0.2) 4.0 (0.0)	12.5 (0.2) 0.5 (0.0)	-0.4 (0.0) -6.3 (0.1)	8.3 (0.1) -3.1 (0.1)	0.4 (0.0) -0.1 (0.0)
0.05	18.4 (0.2) 3.9 (0.0)	19.1 (0.2) 0.3 (0.0)	-0.0 (0.1) -6.3 (0.1)	14.1 (0.2) -3.1 (0.1)	0.8 (0.1) -0.1 (0.0)
0.1	32.6 (0.3) 3.6 (0.0)	35.1 (0.3) 0.0 (0.0)	0.8 (0.1) -6.5 (0.1)	28.0 (0.2) -3.6 (0.1)	1.5 (0.1) -0.1 (0.0)
0.15	46.0 (0.4) 3.2 (0.0)	50.8 (0.4) -0.1 (0.0)	1.6 (0.1) -6.5 (0.1)	41.5 (0.4) -4.2 (0.1)	2.3 (0.1) -0.1 (0.0)
0.2	59.7 (0.4) 2.9 (0.0)	66.4 (0.4) -0.3 (0.0)	2.2 (0.1) -6.4 (0.1)	54.8 (0.4) -4.6 (0.1)	2.8 (0.1) -0.1 (0.0)
0.25	74.4 (0.5) 2.6 (0.0)	83.1 (0.5) -0.3 (0.0)	3.3 (0.1) -6.5 (0.1)	69.3 (0.5) -5.1 (0.1)	4.0 (0.1) -0.1 (0.0)

Table 3 clearly shows the effect of the wrong way risk. For example, if one looks at the second column, one notices that as the possibility of the simultaneous defaults between counterparty and reference credit gets larger, the BVA_p increases significantly due to (1) the counterparty is the rikiest name (2) the higher represented positive correlation makes the spread of the reference entity larger at the counterparty default, thus the option on the residual price of CDS for the investor as payer will be in the money and worth more, but at the counterparty default the investor only gets a fraction of it proportional to the recovery value of the counterparty and (3) at the simultaneous default of the counterparty and the reference credit, that option must be deep into the money, but the payer investor only gets a fraction of it proportional to the recovery value of the counterparty, more BVA_p takes place.

The adjustments BVA_p in Scenario 1 (Base Scenario) and Scenario 2 (Risky Counterparty) are similar, since the possibility of simultaneous defaults between the counterparty and the reference entity are the same if one has medium credit risk and the other one has high risk. The adjustments BVA_p in Scenario 2 are a little bit larger than the corresponding adjustments in Scenario 1, because the counterparty in Scenario 2 is riskier. One looks the adjustments BVA_s in Scenario 1 and Scenario 2, at the simultaneous default between the

counterparty and the reference entity, the option for the investor as the receiver will be out of the money, thus slight adjustments required. In particular, as the dependence parameter $\alpha_{\{C,R\}}$ is larger, less adjustments takes place, but the changes are very small.

In Scenario 3, the values of the adjustments BVA_p have only small changes. For the small dependence parameter $\alpha_{\{C,R\}}$ the adjustment is negative, because the investor is riskier. However, as the dependence parameter $\alpha_{\{C,R\}}$ get larger, this is to say that the possibility of the simultaneous defaults between the counterparty and the reference is increasing, the investor as payer even requires the adjustments although he is risky, see the last three rows at the third column.

An interesting pattern emerges from the fourth column (Risky Ref I). Contrary to earlier works, e.g. [7], one looks BVA_p at the fourth column, one finds that as the possibility of the simultaneous defaults between counterparty and the reference entity is increasing, the BVA_p increases significantly. The reason is, the counterparty has medium credit risk while the reference entity has high risk, thus they have higher possibility for larger $\alpha_{\{C,R\}}$ to default simultaneously, then the investor needs adjustments to against this risk. However, if the counterparty becomes safer while the reference entity is still riskier, then the possibility of the simultaneous defaults between the counterparty and the reference entity will be lower, thus less adjustments will take place as reported in the last column (Risky Ref II).

6 Application to a Market Contract

We apply the methodology to run through the example in [7]. Particularly, we calculate the mark-to-market price of a five-year CDS contract between British Airways (counterparty) and Royal Dutch Shell (investor) on the default of Lehman Brothers (reference entity). Following [7] we consider two CDS contracts. In the first contract Royal Dutch Shell bought a 5-year protection on Lehman Brothers from British Airways on January 5, 2006. In the second contract, Royal Dutch Shell selled a 5-year protection to British Airways on Lehman Brothers on January 5, 2006. We assume that British Airways as the counterparty computed the mark-to-market value of the both contracts on May 1, 2008. We just take the CDS quotes and the calibrated parameters for the CIR process dynamics from [7] which are reported in Tables 4 and 5, respectively. More information about the calibration can be found in [7].

