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Parallel transport observables for connections on finite projective modules over matrix algebras

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Abstract

In this article we investigate parallel transports on finite projective modules over finite matrix algebras. Given a derivation-based differential calculus on the algebra and a connection on the module, we can construct for every derivation X a module parallel transport along the one-parameter group of algebra automorphisms given by the flow of X . This parallel transport morphism is determined uniquely by a differential equation depending on the covariant derivative along X . Based on this, we define a set of basic gauge invariant observables, i.e. functions from the space of connections to complex numbers. For modules equipped with a hermitian structure, we prove that any hermitian connection can be reconstructed up to gauge equivalence from these observables. This solves the gauge copy problem for gauge theory on hermitian finite projective modules over finite matrix algebras, similar to the Wilson loop observables in gauge theories on commutative smooth manifolds.

1 Introduction

In noncommutative geometry, gauge theories are typically described in terms of connections on finite projective modules \mathcal{E} over the noncommutative algebra A , see e.g. Landi's book [1] for an introduction. This description requires the notion of a differential calculus on A , generalizing differential forms and the exterior differential to the noncommutative setting. Depending on the algebra A , there are different choices for a calculus at hand, for example the twist deformed differential calculus [2] in deformation quantization using Drinfel'd twists, the bicovariant differential calculus for quantum groups [3] and the derivation-based differential calculus [4, 5, 6, 7, 8, 9], being particularly useful for finite matrix algebras [10], such as the fuzzy sphere [11].

Connections on finite projective modules, and the noncommutative gauge theories they describe, have been part of many investigations in the past years, leading to interesting results, such as noncommutative instantons in deformation quantization [12] and noncommutative monopoles on fuzzy spaces [13], just to name a few. These works aim at understanding topological invariants constructed from the curvature of the connection, and do not attempt to construct complete sets of observables, i.e. gauge invariant functions from the space of connections to the complex numbers, which allow for the reconstruction of the connection up to gauge equivalence.

In commutative gauge theory the curvature of a connection is not sufficient to extract the complete gauge invariant information of the connection. This is the well-known gauge copy problem. In nonabelian Yang-Mills theories this is already the case if we choose spacetime to be \mathbb{R}^d [14]. For certain gauge groups, the gauge copy problem can be solved by considering the set of all Wilson loops as the basic observables of the gauge theory, see e.g. [15, 16].

There are examples indicating that there is also a gauge copy problem in noncommutative geometry, see e.g. Proposition 2 in this paper, where we review some results of [10, 7, 8, 9]. This means that considering only observables which are constructed from the curvature, we can in general not extract the complete gauge invariant information of the connection. This calls for a suitable generalization of

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parallel transport and Wilson loops to noncommutative geometry. These studies have been initiated for the Moyal-Weyl deformation in earlier works, see e.g. [17, 18].

In our article we will give a purely algebraic definition of parallel transport morphisms on finite projective modules \mathcal{E} over noncommutative algebras A . For simplifying the explicit analysis of these morphisms, we study in the present paper only the case where $A = M_n(\mathbb{C})$ is a finite matrix algebra, equipped with a derivation-based differential calculus, and \mathcal{E} is any finite projective module over A . The generalization of our approach to infinite dimensional algebras is postponed to future work. Given a connection ∇ on \mathcal{E} , we can explicitly construct for every derivation X a parallel transport morphism $\Phi^{(\nabla, X)}$, satisfying a differential equation determined by the covariant derivative ∇_X along X . Using the set of all parallel transport morphisms we define a suitable set of basic gauge invariant observables, i.e. functions from the space of connections to the complex numbers. Given any hermitian connection on a hermitian finite projective module \mathcal{E} over $A = M_n(\mathbb{C})$, we prove that it can be reconstructed up to gauge equivalence from this set of observables.

The structure of this article is as follows: In Section 2 we introduce briefly the basics on derivation-based differential calculi. For a simplified presentation we first discuss in Section 3 parallel transports on the trivial line-bundle over $A = M_n(\mathbb{C})$, i.e. the free module $\mathcal{E} = A$, and then generalize the results in Section 4 to arbitrary finite projective modules.

2 Preliminaries on derivation-based differential calculi

Let A be a unital and associative algebra over \mathbb{C} , which is not necessarily commutative. In [4] a general and purely algebraic framework to associate to A a differential calculus was introduced. This approach was named the derivation-based differential calculus and has been further developed in [5, 6], see also the reviews [7, 8, 9].

The basic idea is to consider the space of derivations $\text{Der}(A)$ of the algebra A , that is the \mathbb{C} -vector space of all \mathbb{C} -linear maps $X : A \rightarrow A$ satisfying, for all $a, b \in A$,

$$X(ab) = X(a)b + aX(b) \quad (2.1)$$

and build up a notion of differential geometry on A by carefully generalizing the algebraic approach to differential geometry of commutative smooth manifolds, see e.g. Koszul's book [19]. We can equip the vector space $\text{Der}(A)$ with the structure of a Lie algebra by defining, for all $X, Y \in \text{Der}(A)$, the Lie bracket by the commutator,

$$[X, Y] := X \circ Y - Y \circ X, \quad (2.2)$$

where \circ denotes the usual composition of endomorphisms $\text{End}_{\mathbb{C}}(A)$. Furthermore, $\text{Der}(A)$ is a module over the center $\mathcal{Z}(A)$ of the algebra A by defining, for all $a \in A$, $f \in \mathcal{Z}(A)$ and $X \in \text{Der}(A)$,

$$(f \cdot X)(a) := fX(a). \quad (2.3)$$

In case A is a $*$ -algebra with involution $*$: $A \rightarrow A$, $a \mapsto a^*$, we say that a derivation $X \in \text{Der}(A)$ is real if, for all $a \in A$,

$$X(a)^* = X(a^*). \quad (2.4)$$

The space of real derivations is denoted by $\text{Der}_{\mathbb{R}}(A)$. It forms a Lie algebra over \mathbb{R} and a module over the real subalgebra $\mathcal{Z}(A)_{\mathbb{R}} := \{a \in \mathcal{Z}(A) : a^* = a\}$.

