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Abstract. Recently Galapon (2009 J. Phys. A: Math. Theor. 42 175201) has posed a question of the existence of delta- convergent sequences that vanish at the support of the limit Dirac delta function and gave an example of sequences of this type. It is a sequence of even functions that don't have a compact support. Motivated by the question in this paper we develop some results concerning delta sequences and show more examples of delta sequences of the type with or without compact support and are even or not even.

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1. Introduction

Recently,from the numerical simulation in quantum theory in [4] Galapon has posed a question of the existence of delta-convergent sequences that vanish at the support of the limit *Dirac delta function* (DDF) and constructed an example of sequences of this type. It is a sequence of even functions that don not have a compact support and have the form like to an empirical sequence. In the sequel for brevity we shall refer to the sequences with the above property as *Galapon delta sequences* (GDS). Motivated by the question in this paper we develop some results concerning delta sequences, namely, we introduce the concept of generating functions of *Dirac delta sequence* (DDS) and use it to design different GDS with or without compact support and are even or not even.

2. Generating function of Dirac delta sequence

In order to design Galapon delta sequences first we cite from [7, Section 3.3] a fundamental theorem, which will serve as a tool for constructing the delta-convergent sequences.

Theorem 2.1 *Let $\alpha(x)$ be a nonnegative, locally integrable function in the $n-$dimensional space $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \alpha(x)dx = 1 \tag{1}$$

Then the sequence of functions

$$f_\varepsilon(x) = \varepsilon^{-n}\alpha(x/\varepsilon) \tag{2}$$

converges to the DDF $\delta(x)$ as $\varepsilon \to 0$ (in the sense of generalized functions [13]).

The Theorem 2.1 was originally stated as an exercise in [12] and afterwards it was proved in [7]. Apparently, Galapon [4] was not aware of this theorem, such that he used another theorem from [5] to prove that his proposed sequence is indeed a DDS.
Remark. Sometimes instead of (2) it is convenient to use the sequence of functions $f^t_\varepsilon(x) := \varepsilon^{-tn}\alpha(x/\varepsilon^t)$.

Definition We shall call the function $\alpha(x)$ in Theorem 2.1 the *generating function* of the Dirac delta sequence $f_\varepsilon(x)$.

Below we list some generating functions and known corresponding DDS $f_\varepsilon(x)$. Alternatively, it is often convenient to use $F_m(x) = m\,\alpha(mx)$ for $m \to \infty$ instead.

A. Some generating functions of DDS without compact supports

The DDS $F_m(x)$ from A1 to A4 are given in [7] as examples and exercises, while A1 and A2 are given in [12].

The graphs of the above generating functions A1-A5 are depicted in Figure 1.

Table 1. Table of not compactly supported generating functions and corresponding DDS.

No	$\alpha(x)$	$f_{\varepsilon(x)}$	$F_m(x)$
A1	$\frac{1}{2\sqrt{\pi}}e^{-x^2/4}$	$\frac{1}{2\sqrt{\pi\varepsilon}}e^{-x^2/4\varepsilon}$	$\frac{1}{2}\sqrt{\frac{m}{\pi}}e^{-mx^2/4}$
A2	$\frac{1}{\pi}\frac{1}{x^2+1}$	$\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$	$\frac{1}{\pi}\frac{m}{m^2x^2+1}$
A3	$\frac{1}{2}e^{-\|x\|}$	$\frac{1}{2\varepsilon}e^{-\|x\|/\varepsilon}$	$\frac{m}{2}e^{-m\|x\|}$
A4	$\frac{1}{\pi}\frac{1}{\cosh(x)}$	$\frac{1}{\pi\varepsilon}\frac{1}{\cosh(x/\varepsilon)}$	$\frac{m}{\pi}\frac{1}{\cosh(mx)}$
A5	$\frac{1}{\pi}\frac{\sin^2(x)}{x^2}$	$\frac{\varepsilon}{\pi}\frac{\sin^2(\frac{x}{\varepsilon})}{x^2}$	$\frac{1}{\pi}\frac{\sin^2(mx)}{mx^2}$