Similar to the procedure by Brigo and Capponi in [7] we evaluate the mark-to-market value as follows:

(1) We take the CDS quotes of the three names on January 5, 2006 in Table 4 and the corresponding parameters of the CIR processes in Table 5, and then calculate the value of the five-year risk-adjusted CDS contract which started at T_a = January 5, 2006 and

Maturity	Royal Dutch Shell	Lehman Brothers	British Airways
1y	4/24	6.8/203	10/151
2y	5.8/24.6	10.2/188.5	23.2/230
3y	7.8/26.4	14.4/166.75	50.6/275
4y	10.1/28.5	18.7/152.25	80.2/305
5y	11.7/30	23.2/145	110/335
6y	15.8/32.1	27.3/136.3	129.5/342
7y	19.4/33.6	30.5/130	142.8/347
8y	20.5/35.1	33.7/125.8	153.6/350.6
9y	21/36.3	36.5/122.6	162.1/353.3
10y	21.4/37.2	38.6/120	168.8/355.5

Table 4: Market spread quotes in basis points for Royal Dutch Shell, Lehman Brothers and British Airways, the notation x/y indicates that x is the CDS spread on January 5, 2006, while y denotes the CDS spread on May 1, 2008.

Credit risk levels (2006/2008)	κ	μ	σ	y_0
British Airways (name 'C')	0.0266/0.6773	0.2582/0.0782	0.0003/0.2242	0.00002/0.00001
Royal Dutch Shell (name 'I')	0.0394/0.1835	0.0219/0.0089	0.0192/0.0057	0.0001/0.003
Lehman Brothers (name 'R')	0.036/7.8788	0.0432/0.0208	0.0533/0.5722	0.0001/0.6611

Table 5: The CIR parameters of British Airways, Royal Dutch Shell and Lehman Brothers calibrated to the market quotes of CDSs in Table 4.

terminated five years later at T_b = January 5, 2011 as:

$$S^A_{T_a} = S_{T_a} - BVA_{T_a} \tag{34}$$

where the $\mathscr{P}_R = 23.2$bp is the five-year spread quote of Lehman Brothers as the reference credit at time T_a, as mentioned before S_{T_a} is the value of the equivalent CDS contract without risk adjustment. All the loss given defaults are set to 0.6.

(2) At T_c = May 1, 2008, Royal Dutch Shell as CDS payer (British Airways in the second CDS contract) calculates the mark-to-market value of the CDS contract. We keep the CIR parameters of Royal Dutch Shell and British Airways the same as the parameters at time T_a and vary the volatility of the CIR process associated with Lehman Brothers, while keeping the other parameters fixed. We use the CDS quotes on May 1, 2008 in Table 4 for λ^*_j in inequality (33) to compute a_j so that the default intensity processes will be recalculated.

(3) Using the recalculated default intensity processes we evaluate BVA_{T_c} which is the risk adjustment of the CDS contract from time T_c for five years, namely until T_a = May 1, 2013. For the risk-adjusted value of the CDS contract starting at T_c and maturing at T_d we have

$$S^A_{T_c} = S_{T_c} - BVA_{T_c}, \tag{35}$$

whereas the five-year spread premium and the loss given defaults are kept.

(4) Finally, we calculate the mark-to-market CDS contract value as

$$MTM_{a,c}(\mathscr{P}_R, \mathscr{L}_{C,I,R}) = S^A_{T_c} - \frac{S^A_{T_a}}{D(T_a, T_c)}. \tag{36}$$

We display the mark-to-market value of the both contracts described above in Figure 2 and 1, respectively. The mark-to-market value of the CDS contract without risk adjustment is also shown in the figures. Besides, we recall the results of this example by Brigo and Capponi in [7] in the figures, e.g. we take the corresponding mark-to-market value of the both contracts described above whereas the correlation among the three names is set equal to 0.

Figure 1 reports the mark-to-market value of the CDS contract between British Airways as payer and Royal Dutch Shell as seller on default of Lehman Brothers under the increasing dependence parameter $\alpha_{\{C,R\}}$. We check the effect of the increasing possibility of simultaneous defaults between Lehman Brothers and British Airways on the value marked to markt by British Airways.

