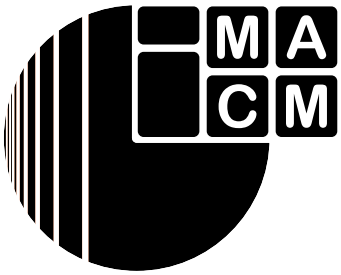

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Abstract

We assume linear systems of differential algebraic equations, which include physical parameters or other parameters. Uncertainties of the parameters are modelled by random variables. We expand the corresponding random-dependent solutions in the polynomial chaos. Approximations of unknown coefficient functions can be obtained by quadrature or sampling schemes. Alternatively, stochastic collocation methods or the stochastic Galerkin approach yield larger coupled systems of differential algebraic equations. We show the equivalence of these types of numerical methods under certain assumptions. The index of the coupled systems is analysed in comparison to the original systems. Sufficient conditions for an identical index are derived. Furthermore, we present results of numerical simulations for an example.

Keywords: differential algebraic equations, initial value problems, index, polynomial chaos, stochastic collocation method, stochastic Galerkin method, quadrature, uncertainty quantification

1. Introduction

The mathematical models of technical applications often represent time-dependent systems of differential algebraic equations (DAEs). For example, corresponding models are applied in the simulation of electric circuits, see [3], and of multibody systems, see [12]. We consider initial value problems of linear systems of DAEs.

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In the systems of DAEs, some parameters often exhibit uncertainties due to modelling errors, measurement errors or imperfections of a manufacturing process, for example. We replace the critical parameters by independent random variables to quantify the uncertainties. Consequently, the solution of a system of DAEs becomes a random process. We expand the random process in the polynomial chaos, which represents a series with unknown time-dependent coefficient functions and given random-dependent basis polynomials, see [1, 2, 15].

The coefficient functions can be computed approximately by a quadrature or sampling method, where the original DAEs have to be resolved many times. Alternatively, stochastic collocation techniques and stochastic Galerkin methods yield larger coupled systems of DAEs satisfied by an approximation of the coefficient functions, which have to be solved just once, see [16, 17]. Under certain assumptions, we show that the collocation approach as well as the Galerkin method are equivalent to a multivariate Gaussian quadrature.

The theoretical and numerical properties of a system of DAEs are characterised by its index, see [4, 6]. Thus we examine the index of the coupled systems in comparison to the original systems of DAEs. Thereby, the equivalence of methods yields sufficient conditions for an identical index. However, we also construct a counterexample, where the index changes. In the previous work [9], the index of the coupled system from the stochastic Galerkin method is analysed for linear systems of DAEs assuming a specific structure of the involved matrices. In this article, we consider the index both for stochastic collocation methods and for the stochastic Galerkin approach, where either no particular structure or another condition on the matrices is assumed. For nonlinear systems of DAEs, numerical methods using the polynomial chaos expansions have been investigated in [7, 8, 11, 12, 13].

The article is organised as follows. In Sect. 2, we introduce the modelling using random parameters, the polynomial chaos expansions and corresponding numerical methods. We analyse the equivalence of numerical techniques and the index of coupled systems in Sect. 3. Finally, numerical simulations of an illustrative example from electrical engineering are presented in Sect. 4.

2. Stochastic Modelling and Numerical Methods

We define the class of problems and outline corresponding numerical techniques now.

2.1. Differential Algebraic Equations with Random Parameters

A linear system of DAEs is considered in the form

$$A(p)x'(t,p) + B(p)x(t,p) = s(t,p), \tag{1}$$

where the matrices $A, B \in \mathbb{R}^{N\times N}$ as well as the source term $s \in \mathbb{R}^N$ depend on parameters $p = (p_1, \ldots, p_Q) \subseteq \Pi$ within a relevant set $\Pi \subseteq \mathbb{R}^Q$. Consequently, the solution $x : [t_0, t_{\text{end}}] \times \Pi \to \mathbb{R}^N$ depends on time as well as the parameters. Let initial value problems

$$x(t_0, p) = x_0(p) \tag{2}$$

be given with a prespecified function x_0, which may depend on the parameters. The choice of initial values (2) has to be consistent with respect to the system (1). Furthermore, we assume that the index of the systems (1) is identical for all $p \in \Pi$.

Now let the chosen parameters exhibit some uncertainties. Hence we replace the parameters by independent random variables

$$p : \Omega \to \Pi, \quad \omega \mapsto (p_1(\omega), \ldots, p_Q(\omega))$$

on some probability space $(\Omega, \mathcal{A}, P)$ to achieve an uncertainty quantification. It follows that the solution of (1) becomes a time-dependent random process. The choice of traditional distributions (Gaussian, uniform, beta, etc.) implies the existence of a probability density function $\rho : \Pi \to \mathbb{R}$. We denote corresponding expected values by

$$\langle f(p) \rangle := \int_\Pi f(p)\rho(p)\, \mathrm{d}p \tag{3}$$

for measurable functions $f : \Pi \to \mathbb{R}$ provided that the integral is finite. We apply the notation (3) also to vector-valued and matrix-valued functions by the separate components. The expected value yields an inner product $\langle f(p)g(p) \rangle$ for two functions $f, g : \Pi \to \mathbb{R}$, which yields the L^2-norm of the probability space.