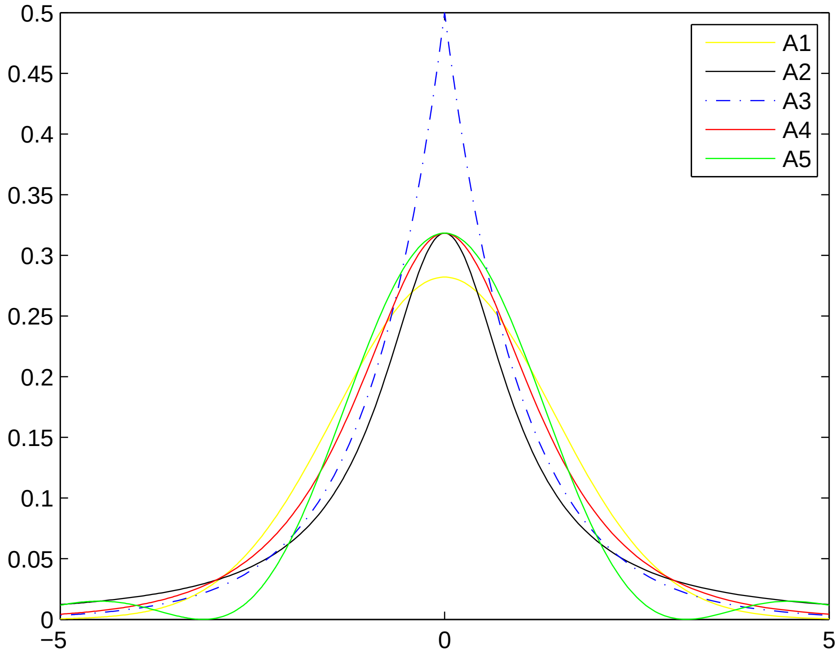

Figure 1. The graphs of the generating functions for the cases A1–A5.

B. Some generating functions of DDS with compact supports

The generating functions B0, B1, B5 are very popular (see e.g. [12], [7]), the functions B2–B4 are B-splines in [8] and B6 can be found in [14]. Notice that the function B4 is infinitely differentiable. The graphs of the above generating functions B1–B5 are depicted in Figure 2.

No.	$\alpha(x)$	$F_m(x)$
B0	$\begin{cases} 1/2, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$\begin{cases} m/2, & \lvert x\rvert \le 1/m \\ 0, & \lvert x\rvert > 1/m \end{cases}$
B1	$\begin{cases} 1-\lvert x\rvert & \le \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$\begin{cases} m(1-m\lvert x\rvert), & \lvert x\rvert \le 1/m \\ 0, & \lvert x\rvert > 1/m \end{cases}$
B2	$\begin{cases} 3(1-x^2)/4, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$\begin{cases} 3m(1-m^2x^2)/4, & \lvert x\rvert \le 1/m \\ 0, & \lvert x\rvert > 1/m \end{cases}$
B3	$\begin{cases} 1-3x^2+2\lvert x\rvert^3, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$\begin{cases} m(1-3m^2x^2+2m^3\lvert x\rvert^3), & \lvert x\rvert \le 1/m \\ 0, & \lvert x\rvert > 1/m \end{cases}$
B4	$\begin{cases} (4-6x^2+3\lvert x\rvert^3)/6, & 0 \le \lvert x\rvert \le 1 \\ (2-\lvert x\rvert)^3/6, & 1 \le \lvert x\rvert \le 2 \\ 0, & \lvert x\rvert > 2 \end{cases}$	$\begin{cases} m(4-6m^2x^2+3m^3\lvert x\rvert^3)/6, & 0 \le \lvert x\rvert \le 1/m \\ m(2-m\lvert x\rvert)^3/6, & 1/m \le \lvert x\rvert \le 2/m \\ 0, & \lvert x\rvert > 2/m \end{cases}$
B5	$\begin{cases} C_1 e^{-1/(1-x^2)}, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$ where $C_1 = \int_{\lvert x\rvert<1} e^{-1/(1-x^2)}dx$	$\begin{cases} C_1 m e^{-1/(1-m^2x^2)}, & \lvert x\rvert \le 1/m \\ 0, & \lvert x\rvert > 1/m \end{cases}$
B6	$\begin{cases} (1+\cos(\pi x))/2, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$\begin{cases} m(1+\cos(m\pi x))/2, & \lvert x\rvert \le 1/m \\ 0, & \lvert x\rvert > 1/m \end{cases}$