That mark-to-marked value of the CDS contract without risk adjustment is 529.3 bps due to the widening of the CDS spread of Lehman Brothers. The risk-adjusted mark-to-marked values of the CDS contract are closed to 529.3 and less sensitivity to the dependence

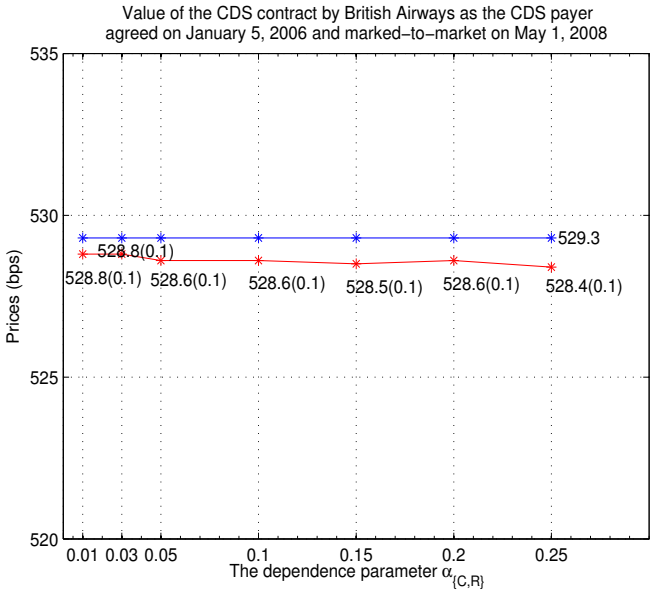

Figure 1: Value of the CDS contract between Royal Dutch Shell and British Airways on default of Lehman Brothers agreed on January 5, 2006 and marked-to-market by British Airways as the CDS payer on May 1, 2008. The blue line is the mark-to-market CDS contract value without risk adjustment and the red line denotes the mark-to-market values of the CDS contract with the adjustment which allows the possibility of simultaneous defaults. The dependence parameters $\alpha_{\{C,I\}} = \alpha_{\{I,R\}} = \alpha_{\{C,I,R\}} = 0.01$, while $\alpha_{\{C,R\}}$ is increasing. The numbers in round brackets represent the Monte-Carlo standard error.

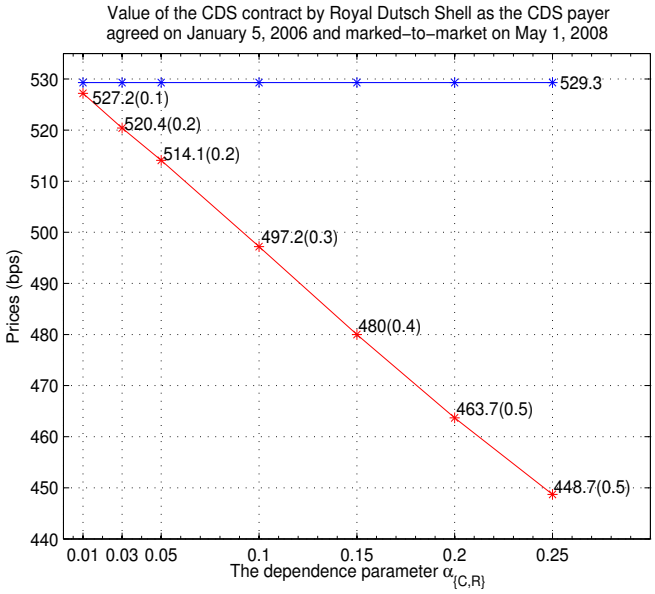

Figure 2: Value of the CDS contract between Royal Dutch Shell and British Airways on default of Lehman Brothers agreed on January 5, 2006 and marked-to-market by Royal Dutch Shell as the CDS payer on May 1, 2008. The blue line is the mark-to-market CDS contract value without risk adjustment and the red line denotes the mark-to-market values of the CDS contract with the adjustment which allows the possibility of simultaneous defaults. The dependence parameters $\alpha_{\{C,I\}} = \alpha_{\{I,R\}} = \alpha_{\{C,I,R\}} = 0.01$, while $\alpha_{\{C,R\}}$ is increasing. The numbers in round brackets represent the Monte-Carlo standard error.

parameter $\alpha_{\{C,R\}}$. As British Airways is the payer and the dependence parameter $\alpha_{\{I,R\}}$ is relatively low (0.01), thus the British Airways BVA_p becomes negligible and consequently the risk-adjusted mark-to-marked value of the CDS contract closed to its value without risk adjustment, however the dependence parameter $\alpha_{\{C,R\}}$ is increasing.