The next step in the construction of a differential geometry on A is the definition of a differential calculus, i.e. a graded differential algebra associated to A .

Definition 1. Let $\mathfrak{g} \subseteq \text{Der}(A)$ be a sub Lie algebra and a sub $\mathcal{Z}(A)$ -module. Let further $\underline{\Omega}_{\mathfrak{g}}^n(A)$, for all $n \in \mathbb{N}$, be the vector space of $\mathcal{Z}(A)$ -multilinear antisymmetric maps from \mathfrak{g}^n to A , $\underline{\Omega}_{\mathfrak{g}}^0(A) := A$ and

$$\underline{\Omega}_{\mathfrak{g}}^{\bullet}(A) := \bigoplus_{n \geq 0} \underline{\Omega}_{\mathfrak{g}}^n(A). \quad (2.5)$$

The vector space $\underline{\Omega}_{\mathfrak{g}}^{\bullet}(A)$ can be equipped with the structure of an \mathbb{N}_0 -graded differential algebra by defining the product, for all $\omega \in \underline{\Omega}_{\mathfrak{g}}^p(A)$, $\eta \in \underline{\Omega}_{\mathfrak{g}}^q(A)$ and $X_1, \dots, X_{p+q} \in \mathfrak{g}$,

$$(\omega \eta)(X_1, \dots, X_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{\text{sign}(\sigma)} \omega(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \eta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}) \quad (2.6a)$$

and the differential (of degree 1), for all $\omega \in \underline{\Omega}_{\mathfrak{g}}^p(A)$ and $X_1, \dots, X_{p+1} \in \mathfrak{g}$,

$$\begin{aligned} d\omega(X_1, \dots, X_{p+1}) &:= \sum_{i=1}^{p+1} (-1)^{i+1} X_i \left(\omega(X_1, \dots, \overset{i}{\dot{}}, \dots, X_{p+1}) \right) \\ &\quad + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i, X_j], \dots, \overset{i}{\dot{}} \dots \overset{j}{\dot{}} \dots, X_{p+1}) . \end{aligned} \quad (2.6b)$$

We call $(\underline{\Omega}_{\mathfrak{g}}^{\bullet}(A), d)$ the \mathfrak{g} -restricted derivation-based differential calculus on A .

Note that the differential calculus $(\underline{\Omega}_{\mathfrak{g}}^{\bullet}(A), d)$ depends on the choice of \mathfrak{g} . The prime example indicating the importance of allowing for a proper sub Lie algebra and sub $\mathcal{Z}(A)$ -module $\mathfrak{g} \subset \text{Der}(A)$ is the fuzzy sphere [11], where $\mathfrak{g} = \mathfrak{su}_2^{\mathbb{C}} \subset \text{Der}(A)$ is the complexification of \mathfrak{su}_2 .

Given a $*$ -algebra A with involution $*$: $A \rightarrow A$, we induce an involution on $\text{Der}(A)$ by defining, for all $X \in \text{Der}(A)$ and $a \in A$, $X^*(a) := X(a^*)^*$. We assume that $\mathfrak{g} \subseteq \text{Der}(A)$ is such that $*$ restricts to an involution on \mathfrak{g} . We induce further a graded involution on $\underline{\Omega}_{\mathfrak{g}}^{\bullet}(A)$ by defining, for all $\omega \in \underline{\Omega}_{\mathfrak{g}}^p(A)$ and $X_1, \dots, X_p \in \mathfrak{g}$, $\omega^*(X_1, \dots, X_p) := \omega(X_1^*, \dots, X_p^*)^*$. As a consequence, we have, for all $\omega \in \underline{\Omega}_{\mathfrak{g}}^p(A)$ and $\eta \in \underline{\Omega}_{\mathfrak{g}}^q(A)$,

$$(\omega \eta)^* = (-1)^{pq} \eta^* \omega^* \quad , \quad (d\omega)^* = d\omega^* . \quad (2.7)$$