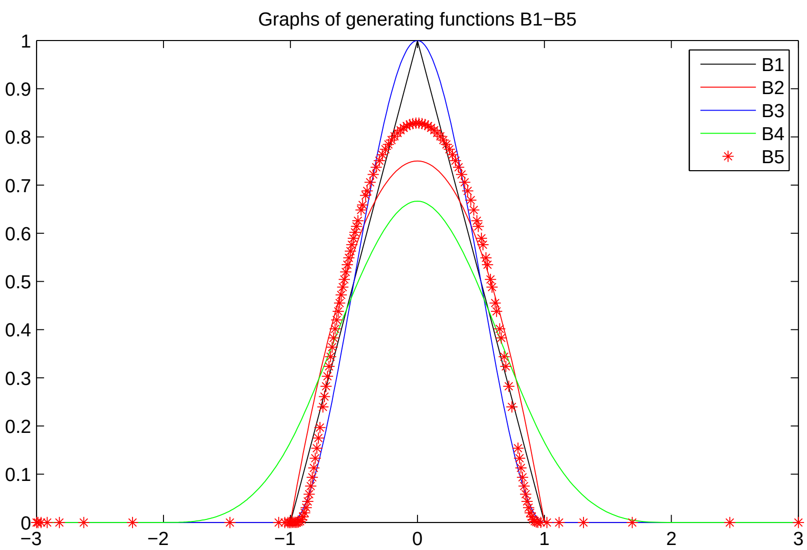

Figure 2. The graphs of the generating functions B1-B5

We remark that all the generating functions Ax and Bx and the corresponding DDS have a maximum at $x = 0$ and are even functions, i.e., symmetric with respect to the y-axis. Below, in the next section, based on Theorem 2.1 we construct DDS that vanish at $x = 0$, i.e. at the support of the DDF.

3. Delta sequences that vanish at the support of the DDF

In order to construct a DDS that vanish at the support of the DDF we shall construct generating functions $\alpha(x)$ that have the above property, which we call the Galapon property.

Using Theorem 2.1 it is easy to verify the following

Theorem 3.1 *If $\alpha(x)$ is a generating function with the support $[-1,1]$ that is continuous and even, then the function*

$$\beta(x) = \sum_{i=1}^{k} c_i a_i(x), \tag{3}$$

where

$$a_i(x) = \Big(\alpha(x+2i-1) + \alpha(x-2i+1)\Big)/2, \qquad c_i = 2^{k-i}/(2^k-1)$$

is a generating function that has the Galapon property and possesses a compact support.

Furthermore, the function

$$\beta(x) = \sum_{i=1}^{\infty} \frac{a_i(x)}{2^i} \tag{4}$$

is a generating function that has the Galapon property and does not possess a compact support.

The Figure 3 and Figure 4 depict the DDS generated by the functions formed by (3) with $k=1$ and $k=2$ from the compactly supported functions B2 and B5, respectively.

Notice that a DDS generated from the functions B2 is not smooth at $x=0$ while a DDS generated from the functions B5 is infinitely differentiable at this point. From Theorem 3.1 we see that in principle it is possible to construct generating functions that are two-sided symmetric, have the Galapon property and do not have a compact support. But due to the construction they do not monotonously tend to zero.

Below we are interested in the design of the unsupported generating functions that monotonously tend to zero which is not an easy task. Concerning this problem we first remark that the DDS introduced by Galapon in [4]

$$h_\nu(n,x) = \frac{1}{2^{2n+1}\Gamma(n+1/2)} \nu^{n+1/2} x^{2n} e^{-\nu x^2/4}$$

tending to DDF as $\nu \to \infty$ for any fixed n has the generating function G1