We consider the second contract, Royal Dutch Shell becomes CDS payer. Unlike the first contract, we observe the significant sensitivity of the risk-adjusted mark-to-marked values of the CDS contract to the dependence parameter $\alpha_{\{C,R\}}$. At the simultaneous default between British Airways and Lehman Brothers, Royal Dutch Shell holds an option which will be deep into the money, but Royal Dutch Shell can not be paid completely due to the default of British Airways. While the possibility of simultaneous defaults between British Airways and Lehman Brothers is increasing, the Royal Dutch Shell BVA_p becomes larger and consequently the mark-to-marked value of the CDS contract for Royal Dutch Shell get reduced as reported in Figure 2.

7 Conclusions

In this work, we discussed the calculating of the bilateral counterparty credit valuation adjustment in the presence of simultaneous defaults. In order to consider the simultaneous defaults we apply the Markov copula model, in which the dependence between defaults and the representation of the wrong way risk by the possibility of the simultaneous default among the three names in a CDS contract. Using a Monte-Carlo scheme we evaluate the BVAs for some scenarios, which include the different situations of credit risk level among the three names in the CDS contract. By analyzing the numerical results we observed the effect of a simultaneous default involved in the CDS contract. Besides, we applied our methodology to evaluate the risk-adjusted mark-to-market of the market CDS contract which is provided in [7]. In comparison with the results in [7], we saw the effect of considering the possibility of simultaneous defaults on the valuation of the counterparty risk in a CDS contract.

The numerical results show that the dependence between defaults only represented by the possibility of simultaneous defaults in the Markov copula model can fully capture wrong-way risk. Besides, the results confirm that the substantial role of considering simultaneous defaults on the valuation of the counterparty risk in CDS contracts. In particular, the effects of the simultaneous defaults on the BVA are not identical for the contracting party as the CDS seller and the CDS payer.

In the market, collateralization is one of the most important techniques of mitigation of counterparty risk. The collateral can be either directly transferred between counterparties themself or held by a third party like a clearing house. A model for the bilateral margin call process and its application in the calculation will be investigated in future work.

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8 Appendix

Here we present the proof of Proposition 2.1.

Proof. : We have that

$$
\begin{aligned}
S_t^A \overset{(4)(5)}{=} \mathbb{E}\Bigg\{ & \mathbb{I}_{\mathfrak{A}} D(t,\tau)(-\mathscr{L}_R) \\
& + \mathbb{I}_{\mathfrak{B}} \left[D(t,\tau) \left(\mathscr{R}_C (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^- \right) \right] \\
& + \mathbb{I}_{\mathfrak{C}} \left[D(t,\tau) \left((S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - \mathscr{R}_I (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^- \right) \right] \\
& + \mathbb{I}_{\mathfrak{D}} \left[D(t,\tau) \left(-(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R) \right) \right] \\
& + \mathbb{I}_{\mathfrak{E}} \left[D(t,\tau) \left(\mathscr{L}_R \right) \right] \\
& + \underbrace{D(t,\tau)(\tau - T_{\gamma(\tau)-1}) \mathscr{P} \mathbb{I}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i \mathscr{P} \mathbb{I}_{\{\tau \geq T_i\}}}_{:=\mathcal{M}_1} \Bigg| \mathcal{G}_t \Bigg\}
\end{aligned}
\tag{37}
$$

We first consider the expression regarding the event $\mathfrak{B}$ inside the above conditional expectation

$$
\mathbb{I}_{\mathfrak{B}} \left[D(t,\tau) \left(\mathscr{R}_C (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^- \right) \right].
$$

Since

$$
\mathscr{R}_C (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ - (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^- = (\mathscr{R}_C - 1)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ + (S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R),
$$

this expression is equal to

$$
\mathbb{I}_{\mathfrak{B}} \left[-D(t,\tau)(1-\mathscr{R}_C)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+ \right] + \underbrace{\mathbb{I}_{\mathfrak{B}} D(t,\tau)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)}_{=:\mathcal{M}_2}.
$$

Similarly, the expression conditional on the event $\mathfrak{C}$ in (37) can be rewritten as

$$
\mathbb{I}_{\mathfrak{C}} \left[D(t,\tau)(1-\mathscr{R}_I)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^- \right] + \underbrace{\mathbb{I}_{\mathfrak{C}} D(t,\tau)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)}_{=:\mathcal{M}_3}.
$$