2.2. Polynomial Chaos Expansions

We assume that each component of the solution x of the random-dependent system (1) exhibits finite second moments at each time $t \in [t_0, t_{\text{end}}]$. It follows that the random process can be expanded in the generalised polynomial chaos (gPC), see [1, 15, 17], i.e.,

$$x(t, p(\omega)) = \sum_{i=0}^{\infty} v_i(t)\Phi_i(p(\omega)) \tag{4}$$

with coefficient functions $v_i : [t_0, t_{\text{end}}] \to \mathbb{R}^N$. The basis functions $(\Phi_i)_{i\in\mathbb{N}}$ with $\Phi_i : \Pi \to \mathbb{R}$ represent a complete set of orthonormal polynomials ($\langle \Phi_i(p)\Phi_j(p)\rangle = \delta_{ij}$ with the Kronecker delta). In the case $Q = 1$, the Gaussian distribution and the uniform distribution imply the Hermite polynomials and the Legendre polynomials, respectively, for example. The multivariate orthonormal polynomials are just the products of the univariate orthonormal polynomials.

The expansion (4) converges in the L^2-norm of the probability space for each fixed time $t \in [t_0, t_{\text{end}}]$. The coefficient functions represent projections of the random process on the basis polynomials, i.e.,

$$v_i(t) = \langle x(t, p)\Phi_i(p)\rangle \qquad \text{for each } i. \tag{5}$$

Since x is unknown a priori, we obtain the problem to determine the coefficient functions now.

2.3. Numerical Methods

The following numerical techniques refer to the gPC expansion (4). In each method, the aim is to compute an approximation of a finite set of coefficients functions $v_0, v_1, \ldots, v_M$.

2.3.1. Quadrature and Sampling

Numerical techniques can be obtained directly from the formula (5) of the coefficient functions. It follows that the coefficient functions represent expected values or, equivalently, probabilistic integrals of the form (3). Using quadrature schemes, the integrals become finite sums

$$\tilde{v}_i(t) = \sum_{k=1}^{K} w_k \, x(t, p^{(k)}) \, \Phi_i(p^{(k)}) \tag{6}$$

for the approximations $\tilde{v}_i$ of (5). Each quadrature method is defined by the nodes $p^{(k)} \in \Pi$ and the weights $w_k \in \mathbb{R}$. Different quadrature schemes may be applied for each i. However, often the same method is applied for all coefficient functions to keep the total number of evaluations of the integrand small. A special case represent sampling methods like Monte-Carlo or quasi Monte-Carlo techniques, where the weights are $w_k = \frac{1}{K}$ for all k.

Hence the application of a quadrature or sampling method requires to solve K separate initial value problems (1),(2) for each tuple of parameters $p^{(k)}$. We can apply corresponding numerical time integrators directly. However, the choice of an appropriate numerical solver depends on the index of the systems (1), see [4].

2.3.2. Stochastic Collocation Techniques

Another class of methods follows from a truncation of the gPC expansion (4)

$$x^M(t, p(\omega)) := \sum_{i=0}^{M} v_i(t)\Phi_i(p(\omega)). \tag{7}$$

Typically, all multivariate polynomials up to some degree are included. Inserting the truncated series (7) in the original DAEs (1) yields the residual

$$r(t,p) := A(p)\sum_{i=0}^{M} v_i'(t)\Phi_i(p) + B(p)\sum_{i=0}^{M} v_i(t)\Phi_i(p) - s(t,p) \tag{8}$$

for each $t \in [t_0, t_{\text{end}}]$. Now we like to determine the unknown coefficient functions such that the residual becomes small in some sense. In a collocation method, the residual vanishes at prespecified nodes, i.e., $r(t, p^{(l)}) = 0$ for $l = 0, 1, \ldots, M$. Thus we obtain a coupled system of DAEs

$$A(p^{(l)})\sum_{i=0}^{M} \tilde{v}_i'(t)\Phi_i(p^{(l)}) + B(p^{(l)})\sum_{i=0}^{M} \tilde{v}_i(t)\Phi_i(p^{(l)}) = s(t, p^{(l)}) \tag{9}$$

for $l = 0, 1, \ldots, M$. The system (9) involves $(M+1)N$ equations for the unknown approximations $\tilde{v}_0, \ldots, \tilde{v}_M$. Corresponding initial values follow from (2) and (5)

$$v_i(t_0) = \langle x_0(p)\Phi_i(p)\rangle \qquad \text{for each } i, \tag{10}$$

where the probabilistic integral is evaluated analytically or by a quadrature scheme, cf. Sect. 2.3.1.

We can solve the initial value problem (9),(10) of the coupled system to compute the approximations of the coefficient functions. Obviously, the computational effort is higher than solving $M+1$ separate initial value problems (1),(2). The stochastic collocation approach is interesting if the system (9) can be decoupled, which is investigated in Sect. 3. Furthermore, a question is if the index of the coupled system (9) is the same as for the original systems (1).

2.3.3. Stochastic Galerkin Method

Again the truncated gPC expansion (7) is applied. To achieve small residuals (8), the Galerkin approach requires the condition

$$\langle r(t,p)\Phi_l(p)\rangle = 0 \qquad \text{for } \; l = 0,1,\ldots,M, \tag{11}$$

which represents an optimal choice of the approximation in some sense. Inserting the formula (8) of the residual into the condition (11) and performing elementary manipulations yields the coupled system

$$\sum_{i=0}^{M} \langle A(p)\Phi_i(p)\Phi_l(p)\rangle \tilde{v}_i'(t) + \langle B(p)\Phi_i(p)\Phi_l(p)\rangle \tilde{v}_i(t) = \langle s(t,p)\Phi_l(p)\rangle \tag{12}$$

for $l = 0,1,\ldots,M$. Corresponding initial values are obtained from (10).