We now restrict our attention to the case relevant for this paper, namely the algebra $A = M_n(\mathbb{C})$ of complex-valued $n \times n$ -matrices with $n \in \mathbb{N}$. Hermitian conjugation structures A as a $*$ -algebra. This model has been studied in detail in [10, 7, 8, 9] and we only summarize the main properties. The center of A is isomorphic to \mathbb{C} , via $\mathbb{C} \rightarrow \mathcal{Z}(A)$, $\lambda \mapsto \lambda \mathbb{1}$, with $\mathbb{1} \in A$ denoting the unit. Furthermore, all derivations $X \in \text{Der}(A)$ are inner, meaning that the Lie algebra of traceless complex-valued matrices $\mathfrak{sl}_n \subseteq A$ is isomorphic (as a Lie algebra) to $\text{Der}(A)$ via the map $\text{ad} : \mathfrak{sl}_n \rightarrow \text{Der}(A)$, $\gamma \mapsto \text{ad}_{\gamma}$, where $\text{ad}_{\gamma} \in \text{Der}(A)$ is the derivation defined by, for all $a \in A$, $\text{ad}_{\gamma}(a) := [\gamma, a]$. We denote the inverse of the map ad by $\theta : \text{Der}(A) \rightarrow \mathfrak{sl}_n \subset A$. With this identification, the Lie algebra (over \mathbb{R}) of real derivations $\text{Der}_{\mathbb{R}}(A)$ is isomorphic to the real sub Lie algebra $\mathfrak{su}_n \subset \mathfrak{sl}_n$ of antihermitian and traceless matrices. For the \mathfrak{g} -restricted derivation-based differential calculus we have $\underline{\Omega}_{\mathfrak{g}}^{\bullet}(A) \simeq A \otimes_{\mathbb{C}} \bigwedge^{\bullet} \mathfrak{g}^*$, where \mathfrak{g}^* is the \mathbb{C} -vector space dual of \mathfrak{g} . There is a canonical one-form given by restricting the map $\theta : \text{Der}(A) \rightarrow \mathfrak{sl}_n \subset A$ to $\mathfrak{g} \subseteq \text{Der}(A)$. For notational simplicity we use the same symbol θ for the canonical one-form $\underline{\Omega}_{\mathfrak{g}}^1(A) \ni \theta : \mathfrak{g} \rightarrow A$. The one-form $\theta \in \underline{\Omega}_{\mathfrak{g}}^1(A)$ satisfies the Maurer-Cartan equation $d\theta - \theta^2 = 0$ and its adjoint action on A is the differential, for all $a \in A$, $da = [\theta, a]$.

3 Trivial line-bundle over $A = M_n(\mathbb{C})$

For a simplified presentation we first consider the case where the finite projective module is A itself, $\mathcal{E} = A = M_n(\mathbb{C})$. The general case is discussed in Section 4. The right A -action on \mathcal{E} is simply given by matrix multiplication from the right, $\mathcal{E} \times A \rightarrow \mathcal{E}$, $(s, a) \mapsto sa$. The endomorphism algebra of the right A -module \mathcal{E} is $\text{End}_A(\mathcal{E}) \simeq A$, which acts on \mathcal{E} by matrix multiplication from the left, $A \times \mathcal{E} \rightarrow \mathcal{E}$, $(a, s) \mapsto as$.

We equip the right A -module \mathcal{E} with a hermitian structure $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow A$ by defining, for all $s, t \in \mathcal{E}$,

$$\langle s, t \rangle := s^* t , \quad (3.1)$$

where $*$ acts on $\mathcal{E} = A$ by hermitian conjugation. We say that an endomorphism $T \in \text{End}_A(\mathcal{E})$ is hermitian if, for all $s, t \in \mathcal{E}$, $\langle s, T(t) \rangle = \langle T(s), t \rangle$. Employing the isomorphism $\text{End}_A(\mathcal{E}) \simeq A$, hermitian endomorphisms correspond to hermitian matrices. Analogously, antihermitian endomorphisms correspond to antihermitian matrices.

In general, a connection on a right A -module \mathcal{E} is a \mathbb{C} -linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_A \underline{\Omega}_{\mathfrak{g}}^1(A)$ satisfying, for all $s \in \mathcal{E}$ and $a \in A$,

$$\nabla(sa) = (\nabla s)a + s \otimes_A da . \quad (3.2)$$

Since $\underline{\Omega}_{\mathfrak{g}}^1(A) \simeq A \otimes_{\mathbb{C}} \mathfrak{g}^*$ there is the following isomorphism of right A -modules

$$\mathcal{E} \otimes_A \underline{\Omega}_{\mathfrak{g}}^1(A) \simeq \mathcal{E} \otimes_A (A \otimes_{\mathbb{C}} \mathfrak{g}^*) \simeq \mathcal{E} \otimes_{\mathbb{C}} \mathfrak{g}^* . \quad (3.3)$$

Thus, we can equivalently regard a connection as a \mathbb{C} -linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathbb{C}} \mathfrak{g}^*$ satisfying, for all $s \in \mathcal{E}$ and $a \in A$, $\nabla(sa) = (\nabla s)a + s da$. The space of all connections on \mathcal{E} is denoted by $\text{Con}_A(\mathcal{E})$ and it forms an affine space over $\text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes_{\mathbb{C}} \mathfrak{g}^*) \simeq \text{End}_A(\mathcal{E}) \otimes_{\mathbb{C}} \mathfrak{g}^* \simeq A \otimes_{\mathbb{C}} \mathfrak{g}^*$.

Associated to $\nabla \in \text{Con}_A(\mathcal{E})$ there is the covariant derivative along any $X \in \mathfrak{g}$ defined by the \mathbb{C} -linear map $\nabla_X : \mathcal{E} \rightarrow \mathcal{E}$, $s \mapsto \nabla_X s = (\nabla s)(X)$, where the last term denotes the canonical evaluation of $\mathcal{E} \otimes_{\mathbb{C}} \mathfrak{g}^*$ on \mathfrak{g} . The covariant derivative has the following properties, for all $f, h \in \mathcal{Z}(A)$, $X, Y \in \mathfrak{g}$, $s \in \mathcal{E}$ and $a \in A$,

$$\nabla_f X + h Y s = f \nabla_X s + h \nabla_Y s, \quad \nabla_X(sa) = (\nabla_X s)a + s X(a) . \quad (3.4)$$

From the covariant derivative ∇_X along all $X \in \mathfrak{g}$ we can reconstruct the connection $\nabla \in \text{Con}_A(\mathcal{E})$.