It is obvious that $\mathbb{I}_{\{\tau=\tau_R\}}S_\tau = 0$, and therefore we observe $\mathcal{M}_2, \mathcal{M}_3$ and the last two expressions respectively regarding the event $\mathfrak{D}$ and $\mathfrak{E}$ in (37) together as follows,

$$D(t,\tau)\mathbb{I}_{\{\tau=\tau_R\}}(-\mathscr{L}_R)\left(\mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{C}} - \mathbb{I}_{\mathfrak{D}} - \mathbb{I}_{\mathfrak{E}}\right) + D(t,\tau)\mathbb{I}_{\{\tau\neq\tau_R\}}S_\tau\left(\mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{C}} - \mathbb{I}_{\mathfrak{D}}\right).$$

Observing that $\mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{C}} - \mathbb{I}_{\mathfrak{D}} - \mathbb{I}_{\mathfrak{E}} = 0$, we can rewrite the terms inside (37) as

$$\begin{aligned}\Pi^A(t,T) =& \mathbb{I}_{\mathfrak{B}}\left[-D(t,\tau)(1-\mathscr{R}_C)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+\right] \\ &+ \mathbb{I}_{\mathfrak{C}}\left[D(t,\tau)(1-\mathscr{R}_I)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^-\right] \\ &+ \mathbb{I}_{\mathfrak{A}}\left(-D(t,\tau)\mathcal{L}_R + \mathcal{M}_1\right) + \mathbb{I}_{\{\tau>T\}}\mathcal{M}_1 \\ &+ \mathbb{I}_{\{\tau\neq\tau_R\}}\left(\mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{C}} - \mathbb{I}_{\mathfrak{D}}\right)\left(D(t,\tau)S_\tau + \mathcal{M}_1\right).\end{aligned}$$

Now, by comparing $\mathcal{M}_1$ with (1) we get

$$\begin{aligned}S_t^A =& \mathbb{E}\left\{-\mathbb{I}_{\mathfrak{B}}D(t,\tau)(1-\mathscr{R}_C)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+\middle|\,\mathcal{G}_t\right\} \\ &+ \mathbb{E}\left\{\mathbb{I}_{\mathfrak{C}}D(t,\tau)(1-\mathscr{R}_I)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^-\middle|\,\mathcal{G}_t\right\} \\ &+ \mathbb{E}\left\{\left(\mathbb{I}_{\mathfrak{A}} + \mathbb{I}_{\{\tau>T\}}\right)\Pi(t,T)\middle|\,\mathcal{G}_t\right\} \\ &+ \mathbb{E}\left\{\mathbb{I}_{\{\tau\neq\tau_R\}}\left(\mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{C}} - \mathbb{I}_{\mathfrak{D}}\right)\left(D(t,\tau)\mathbb{E}\{\Pi(\tau,T)|\mathcal{G}_\tau\} + \Pi(t,\tau)\right)\middle|\,\mathcal{G}_t\right\}.\end{aligned} \tag{38}$$

Using $\mathbb{E}\{\mathbb{E}\{\cdot|\mathcal{G}_\tau\}|\mathcal{G}_t\} = \mathbb{E}\{\cdot|\mathcal{G}_t\}$ for $t < \tau$, the last expression in (38) equals

$$\mathbb{E}\left\{\mathbb{I}_{\{\tau\neq\tau_R\}}\left(\mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{C}} - \mathbb{I}_{\mathfrak{D}}\right)\Pi(\tau,T)\middle|\,\mathcal{G}_t\right\}.$$

We see that

$$\mathbb{I}_{\{\tau\leq t\}} + \mathbb{I}_{\{\tau>T\}} + \mathbb{I}_{\mathfrak{A}} + \mathbb{I}_{\{\tau\neq\tau_R\}}\left(\mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{C}} - \mathbb{I}_{\mathfrak{D}}\right) = 1$$

and the events in the terms of the sum are exclusive, we get

$$\begin{aligned}S_t^A = S_t &- \mathbb{E}\left\{\mathbb{I}_{\mathfrak{B}}D(t,\tau)(1-\mathscr{R}_C)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^+\middle|\,\mathcal{G}_t\right\} \\ &+ \mathbb{E}\left\{\mathbb{I}_{\mathfrak{C}}D(t,\tau)(1-\mathscr{R}_I)(S_\tau - \mathbb{I}_{\{\tau=\tau_R\}}\mathscr{L}_R)^-\middle|\,\mathcal{G}_t\right\}\end{aligned} \tag{39}$$

□

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