Again initial value problems (12),(10) of a coupled system of DAEs have to be solved to compute the approximations $\tilde{v}_0,\ldots,\tilde{v}_M$. In general, the accuracy of the approximations is better in the Galerkin method than in the collocation method for the case of identical M. Once again, a question is if the index of the coupled system (12) is the same as for the original systems (1).

3. Index Analysis

The properties of a system of DAEs are characterised by its index. Different index concepts exist, see [4, 6]. Nevertheless, the concepts coincide in case of linear DAEs. We investigate the index of the coupled systems (9) and (12) in comparison to the original systems (1).

3.1. Choice of Polynomial Spaces

We consider expansions (4) with orthonormal basis polynomials in the following. The multivariate basis polynomials are the products of the univariate orthonormal polynomials, i.e.,

$$\Phi_{i_1,i_2,\ldots,i_Q}(p) = \Psi^1_{i_1}(p_1)\Psi^2_{i_2}(p_2)\cdots\Psi^Q_{i_Q}(p_Q), \tag{13}$$

where Ψ^j_i is a univariate polynomial of degree i.

We define two types of finite sets of basis polynomials and their corresponding spans

$$\begin{aligned} \mathcal{M}_R &:= \operatorname{span}\left\{\Phi_{i_1,\ldots,i_Q} : i_1+\cdots+i_Q \le R\right\}, \\ \mathcal{N}_R &:= \operatorname{span}\left\{\Phi_{i_1,\ldots,i_Q} : i_1,\ldots,i_Q \le R\right\}. \end{aligned} \tag{14}$$

The set $\mathcal{M}_R$ represents all polynomials up to degree R as in a multidimensional Taylor expansion. The basis of the set $\mathcal{N}_R$ is sometimes called a tensor-product basis. Obviously, it holds that $\mathcal{M}_R \subseteq \mathcal{N}_R$ for each R. The two sets coincide only in case of $Q = 1$. Furthermore, the corresponding dimensions read

$$\dim(\mathcal{M}_R) = \frac{(R+Q)!}{R!Q!}, \qquad \dim(\mathcal{N}_R) = (R+1)^Q.$$

A different behaviour of the stochastic Galerkin method for the two different types of polynomial spaces (14) has been observed in case of linear hyperbolic equations, see [10].

In the analysis, we restrict to domains $\Pi = I_1 \times \cdots \times I_Q$ with compact or infinite intervals I_j. Let the corresponding random distributions be chosen via $\rho(p) = \rho_1(p_1)\cdots\rho_Q(p_Q)$ such that a Gaussian quadrature with optimal order exists on each interval I_j for $j = 1,\ldots,Q$. Given a multivariate Gaussian quadrature with K nodes in each dimension and corresponding weights w_k, it holds that

$$\langle\Theta(p)\rangle = \sum_{k=1}^{K^Q} w_k\Theta(p^{(k)}) \quad \text{for all } \Theta(p) \in \mathcal{N}_{2K-1},$$

since the multivariate quadrature decomposes into Q univariate Gaussian quadratures.

For the matrix $A(p)$ in the system (1), we will sometimes assume the structure

$$A(p) = A_0 + \sum_i p_iA_i + \sum_{i<j} p_ip_jA_{ij} + \sum_{i<j<k} p_ip_jp_kA_{ijk} + \cdots + p_1\cdots p_QA_{1\cdots Q} \tag{15}$$

with constant matrices and, likewise, for the matrix $B(p)$ in the systems (1). This assumption is often satisfied in the applications. The condition (15) is equivalent to $A(p), B(p) \in \mathcal{N}_1$. The structure (15) differs from the assumption on $A(p), B(p)$ in the previous work [9]. None of the two assumptions implies the other, i.e., they just agree for a common subset of problems.

3.2. Equivalence of Numerical Methods

In this subsection, we apply and extend results from [12] on numerical techniques for nonlinear systems of DAEs to our case of linear systems. Equivalence of numerical techniques means that the resulting solutions are exactly the same. We apply the vectors $V := (v_0^\top, \ldots, v_M^\top)^\top$ to collect the first $M+1$ coefficient functions of a gPC expansion (4).

We consider the stochastic collocation method from Sect. 2.3.2. It holds that the following result is valid.

Lemma 1. *Let a basis $\Phi_0, \ldots, \Phi_M$ and collocation points $p^{(0)}, \ldots, p^{(M)}$ be given. If the matrix $C := (\Phi_j(p^{(l)}))_{lj}$ of order $M+1$ is regular, then the solution of the coupled system (9) is*

$$\tilde{V}_{\mathrm{col}}(t) = (C^{-1} \otimes I_N) X(t)$$

with $X(t) := (x(t, p^{(0)})^\top, \ldots, x(t, p^{(M)})^\top)^\top$ using the Kronecker product and the identity matrix I_N.

The proof can be obtained by mimicking the steps in Sect. 5.2 of [12]. Lemma 1 shows that the system (9) of the collocation method can be decoupled. Consequently, we will never solve the coupled system (9) in case of a regular matrix C. Instead, the original systems (1) are resolved separately for each collocation point.

Concerning the regularity of the transformation matrix, the following well-known property will be used.