It is well-known [10, 7, 8, 9] that there exists a connection on \mathcal{E} , the so-called canonical connection, defined by, for all $s \in \mathcal{E}$,

$$\nabla^\theta s := -s \theta , \quad (3.5)$$

where $\theta \in \underline{\Omega}_{\mathfrak{g}}^1(A)$ is the canonical one-form. Since $\text{Con}_A(\mathcal{E})$ is an affine space over $A \otimes_{\mathbb{C}} \mathfrak{g}^*$, any connection $\nabla \in \text{Con}_A(\mathcal{E})$ can be written as, for all $s \in \mathcal{E}$,

$$\nabla s = \nabla^\theta s + B s = -s \theta + B s , \quad (3.6)$$

where $B \in A \otimes_{\mathbb{C}} \mathfrak{g}^*$ is the gauge potential relative to ∇^θ .

A connection is compatible with a hermitian structure if, for all $s, t \in \mathcal{E}$ and $X \in \mathfrak{g}$,

$$X(\langle s, t \rangle) = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle . \quad (3.7)$$

The canonical connection ∇^θ is compatible with the hermitian structure (3.1) and a general connection $\nabla \in \text{Con}_A(\mathcal{E})$ is compatible if and only if the gauge potential $B \in A \otimes_{\mathbb{C}} \mathfrak{g}^*$ relative to ∇^θ is antihermitian.

For all $X, Y \in \mathfrak{g}$ we define the curvature endomorphism $F(X, Y) \in \text{End}_A(\mathcal{E})$ by, for all $s \in \mathcal{E}$,

$$F(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s . \quad (3.8)$$

Decomposing ∇ as in (3.6) into the canonical connection and a gauge potential one obtains after also using the isomorphism $\text{End}_A(\mathcal{E}) \simeq A$ the curvature 2-form

$$F(X, Y) = [B(X), B(Y)] - B([X, Y]) . \quad (3.9)$$

Notice that the connection is flat, i.e. of vanishing curvature, if and only if the map $B : \mathfrak{g} \rightarrow A$ is a Lie algebra representation of \mathfrak{g} . The curvature 2-form of a connection compatible with the hermitian structure (3.1) is antihermitian.

We next introduce gauge transformations. Let us denote by $\mathcal{U}_{\mathcal{E}} \subseteq \text{End}_A(\mathcal{E})$ the group of unitary endomorphisms of the module \mathcal{E} , i.e. for all $u \in \mathcal{U}_{\mathcal{E}}$ and all $s, t \in \mathcal{E}$,

$$\langle u(s), u(t) \rangle = \langle s, t \rangle . \quad (3.10)$$

Employing the isomorphism $\text{End}_A(\mathcal{E}) \simeq A = M_n(\mathbb{C})$ we find that the group $\mathcal{U}_{\mathcal{E}}$ is isomorphic to the group of unitary $n \times n$ -matrices, $\mathcal{U}_{\mathcal{E}} \simeq U_n \subseteq M_n(\mathbb{C})$. We define an action of $\mathcal{U}_{\mathcal{E}}$ on the affine space of connections

$$\mathcal{U}_{\mathcal{E}} \times \text{Con}_A(\mathcal{E}) \rightarrow \text{Con}_A(\mathcal{E}), \quad (u, \nabla) \mapsto \nabla^u = (u \otimes_{\mathbb{C}} \text{id}_{\mathfrak{g}^*}) \circ \nabla \circ u^* . \quad (3.11)$$

For the covariant derivative this action simply reads, for all $X \in \mathfrak{g}$, $(u, \nabla_X) \mapsto u \circ \nabla_X \circ u^*$, and the curvature transforms, for all $X, Y \in \mathfrak{g}$, as $(u, F(X, Y)) \mapsto u F(X, Y) u^*$, where we have implicitly used the isomorphism $\mathcal{U}_{\mathcal{E}} \simeq U_n$. Notice that the canonical connection ∇^θ is gauge invariant and that the gauge potential $B \in A \otimes_{\mathbb{C}} \mathfrak{g}^*$ (relative to ∇^θ) transforms, for all $X \in \mathfrak{g}$, as

$$(u, B(X)) \mapsto u B(X) u^* . \quad (3.12)$$

The following proposition taken from [10, 7, 8, 9] provides the main motivation for our studies on parallel transport observables to be presented below.

Proposition 2. *Let $A = M_n(\mathbb{C})$ and let $\mathcal{E} = A$ be equipped with the hermitian structure (3.1). Then gauge equivalence classes of flat hermitian connections on \mathcal{E} are in 1-to-1 and onto correspondence with unitary inequivalent Lie algebra representations $B : \mathfrak{g} \rightarrow A$.*

Proof. Follows immediately from (3.9) and (3.12). \square

Considering the prime example of the fuzzy sphere [11], where $A = M_n(\mathbb{C})$, $n = 2j + 1$ (j being the maximal spin) and $\mathfrak{g} = \mathfrak{su}_2^{\mathbb{C}}$, there are the unitary inequivalent representations $B : \mathfrak{su}_2^{\mathbb{C}} \rightarrow M_n(\mathbb{C})$ of spin $s = 0, \frac{1}{2}, \dots, j$. Since all of them have the same curvature, namely zero, we can not reconstruct the gauge equivalence class of the connection from curvature observables, i.e. gauge invariant functions $\text{Con}_A(\mathcal{E}) \rightarrow \mathbb{C}$ constructed out of F .