$$\alpha_n(x) = \frac{1}{2^{2n+1}\Gamma(n+1/2)} x^{2n} e^{-x^2/4}. \tag{5}$$

Except for this generating function of DDS due to the formula

$$\int_0^\infty x^n e^{-x} dx = n! \tag{6}$$

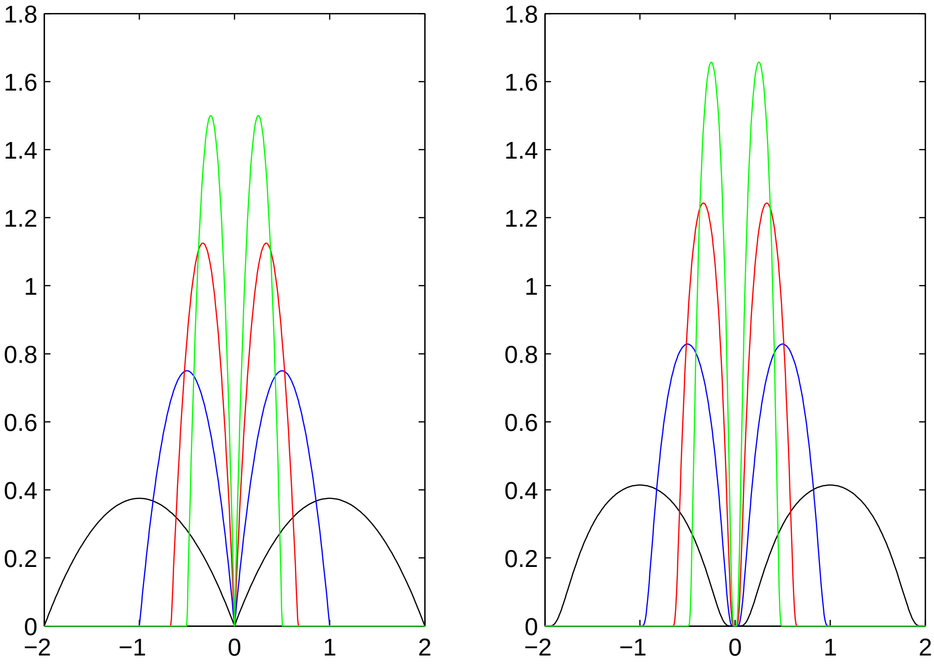

Figure 3. Graphs of the DDS generated by functions B2 (left) and B5 (right) using (3) with $k = 1$.

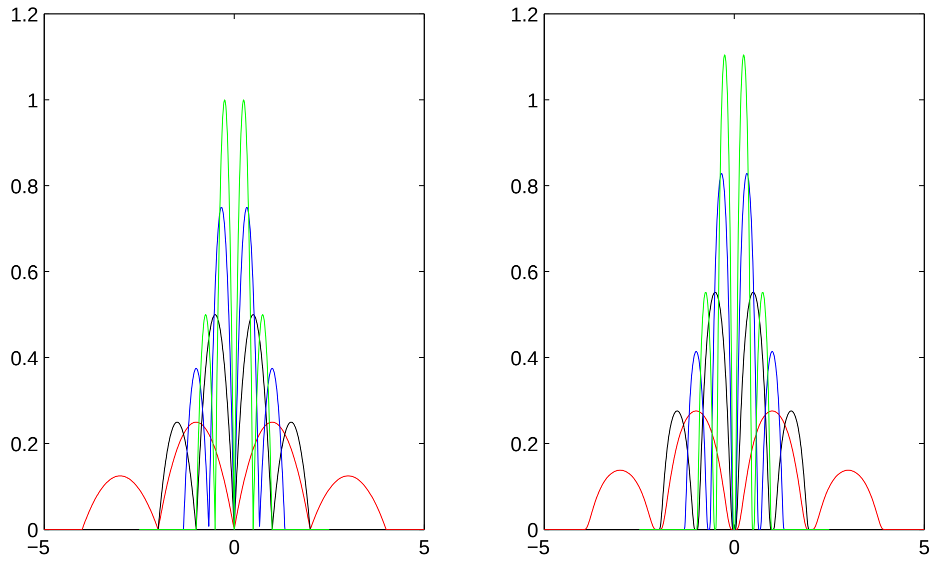

Figure 4. Graphs of the DDS generated by functions B2 (left) and B5 (right) using (3) with $k = 2$.

[3, formula 9.64, page 363] or [7, formula (14), page 383]) we conclude that the function

$$\alpha_n(x) = \frac{x^{2n}e^{-|x|}}{2(2n)!} \tag{7}$$

is also a generating function of DDS with the Galapon property, which we denote by G2. By the change of variables $x = y^2$ in the integral (6) we obtain

$$\int_0^\infty y^{2n+1}e^{-y^2}dy = \frac{n!}{2}.$$

Therefore, the function

$$\alpha_n(x) = 2x^{2n+1}e^{-x^2}/n!$$

is a generating function of DDS with the Galapon property.

The graphs of DDS generated by the functions G1 and G2 for $n = 1$ are given in Figure 5.