Lemma 2. *Let the domain Π and the probability distributions be chosen such that a Gaussian quadrature of optimal order exists. If $p^{(0)}, \ldots, p^{((R+1)^Q-1)}$ are the nodes of this quadrature and $\Phi_0, \ldots, \Phi_{(R+1)^Q-1}$ represent the orthonormal basis of $\mathcal{N}_R$, then the matrix $C := (\Phi_j(p^{(l)}))_{lj}$ is regular.*

This statement can be shown by considering the products of the evaluations of the univariate polynomials from (13) and the products of the weights of the involved univariate Gaussian quadratures.

Now we achieve a first result on the equivalence of methods.

Theorem 1. *Let the domain* Π *and the probability distributions be chosen such that a Gaussian quadrature of optimal order exists. It follows that the stochastic collocation method using the basis of* $\mathcal{N}_R$ *and the* $(R+1)^Q$ *nodes of the quadrature is equivalent to the scheme (6) based directly on the Gaussian quadrature.*

Proof:

We define $K := (R+1)^Q$. Let $p^{(l)}$ and w_l for $l = 1, \dots, K$ be the nodes and the weights, respectively, of the multivariate Gaussian quadrature. The orthonormal basis of $\mathcal{N}_R$ is used to define the transformation matrix $C := (\Phi_i(p^{(l)}))_{li} \in \mathbb{R}^{K \times K}$. Furthermore, we introduce $W := \operatorname{diag}(w_1, \dots, w_K)$. Let $\tilde{V}_{\text{col}}, \tilde{V}_{\text{qdr}} \in \mathbb{R}^{KN}$ be the approximations from the coupled system (9) and the quadrature (6), respectively. It follows that

$$X(t) = (C \otimes I_N)\tilde{V}_{\text{col}}(t)$$

with $X(t) := (x(t, p^{(1)})^\top, \dots, x(t, p^{(K)})^\top)^\top$ due to Lemma 1. The quadrature (6) can be written in the form

$$\tilde{V}_{\text{qdr}}(t) = ((C^\top W) \otimes I_N)X(t) = ((C^\top W C) \otimes I_N)\tilde{V}_{\text{col}}(t).$$

To achieve $\tilde{V}_{\text{col}} = \tilde{V}_{\text{qdr}}$, it remains to show that $C^\top W C = I_K$. This property follows from

$$(C^\top W C)_{ij} = \sum_{l=1}^{K} w_l \Phi_i(p^{(l)})\Phi_j(p^{(l)}) = \langle \Phi_i(p)\Phi_j(p) \rangle = \delta_{ij}$$

due to the exactness of the Gaussian quadrature in case of $\Phi_i, \Phi_j \in \mathcal{N}_R$. $\square$

Lemma 2 and Theorem 1 apply the space $\mathcal{N}_R$ for the basis polynomials, whereas the matrices $A(p), B(p)$ from (1) are arbitrary. The following theorem holds for the specific structure (15) of these matrices.

Theorem 2. *Let the domain* Π *and the probability distributions be chosen such that a Gaussian quadrature of optimal order exists. If it holds that* $A(p), B(p) \in \mathcal{N}_1$ *and* $s(t,p) \in \mathcal{N}_R$*, then the stochastic collocation method using the* $(R+1)^Q$ *nodes of the quadrature is equivalent to the stochastic Galerkin method provided that the same basis of* $\mathcal{N}_R$ *is applied.*

Proof:

Consider the coupled system (9) from the stochastic collocation method, where the collocation points are chosen as the $M+1=(R+1)^Q$ nodes of the multivariate Gaussian quadrature. We multiply the the lth equation by $w_l\Phi_j(p^{(l)})$

$$\begin{aligned}&\sum_{i=0}^{M} w_l A(p^{(l)})\Phi_i(p^{(l)})\Phi_j(p^{(l)})\tilde{v}_i'(t) + w_l B(p^{(l)})\Phi_i(p^{(l)})\Phi_j(p^{(l)})\tilde{v}_i(t)\\ &= w_l s(t,p^{(l)})\Phi_j(p^{(l)}).\end{aligned}$$

Summing up over all $l=0,1,\ldots,M$ yields

$$\begin{aligned}&\sum_{i=0}^{M}\left(\sum_{l=0}^{M} w_l A(p^{(l)})\Phi_i(p^{(l)})\Phi_j(p^{(l)})\right)\tilde{v}_i'(t)\\ &\quad+\left(\sum_{l=0}^{M} w_l B(p^{(l)})\Phi_i(p^{(l)})\Phi_j(p^{(l)})\right)\tilde{v}_i(t) = \sum_{l=0}^{M} w_l s(t,p^{(l)})\Phi_j(p^{(l)})\end{aligned} \tag{16}$$

for each $j=0,1,\ldots,M$. The Gaussian quadrature reproduces the exact integrals for polynomials in $\mathcal{N}_{2R+1}$. The assumptions on the matrices, the source term and basis polynomials guarantee this property. Thus a solution of the system (9) also solves the system (12).