The problem of not being able to extract the gauge equivalence class of the connection from curvature observables also exists in the commutative setting, where A is the algebra of smooth functions on a manifold \mathcal{M} and \mathcal{E} is the A -module of smooth sections of a hermitian vector bundle, see e.g. [14, 15]. The solution there is to consider parallel transports along curves: Given a curve $\gamma : [0, 1] \rightarrow \mathcal{M}$ starting at x and ending at y , a connection on \mathcal{E} determines a parallel transport morphism $h_\gamma : E_x \rightarrow E_y$ from the fibre over x to the fibre over y . For closed curves, i.e. $y = x$, the trace over the endomorphism $h_\gamma : E_x \rightarrow E_x$ is gauge invariant and it can be shown that for certain gauge groups, these Wilson loops determine the connection up to gauge equivalence, see e.g. [15, 16] and references therein. A naive generalization of the concept of parallel transport along curves to noncommutative geometry seems to be problematic, since firstly we in general do not have the notion of curves and secondly there is no fibre over a “point” due to the lack of localization. See, however, [17, 18] for an interesting definition of Wilson loop observables for noncommutative gauge theory on Moyal-Weyl deformed spaces.

The plan for the rest of this section is as follows: We first give a definition for parallel transports along one-parameter groups of algebra automorphisms on modules \mathcal{E} over general associative and unital algebras A . In the second step we show that for $A = M_n(\mathbb{C})$, equipped with a \mathfrak{g} -restricted derivation-based differential calculus, and $\mathcal{E} = A$, equipped with a connection, such parallel transport morphisms can be constructed explicitly. In the last step we prove that a hermitian connection can be reconstructed up to gauge equivalence from a certain set of gauge invariant observables constructed from module parallel transports.

Definition 3. Let A be an associative and unital algebra and \mathcal{E} a right A -module.

- 1.) A one-parameter group of automorphisms of A is a map $\varphi : \mathbb{R} \times A \rightarrow A$, $(\tau, a) \mapsto \varphi(\tau, a) = \varphi_\tau(a)$, such that
 - (i) $\varphi_\tau(ab) = \varphi_\tau(a)\varphi_\tau(b)$, for all $\tau \in \mathbb{R}$ and $a, b \in A$
 - (ii) $\varphi_0 = \text{id}_A$
 - (iii) $\varphi_{\tau+\sigma} = \varphi_\tau \circ \varphi_\sigma$, for all $\tau, \sigma \in \mathbb{R}$
- 2.) Let $\varphi : \mathbb{R} \times A \rightarrow A$ be a one-parameter group of automorphisms of A . A parallel transport on \mathcal{E} along φ is a map $\Phi : \mathbb{R} \times \mathcal{E} \rightarrow \mathcal{E}$, $(\tau, s) \mapsto \Phi(\tau, s) = \Phi_\tau(s)$, such that
 - (i) $\Phi_\tau(sa) = \Phi_\tau(s)\varphi_\tau(a)$, for all $\tau \in \mathbb{R}$, $s \in \mathcal{E}$ and $a \in A$
 - (ii) $\Phi_0 = \text{id}_{\mathcal{E}}$
 - (iii) $\Phi_{\tau+\sigma} = \Phi_\tau \circ \Phi_\sigma$, for all $\tau, \sigma \in \mathbb{R}$

If A and \mathcal{E} are equipped with a smooth structure, the maps φ and Φ are defined to be smooth.

Let us investigate these structures in detail for $A = M_n(\mathbb{C})$ and $\mathcal{E} = A$. In this case there is a bijection between smooth one-parameter groups of automorphism and derivations $\text{Der}(A)$ given by the matrix exponential ($\text{Der}(A) \subset \text{End}_{\mathbb{C}}(A) \simeq M_{n^2}(\mathbb{C})$ can be seen as complex-valued $n^2 \times n^2$ -matrices). We associate to any $X \in \text{Der}(A)$ the one-parameter group of automorphisms $\varphi^X : \mathbb{R} \times A \rightarrow A$, $(\tau, a) \mapsto \varphi_\tau^X(a) = e^{\tau X}(a)$. The map φ^X is uniquely specified by the differential equation

$$\frac{d}{d\tau} \varphi_\tau^X = X \circ \varphi_\tau^X, \quad (3.13)$$

together with the initial condition $\varphi_0^X = \text{id}_A$, which is already part of Definition 3. The derivation X can be reconstructed from φ^X by taking the τ -derivative at $\tau = 0$. Furthermore, using the canonical

one-form $\theta \in \underline{\Omega}_{\mathfrak{g}}^1(A)$, we can express, for all $X \in \mathfrak{g} \subseteq \text{Der}(A)$ and $a \in A$, the derivation as the difference between left and right multiplication by $\theta(X)$, $X(a) = [\theta(X), a] = \theta(X)a - a\theta(X)$. Since left and right multiplication commutes, the exponential becomes, for all $X \in \mathfrak{g}$, $a \in A$ and $\tau \in \mathbb{R}$,

$$\varphi_\tau^X(a) = e^{\tau\theta(X)} a e^{-\tau\theta(X)}. \quad (3.14)$$

Notice that φ^X is a one-parameter group of $*$ -automorphism if and only if X is a real derivation.