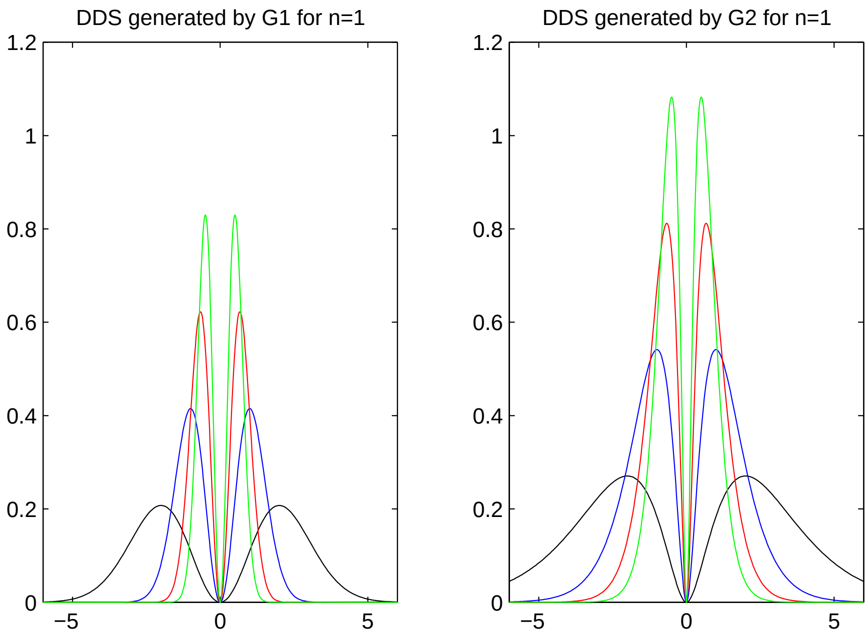

Figure 5. Graphs of DDS generated by the functions G1 and G2 for $n = 1$

Further, from the formula (6) by the change of variables $x = 1/y$ it is easy to obtain the following formula

$$\int_0^\infty \frac{y^{-2n}e^{-1/|y}}{(2(n-1))!}dx = 1.$$

Hence, the function

$$\alpha(x) = \begin{cases} \dfrac{x^{-2n}e^{-1/|x|}}{2(2(n-1))!}, & x \neq 0 \\ 0, \quad x = 0 \end{cases} \tag{8}$$

is a generating function of DDS with the Galapon property.

It is interesting to remark that all the above DDS are even functions. But we can construct DDS that are not even functions. For this purpose we see that if $\alpha(x)$ is an even generating function then the function

$$\gamma(x) = \begin{cases} 2\alpha(x), & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{9}$$

also generates a DDS that is not even and vanish at $x = 0$.

An example of such generating function is

$$\gamma(x) = \begin{cases} x^{-2}e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

which is obtained from (8) for $n = 1$. This function also can be found in page 14 of [9]. We shall refer to this generating function as G3. The graphs of the DDS generated by the function (9) for $B2$ and by G3 are depicted in Figure 6.

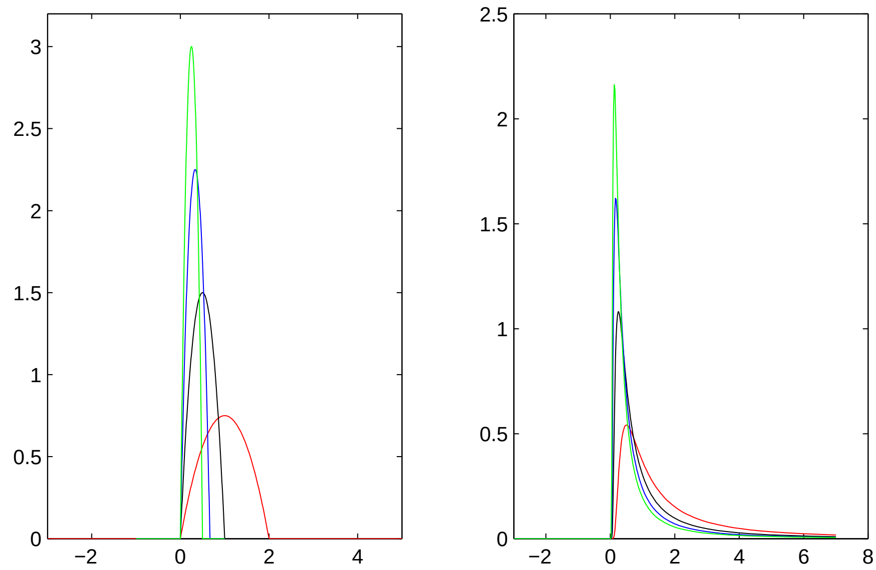

Figure 6. Graphs of the one-sided DDS generated by the function (9) for $B2$ (left) and by the function G3 (right)

4. Concluding remarks

In this short note we introduce the concept of generating function of DDS and use it to systematize some known DDS with or without compact support. Based on this concept we propose a method to design generating functions of DDS that vanish at the support of the Dirac delta function and are symmetric or not symmetric.

At present the solution of differential equations containing singular source terms attracts the attention of many researchers (see e.g. [1], [10], [11] and [14] and the bibliography therein). The central problem there is the approximation of the delta function which usually is called as its regularization. The question of how to choose a method of regularization of the delta function is very important for reaching a high accuracy.

In the future we will devote our attention to the regularization of the delta function describing the location and the time of an accident of oil pollution in order to justify and improve our recent results in [2].

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