Vice versa, the system (12) coincides with (16) under these assumptions. We rewrite the system (16) as

$$\sum_{l=0}^{M} w_l\Phi_j(p^{(l)})\left[\sum_{i=0}^{M}\left(A(p^{(l)})\Phi_i(p^{(l)})\tilde{v}_i'(t)+B(p^{(l)})\Phi_i(p^{(l)})\tilde{v}_i(t)\right)-s(t,p^{(l)})\right]=0$$

for $j=0,1,\ldots,M$. Since the matrix $(w_l\Phi_j(p^{(l)}))_{lj}$ is regular due to Lemma 2, the system can be transformed into (9). It follows that a solution of the system (12) also satisfies the system (9). □

Theorem 1 and Theorem 2 imply the following conclusion directly.

Corollary 1. *Under the assumptions of Theorem 2, the stochastic Galerkin method with polynomial space $\mathcal{N}_R$ is equivalent to a multivariate Gaussian quadrature using $(R+1)^Q$ nodes.*

The above theorems do not hold in case of the polynomial space $\mathcal{M}_R$ from (14) for $R\geq 2$, since a quadrature scheme with a number of nodes equal to the dimension of $\mathcal{M}_R$ does not achieve the required optimal order in general. Nevertheless, the results also hold in case of $\mathcal{M}_1$.

3.3. Conclusions on the Index

We employ the results from the previous subsection to obtain statements on the index of the coupled systems in the numerical methods.

Corollary 2. *Let the domain* Π *and the probability distributions be chosen such that a Gaussian quadrature of optimal order exists. If the systems of DAEs (1) exhibit the index* k *for all* $p \in \Pi$*, then the coupled system (9) of the stochastic collocation method based on the* $(R+1)^Q$ *nodes of the quadrature and the basis of* $\mathcal{N}_R$ *inherits the same index* k*.*

Proof:

Lemma 1 and Lemma 2 imply that the coupled system (9) is equivalent to a decoupled system with subsystems (1) evaluated at the nodes $p^{(l)} \in \Pi$ of the Gaussian quadrature. Since the index of each subsystem is identical to k, also the index of the system (9) results to k. □

Corollary 3. *Let the domain* Π *and the probability distributions be chosen such that a Gaussian quadrature of optimal order exists. If it holds that* $A(p), B(p) \in \mathcal{N}_1$*,* $s(t,p)$ *arbitrary and the systems of DAEs (1) exhibit the index* k *for all* $p \in \Pi$*, then the coupled system (12) of the stochastic Galerkin method using the basis of* $\mathcal{N}_R$ *has the same index* k*.*

Proof:

In the proof of Theorem 2, the steps show that the matrices in the system (12) of the Galerkin method are identical to the matrices in the system (9) of the collocation method with the $(R+1)^Q$ nodes from the Gaussian quadrature. The index is determined by the matrices only, i.e., the index is independent of the source term. Hence the index of both systems coincides. Now Corollary 2 allows for the conclusion. □

The above corollaries represent sufficient conditions such that the coupled systems inherit the index of the original systems. The conditions are not necessary. However, the index is not always preserved as demonstrated by the counterexample in the next subsection.

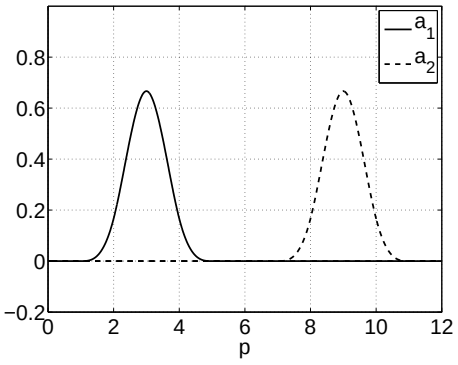

Figure 1: Functions in counterexample chosen as cubic B-splines.

3.4. Counterexample

We apply just a single random parameter ($Q = 1$) in the following. The index of a system of DAEs is often related to the regularity of matrices. We construct an example in the case $N = 2$, where regularity of a matrix is gained.

We consider the system (1) with the matrix

$$A(p) = \begin{pmatrix} a_1(p) & 0 \\ 0 & a_2(p) \end{pmatrix} \tag{17}$$

and the simple choice $B(p) \equiv I_2$ (identity matrix). Now we choose two functions a_j with compact supports $[\alpha_j, \beta_j]$ for $j = 1, 2$. Let $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$. It follows that the matrix (17) is singular for all $p \in \mathbb{R}$. We assume that the functions are strictly positive within the interior of their supports. For example, B-splines are feasible as depicted in Fig. 1. Arbitrarily smooth functions can be constructed.

In case of $B(p) \equiv I_2$, the index of the linear DAEs (1) depends just on the Jordan form $J(p)$ of the matrix $A(p)$, see [14]. We obtain

$$J(p) = \begin{cases} \begin{pmatrix} \gamma(p) & 0 \\ 0 & 0 \end{pmatrix} & \text{with } \gamma(p) \neq 0 \text{ for } p \in (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2), \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Note that the Jordan form is unique except for permutations of Jordan blocks. It follows that the nilpotency index of the corresponding linear system of DAEs is $k = 1$ for all $p \in \Pi$ and an arbitrary choice of $\Pi \subseteq \mathbb{R}$.