Proposition 4. *Let $A = M_n(\mathbb{C})$, $\mathcal{E} = A$ and $\nabla \in \text{Con}_A(\mathcal{E})$. Then there exists for all $X \in \mathfrak{g} \subseteq \text{Der}(A)$ a unique parallel transport $\Phi^{(\nabla, X)} : \mathbb{R} \times \mathcal{E} \rightarrow \mathcal{E}$ along φ^X satisfying the differential equation*

$$\frac{d}{d\tau} \Phi_\tau^{(\nabla, X)} = \nabla_X \circ \Phi_\tau^{(\nabla, X)}. \quad (3.15)$$

It is given by, for all $X \in \mathfrak{g}$, $s \in \mathcal{E}$ and $\tau \in \mathbb{R}$,

$$\Phi_\tau^{(\nabla, X)}(s) = e^{\tau B(X)} s e^{-\tau\theta(X)}, \quad (3.16)$$

where $B \in A \otimes_{\mathbb{C}} \mathfrak{g}^*$ is the gauge potential relative to the canonical connection ∇^θ .

Proof. The differential equation (3.15) subject to the initial condition $\Phi_0^{(\nabla, X)} = \text{id}_{\mathcal{E}}$, which is already part of Definition 3, has a unique solution given by

$$\Phi_\tau^{(\nabla, X)} = e^{\tau \nabla_X}, \quad (3.17)$$

where we regard $\nabla_X : \mathcal{E} \rightarrow \mathcal{E}$ as an element of $\text{End}_{\mathbb{C}}(\mathcal{E}) \simeq M_{n^2}(\mathbb{C})$. Using the decomposition (3.6) we have, for all $X \in \mathfrak{g}$ and $s \in \mathcal{E}$, $\nabla_X(s) = -s\theta(X) + B(X)s$. Thus, for all $X \in \mathfrak{g}$, $s \in \mathcal{E}$ and $\tau \in \mathbb{R}$,

$$\Phi_\tau^{(\nabla, X)}(s) = e^{\tau B(X)} s e^{-\tau\theta(X)}. \quad (3.18)$$

From (3.16) and (3.14) one easily checks that $\Phi^{(\nabla, X)}$ is a parallel transport along φ^X , i.e. that it satisfies the conditions posed in Definition 3. \square

Notice that from $\Phi^{(\nabla, X)}$ the covariant derivative ∇_X can be reconstructed by taking the τ -derivative at $\tau = 0$. Thus, the connection can be reconstructed from the set $\{\Phi^{(\nabla, X)}\}_{X \in \mathfrak{g}}$.

We come to the definition of a class of gauge invariant observables, i.e. gauge invariant functions $\text{Con}_A(\mathcal{E}) \rightarrow \mathbb{C}$, from which we can reconstruct up to gauge equivalence the connection ∇ . We can construct for any $\tau \in \mathbb{R}$ and $X \in \mathfrak{g}$ an endomorphism $\widehat{\Phi}_\tau^{(\nabla, X)} \in \text{End}_A(\mathcal{E})$ by composing $\Phi_\tau^{(\nabla, X)}$ with the right multiplication of $e^{\tau\theta(X)} \in A$, for all $\tau \in \mathbb{R}$, $X \in \mathfrak{g}$ and $s \in \mathcal{E}$,

$$\widehat{\Phi}_\tau^{(\nabla, X)}(s) := \Phi_\tau^{(\nabla, X)}(s) e^{\tau\theta(X)} \stackrel{(3.16)}{=} e^{\tau B(X)} s. \quad (3.19)$$

Applying the trace $\text{Tr}_{\text{End}} : \text{End}_A(\mathcal{E}) \rightarrow \mathbb{C}$, $T \mapsto \text{Tr}_{\text{End}}(T) = \text{Tr}_A(T(\mathbb{1}))$ on $\widehat{\Phi}_\tau^{(\nabla, X)}$, we obtain the gauge invariant observables

$$\mathcal{W}_{(\tau, X)} : \text{Con}_A(\mathcal{E}) \rightarrow \mathbb{C}, \quad \nabla \mapsto \mathcal{W}_{(\tau, X)}(\nabla) = \text{Tr}_{\text{End}}(\widehat{\Phi}_\tau^{(\nabla, X)}) = \text{Tr}_A(e^{\tau B(X)}), \quad (3.20)$$

labelled by $(\tau, X) \in \mathbb{R} \times \mathfrak{g}$. Note that for defining the observables (3.20) we *did not* require the flow φ^X to be periodic in τ . This is similar to the gauge invariant *open* Wilson lines discovered in noncommutative gauge theory on the Moyal-Weyl space, see e.g. [17, 18].

We now will prove that we can reconstruct the connection up to gauge equivalence from the set of observables

$$\mathcal{O} := \left\{ \mathcal{W}_X := \mathcal{W}_{(1, X)} : X \in \mathfrak{g}_{\mathbb{R}} \right\}, \quad (3.21)$$

labelled by the real sub Lie algebra $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$ of real derivations.