Now we investigate the coupled system (12) from the stochastic Galerkin method. The system includes the symmetric matrix $\hat{A} = (\langle A(p)\Phi_i(p)\Phi_l(p)\rangle)_{il}$ of size $2(M+1)$. After a permutation of rows and columns described by a regular matrix P, we achieve the structure

$$P\hat{A}P = \begin{pmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{pmatrix}$$

with the symmetric minors $\hat{A}_j = (\langle a_j(p)\Phi_i(p)\Phi_l(p)\rangle)_{il}$ of size $M+1$ for $j = 1, 2$. Let a probability density function ρ be chosen such that $\rho(p) > 0$ for all $p \in (\alpha_1, \beta_2)$, for example, an arbitrary Gaussian distribution or a uniform distribution in $[\alpha_1, \beta_2]$. Since a function a_j is positive in the interior of its support, it follows that the matrix $\hat{A}_j$ is positive definite, see the proof of Theorem 2.1 in [5]. Hence the matrix $\hat{A}$ is regular and the coupled system (12) exhibits the index $k = 0$ in contrast to the index $k = 1$ of the original systems (1).

A motivation for the heterogeneity of the index in this example can be given by observing the kernels of the matrix (17). It holds that

$$\operatorname{kern}(A(p)) = \begin{cases} \operatorname{span}(1,0)^\top & \text{for } p \in (\alpha_2, \beta_2), \\ \operatorname{span}(0,1)^\top & \text{for } p \in (\alpha_1, \beta_1), \\ \mathbb{R}^2 & \text{otherwise.} \end{cases}$$

Thus the kernel does not depend continuously on p, since the kernel changes its dimension.

In this example, Corollary 3 is not applicable, since the assumption $A(p) \in \mathcal{N}_1$ is not satisfied. Furthermore, the above counterexample does not contradict the sufficient conditions for an identical index proven in [9]. Theorem 2 of [9] would require the assumption $a_1 \equiv a_2$, whereas the choice of different functions a_1, a_2 is essential in the example (17).

It is important to note that this counterexample is independent of the finite choice from the basis functions $(\Phi_i)_{i\in\mathbb{N}}$. An arbitrary subset of polynomials can be applied. Other (non-polynomial) basis functions yield the same result provided that each basis function is not identical to zero within the supports of the functions a_j for $j = 1, 2$.

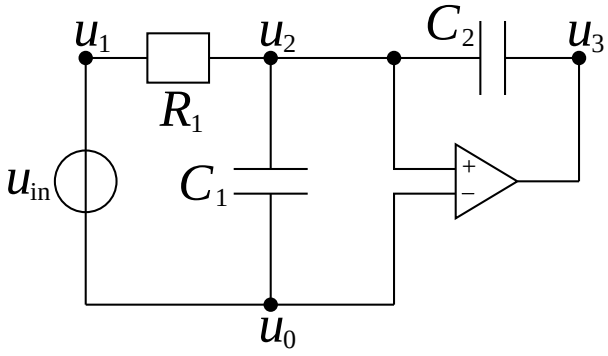

Figure 2: Electric circuit of a Miller integrator.

4. Numerical Simulation of an Illustrative Example

The electric circuit of a Miller integrator amplifies an input signal u_{in} to an output signal u_3, see Fig. 2. For specific choices of the involved physical parameters, the output is approximately the integral of the input. However, we apply parameters in other ranges now.

A mathematical model of the Miller integrator has been derived in [3]. Modified nodal analysis yields a linear system of DAEs

$$\begin{aligned} A_C C A_C^\top u'(t) + A_R G A_R^\top u(t) + A_V \jmath_V(t) &= 0 \\ A_V^\top u(t) - z(u(t), t) &= 0 \end{aligned} \tag{18}$$

for the unknown node voltages $u = (u_1, u_2, u_3)^\top$ and unknown branch currents $\jmath_V = (\jmath_1, \jmath_2)^\top$. The involved matrices read

$$A_C = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad A_V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

and $G = (\frac{1}{R_1})$. In the modelling, a voltage controlled source z is introduced as

$$z(u, t) = \begin{pmatrix} u_{\text{in}}(t) \\ a u_2 \end{pmatrix}.$$

In [3], an analysis of the index shows that this linear system exhibits

$$\begin{aligned} &\text{index } k = 2 \quad \text{for } a \neq 1 + \tfrac{C_1}{C_2}, \\ &\text{index } k = 3 \quad \text{for } a = 1 + \tfrac{C_1}{C_2}. \end{aligned}$$

The physical parameters are chosen as $C_1 = 10^{-10}$ F, $C_2 = 5 \cdot 10^{-11}$ F, $a = 2$, $R_1 = 10^3\ \Omega$. We select a harmonic oscillation as input signal, i.e.,

$$u_{\text{in}}(t) = \sin\left(\tfrac{2\pi}{T}t\right) \qquad \text{with} \quad T = 10^{-6}\ \text{s}.$$

The output voltage is $u_{\text{out}}(t) = u_3(t)$. Furthermore, we choose the initial values $x_0 = 0$ in (2).

In a stochastic modelling, we arrange random distributions for the amplification factor and the two capacitances by

$$\widetilde{a}(p_1) := a(1 + 0.1p_1), \quad \widetilde{C}_1(p_2) := C_1(1 + 0.1p_2), \quad \widetilde{C}_2(p_3) := C_2(1 + 0.1p_3)$$

with independent uniformly distributed random variables $p_j \in [-1, 1]$ for $j = 1, 2, 3$. Thus variations of 10% are considered in each parameter. The index of the system (18) is two for all realisations of the random parameters. Furthermore, the system (18) can be written in the form (1), where the matrices $A(p), B(p)$ exhibit the structure (15).

In the following, initial value problems of DAEs are always solved by the RADAU5 method, see [4]. In case of DAEs of index two, the order of this time integrator reduces to three. We apply equidistant step sizes in time. The total interval of the simulations is always $[0, 4T]$.

Now we use the stochastic Galerkin method to solve the model including the random parameters. Thereby, we choose the space $\mathcal{M}_3$ from (14), where the dimension is $M + 1 = 20$. Thus the coupled system (12) is solved with initial values equal to zero. Although the conditions from Sect. 3 do not apply to this case, numerical computations confirm that the coupled system inherits the index two. Fig. 3 illustrates the resulting expected values and standard deviations of the five components in the Miller integrator. All coefficient functions corresponding to the output signal are shown in Fig. 4. We observe that the magnitude of the coefficients decreases exponentially for increasing degree, which is a typical behaviour for processes with high smoothness in the parameter space.

We like to compare the efficiency of different methods. We apply a Gaussian quadrature with a grid of size 4^3, i.e., 64 separate systems (18) are resolved. Theorem 1 and Corollary 1 imply that the reconstruction (6) of the coefficient functions for $\mathcal{N}_3$ in (14) is equivalent to the corresponding collocation method as well as Galerkin method. The associated coupled systems (9) and (12) have the index two due to Corollary 2 and Corollary 3. In

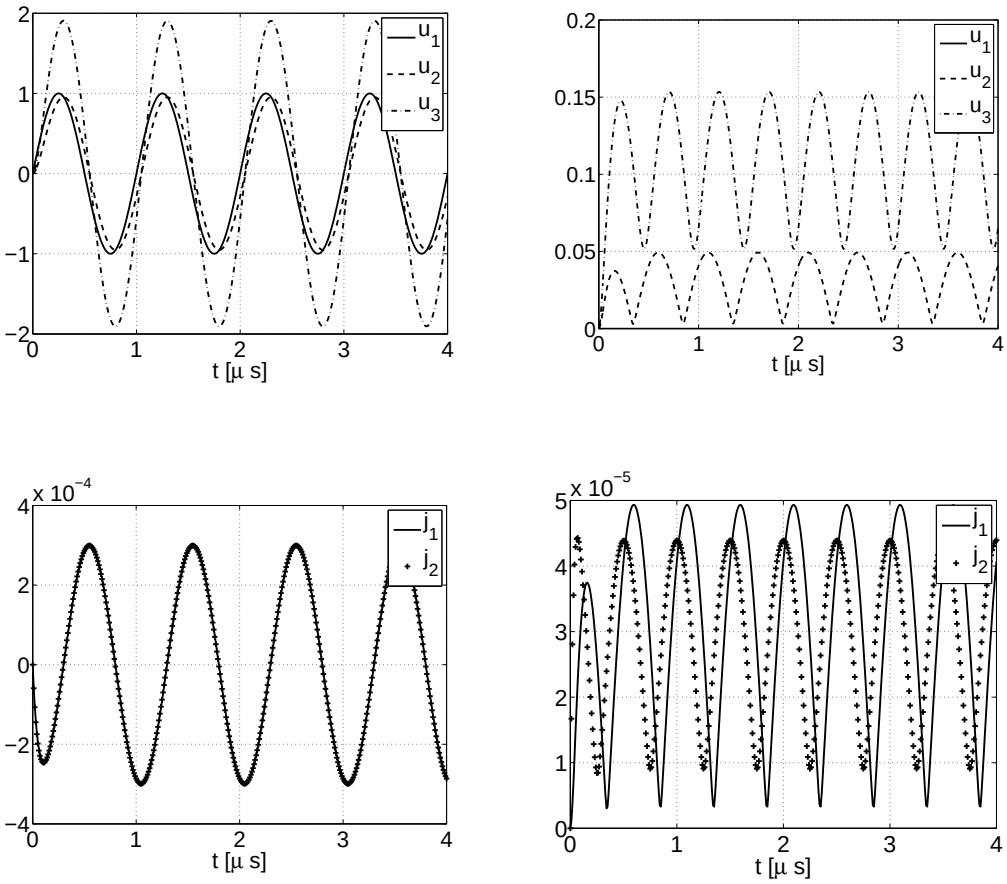

Figure 3: Expected values (left) and standard deviations (right) of the solution of the Miller integrator with random parameters.