Theorem 5. *Let $A = M_n(\mathbb{C})$ and let $\mathcal{E} = A$ be equipped with the hermitian structure (3.1). Then any hermitian connection $\nabla \in \text{Con}_A(\mathcal{E})$ can be reconstructed up to gauge equivalence (3.11) from the set of observables \mathcal{O} (3.21).*

Proof. Since we can reconstruct ∇ from the set $\{\nabla_X : X \in \mathfrak{g}_{\mathbb{R}}\}$ it is sufficient to show that we can reconstruct for every $X \in \mathfrak{g}_{\mathbb{R}}$ the covariant derivative ∇_X up to gauge equivalence from \mathcal{O} . Let $X \in \mathfrak{g}_{\mathbb{R}}$ be fixed and consider the corresponding one-parameter family of observables $\mathcal{W}_{\tau X}$, $\tau \in \mathbb{R}$. Since X is a real derivation and the gauge potential B relative to ∇^θ of any hermitian connection is antihermitian, we have that $B(X)^* = -B(X) \in M_n(\mathbb{C})$ is an antihermitian matrix. We can find a unitary matrix $u \in U_n \subset M_n(\mathbb{C})$ such that $B(X) = u D u^*$, where D is a diagonal matrix with imaginary entries $D_{jj} = i \lambda_j$, $\lambda_j \in \mathbb{R}$ for all $j = 1, \dots, n$. The observable $\mathcal{W}_{\tau X}$ thus becomes

$$\mathcal{W}_{\tau X}(\nabla) = \text{Tr}_A(e^{\tau D}) = \sum_{j=1}^n e^{i\tau \lambda_j}, \quad (3.22)$$

and we can reconstruct the spectrum of $B(X)$, i.e. $\{\lambda_j : j = 1, \dots, n\}$, via Fourier transformation in τ of $\mathcal{W}_{\tau X}$. The proof follows by noting that two antihermitian matrices $B(X), \tilde{B}(X) \in A$ are unitary equivalent (i.e. the corresponding covariant derivatives are gauge equivalent) if and only if they have the same spectrum. \square

4 General finite projective modules over $A = M_n(\mathbb{C})$

We now generalize the results of Section 3 to general finite projective modules over $A = M_n(\mathbb{C})$. Remember that any finite projective right A -module \mathcal{E} is isomorphic to a right A -module of the form pA^N , with $N \in \mathbb{N}$ being the dimension of an underlying free module and $p \in M_N(A)$ being a projector, i.e. $p^2 = p$. For our studies the following lemma turns out to be useful.

Lemma 6. *Let $A = M_n(\mathbb{C})$ and \mathcal{E} be any finite projective right A -module. Then there exists a $m \in \mathbb{N}$, such that $\mathcal{E} \simeq M_{m,n}(\mathbb{C})$, with $M_{m,n}(\mathbb{C})$ being the space of complex-valued $m \times n$ -matrices. The right A -action on the module $M_{m,n}(\mathbb{C})$ is given by matrix multiplication from the right, $M_{m,n}(\mathbb{C}) \times M_n(\mathbb{C}) \rightarrow M_{m,n}(\mathbb{C})$, $(s, a) \mapsto s a$.*

Proof. Let \mathcal{E} be any right A -module. Then there exists a $N \in \mathbb{N}$ and a projector $p \in M_N(A) \simeq \text{End}_A(A^N)$, with $p^2 = p$, such that $\mathcal{E} \simeq pA^N$. The right A -module structure on the free module A^N (and also on the projective module pA^N , since it is preserved by the projection) is given by component-wise matrix multiplication from the right, i.e. $A^N \times A \rightarrow A^N$, $(\bar{a}, b) \mapsto \bar{a} b = (a_1 b, \dots, a_N b)^T$, with T denoting the transposition operation. For $A = M_n(\mathbb{C})$ we have to following vector space isomorphisms

$$A^N \simeq A \otimes_{\mathbb{C}} \mathbb{C}^N \simeq \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^N \simeq (\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^N) \otimes_{\mathbb{C}} \mathbb{C}^n \simeq \mathbb{C}^{nN} \otimes_{\mathbb{C}} \mathbb{C}^n \simeq M_{nN,n}(\mathbb{C}). \quad (4.1)$$

The induced right A -module structure on $M_{nN,n}(\mathbb{C})$ is given by matrix multiplication from the right, $M_{nN,n}(\mathbb{C}) \times M_n(\mathbb{C}) \rightarrow M_{m,n}(\mathbb{C})$, $(s, a) \mapsto s a$. Furthermore, the induced left $\text{End}_A(A^N) \simeq M_N(A)$ -module structure is given by matrix multiplication of $nN \times nN$ -matrices $M_{nN}(\mathbb{C})$ from the left, $M_{nN}(\mathbb{C}) \times M_{nN,n}(\mathbb{C}) \rightarrow M_{nN,n}(\mathbb{C})$, $(T, s) \mapsto T s$. The projector $p \in M_N(A)$ becomes under this identification a complex-valued $nN \times nN$ -matrix (denoted by the same symbol) and we denote by $V = p\mathbb{C}^{nN}$ the vector space obtained by this projection acting on \mathbb{C}^{nN} . Being a finite dimensional complex vector space, we have $V \simeq \mathbb{C}^m$ for some $m \in \mathbb{N}$ with $m \leq nN$. Thus,

$$pA^N \simeq (p\mathbb{C}^{nN}) \otimes_{\mathbb{C}} \mathbb{C}^n \simeq \mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^n \simeq M_{m,n}(\mathbb{C}). \quad (4.2)$$

\square

As a result of Lemma 6 is it sufficient to study the case $\mathcal{E} = M_{m,n}(\mathbb{C})$, $m \in \mathbb{N}$, in order to cover (up to isomorphism) all finite projective modules over $A = M_n(\mathbb{C})$. The endomorphism algebra $\text{End}_A(\mathcal{E})$ of \mathcal{E} is isomorphic to the complex-valued $m \times m$ -matrices, $\text{End}_A(\mathcal{E}) \simeq M_m(\mathbb{C})$, which act on \mathcal{E} simply by matrix multiplication from the left, $M_m(\mathbb{C}) \times M_{m,n}(\mathbb{C}) \rightarrow M_{m,n}(\mathbb{C})$, $(T, s) \mapsto T s$.