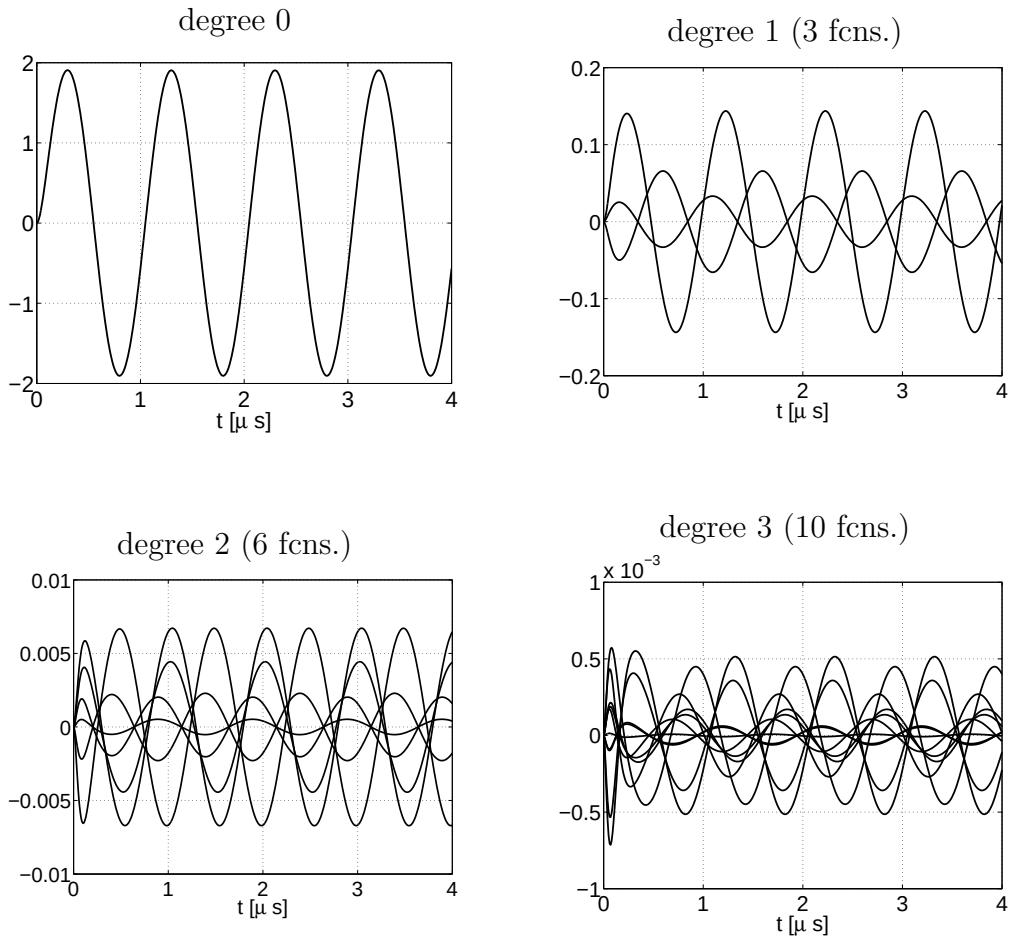

Figure 4: Coefficient functions of gPC expansion for output voltage of the Miller integrator with random parameters.

Table 1: Maximum absolute errors in methods (maximum for all grid points in time interval $[0, 4T]$ and all coefficient functions $v_0, \ldots, v_{19}$).

component	u_1	u_2	u_3	j_1	j_2
Galerkin	$4 \cdot 10^{-15}$	$6 \cdot 10^{-6}$	$2 \cdot 10^{-5}$	$6 \cdot 10^{-9}$	$2 \cdot 10^{-7}$
quadrature	$4 \cdot 10^{-15}$	$3 \cdot 10^{-7}$	$8 \cdot 10^{-7}$	$3 \cdot 10^{-10}$	$7 \cdot 10^{-7}$

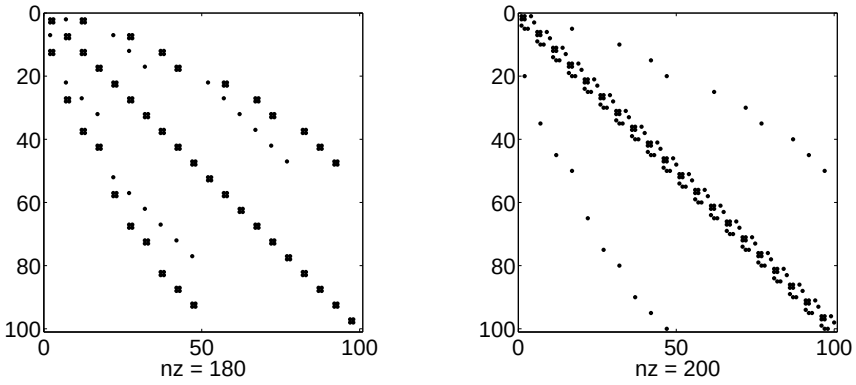

Figure 5: Sparse structure of the matrix A_{gal} (left) and the matrix B_{gal} (right) in the stochastic Galerkin method (nz: number of non-zero elements).

the software package MATLAB, the CPU times are 0.25 for the Galerkin approach with $\mathcal{M}_3$ and $64 \cdot 0.18 = 11.5$ for the Gaussian quadrature. Thus the Galerkin method is faster. To estimate the errors, we compute a reference solution by a Gaussian quadrature with a grid of size 8^3. Moreover, the time step size is halved. Table 1 shows the maximum differences with respect to the reference solution. It follows that the quadrature is more accurate than the Galerkin method. In conclusion, the two approaches exhibit roughly the same efficiency in this example.

It is interesting to understand why the CPU time for the Galerkin method with a coupled system of dimension 100 is just slightly higher than the CPU time for a single original system of dimension 5. The main reason for this efficiency is the sparse and banded structure of the matrices in the coupled system shown in Fig. 5, which is a consequence of the orthogonality of the basis polynomials. It follows that the LU-decompositions of the matrices in the time integration are also sparse and thus cheap.

5. Conclusions

Three classes of numerical methods can be used to solve differential algebraic equations with random parameters: quadrature schemes, the stochastic collocation and the stochastic Galerkin method. We showed the equivalence of two or more classes in specific cases of linear systems, where a multivariate Gaussian quadrature and a space of polynomials with tensor-product basis is considered. We derived sufficient conditions for an identical index of the linear systems of differential algebraic equations in the methods. An open question is the behaviour of the index in case of a smaller space of polynomials as used in truncated Taylor expansions. Furthermore, nonlinear systems of differential algebraic equations have to be investigated in this context.

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