Analogously to (3.1), we equip $\mathcal{E} = M_{m,n}(\mathbb{C})$ with the hermitian structure $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow A$ defined by, for all $s, t \in \mathcal{E}$,

$$\langle s, t \rangle := s^* t, \quad (4.3)$$

where $*$ acts on \mathcal{E} by hermitian conjugation. Under the isomorphism $\text{End}_A(\mathcal{E}) \simeq M_m(\mathbb{C})$ (anti)hermitian endomorphisms correspond to (anti)hermitian matrices.

The discussion of connections on $\mathcal{E} = A = M_n(\mathbb{C})$ in Section 3 generalizes trivially to the case $\mathcal{E} = M_{m,n}(\mathbb{C})$. In particular, there is a canonical connection defined by, for all $s \in \mathcal{E}$,

$$\nabla^\theta s := -s\theta, \quad (4.4)$$

where $\theta \in \underline{\Omega}_{\mathfrak{g}}^1(A)$ is the canonical one-form. Due to the affine space structure of $\text{Con}_A(\mathcal{E})$, we find for any connection $\nabla \in \text{Con}_A(\mathcal{E})$ a gauge potential $B \in M_m(\mathbb{C}) \otimes_{\mathbb{C}} \mathfrak{g}^* \simeq \text{End}_A(\mathcal{E}) \otimes_{\mathbb{C}} \mathfrak{g}^*$ relative to ∇^θ , such that, for all $s \in \mathcal{E}$,

$$\nabla s = \nabla^\theta s + B s = -s\theta + B s. \quad (4.5)$$

The canonical connection ∇^θ is compatible with the hermitian structure (4.3) and a generic connection $\nabla \in \text{Con}_A(\mathcal{E})$ is hermitian, if and only if the gauge potential B relative to ∇^θ is antihermitian. Gauge transformations of \mathcal{E} are defined analogously to (3.11) and can be identified with unitary $m \times m$ -matrices, $\mathcal{U}_{\mathcal{E}} \simeq U_m \subset M_m(\mathbb{C})$. For any $X \in \mathfrak{g}$ the gauge potential relative to ∇^θ transforms as

$$(u, B(X)) \mapsto u B(X) u^*. \quad (4.6)$$

Proposition 4 on the existence of parallel transports generalizes easily to the case $\mathcal{E} = M_{m,n}(\mathbb{C})$, resulting in the maps, for all $X \in \mathfrak{g}$,

$$\Phi^{(\nabla, X)} : \mathbb{R} \times \mathcal{E} \rightarrow \mathcal{E}, \quad (\tau, s) \mapsto \Phi_\tau^{(\nabla, X)}(s) = e^{\tau B(X)} s e^{-\tau \theta(X)}. \quad (4.7)$$

Defining the endomorphisms $\widehat{\Phi}_\tau^{(\nabla, X)} \in \text{End}_A(\mathcal{E}) \simeq M_m(\mathbb{C})$ analogously to (3.19) by, for all $\tau \in \mathbb{R}$, $X \in \mathfrak{g}$ and $s \in \mathcal{E}$,

$$\widehat{\Phi}_\tau^{(\nabla, X)}(s) := \Phi_\tau^{(\nabla, X)}(s) e^{\tau \theta(X)} = e^{\tau B(X)} s, \quad (4.8)$$

we obtain the gauge invariant observables, for all $\tau \in \mathbb{R}$ and $X \in \mathfrak{g}$,

$$\mathcal{W}_{(\tau, X)} : \text{Con}_A(\mathcal{E}) \rightarrow \mathbb{C}, \quad \nabla \mapsto \mathcal{W}_{(\tau, X)}(\nabla) = \text{Tr}_{\text{End}}(\widehat{\Phi}_\tau^{(\nabla, X)}) = \text{Tr}_{M_m(\mathbb{C})}(e^{\tau B(X)}). \quad (4.9)$$

Analogously to Theorem 5 we obtain that also in the general finite projective module case we can reconstruct the connection up to gauge equivalence from the set of observables

$$\mathcal{O} := \left\{ \mathcal{W}_X := \mathcal{W}_{(1, X)} : X \in \mathfrak{g}_{\mathbb{R}} \right\}, \quad (4.10)$$

labelled by the real sub Lie algebra $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$ of real derivations.

Theorem 7. *Let $A = M_n(\mathbb{C})$ and let \mathcal{E} be a finite projective right A -module, equipped with the hermitian structure (4.3). Then any hermitian connection $\nabla \in \text{Con}_A(\mathcal{E})$ can be reconstructed up to gauge equivalence (4.6) from the set of observables \mathcal{O} (4.10).*

Proof. The proof is a straightforward generalization of the one of Theorem 5. Just replace the $n \times n$ -matrices $B(X)$ by $m \times m$ -matrices and U_n gauge transformations by U_m gauge transformations. \square

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