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Abstract

We consider dynamical systems consisting of autonomous ordinary differential equations. Uncertainties of involved parameters are modelled by random variables. The random-dependent models are resolved by a stochastic Galerkin method based on the expansions of the polynomial chaos. We focus on the equilibria of the dynamical systems, which are defined as solutions of a nonlinear system of algebraic equations. The stochastic Galerkin method yields a larger coupled system. We prove the convergence of the Galerkin scheme under certain assumptions. Moreover, the stability of the equilibria for the coupled system is analysed with respect to the stability of the equilibria for the original dynamical systems. We present numerical simulations of corresponding examples, which confirm the convergence and stability properties.

Key words: ordinary differential equations, equilibrium, stability, random variables, polynomial chaos, stochastic Galerkin method.

AMS-classification: 65L20, 65L07, 65C20

1 Introduction

Mathematical modelling often yields dynamical systems in form of ordinary differential equations (ODEs) or differential algebraic equations (DAEs), see [3]. In case of autonomous dynamical systems, equilibria represent corresponding stationary solutions, which are defined by a nonlinear system of algebraic equations. Transient solutions of the dynamical system remain in a neighbourhood of an equilibrium provided that the equilibrium is stable. The dynamical systems typically include technical parameters. Uncertainties of such parameters appear due to measurement errors, modelling errors or other reasons. Consequently, the time-dependent solutions as well as the equilibria of the dynamical system inherit some uncertainty.

We consider autonomous systems of ODEs. Uncertainties of involved parameters are modelled by the introduction of a finite number of random variables. An uncertainty quantification requires to compute the statistics of the random-dependent ODEs, which can be done by methods of Monte-Carlo type, for example. Alternatively, we apply techniques based on the expansions of the polynomial chaos. The corresponding coefficients can be determined by either a stochastic collocation scheme or a stochastic Galerkin method, see [4, 18, 21].

We investigate the stochastic Galerkin method in case of autonomous ODEs with random parameters. Techniques of the polynomial chaos have already been applied successfully to ODEs, see [1, 2, 7, 8], and DAEs, see [10, 11, 12, 13]. However, the convergence of the stochastic Galerkin method for an increasing polynomial degree has not been proved yet. We focus on the equilibria of the dynamical systems. Thus each equilibrium depends on the random parameters. The stochastic Galerkin method yields a larger coupled system of algebraic equations satisfied by an approximation of the random-dependent equilibria. We analyse the convergence of the stochastic Galerkin method using the famous Theorem of Newton-Kantorovich. Thereby, sufficient conditions for the convergence are specified. The assumptions can be guaranteed in case of symmetric Jacobian matrices combined with a sufficiently fast convergence of the polynomial chaos expansions. Furthermore, we investigate the stability of the equilibria of the larger coupled system from the stochastic Galerkin method with respect to the stability of the equilibria of the original dynamical systems. Thereby, results of Sonday et al. [15] concerning random-dependent Jacobian matrices are employed within our particular problem.

The article is organised as follows. In Sect. 2, we apply the stochastic Galerkin method to systems of ODEs with random parameters. The problem of the determination of random-dependent equilibria is defined. In Sect. 3, the existence and convergence of the approximations by the stochastic Galerkin method is investigated. We analyse the stability of the corresponding equilibria in Sect. 4. Finally, numerical simulation of two test examples are presented.

2 Problem definition

Random-dependent dynamical systems are introduced now, where the stochastic Galerkin method is used for the determination of a numerical solution.

2.1 Dynamical systems with random parameters

We consider autonomous dynamical systems of the form

$$y'(t,p) = f(y(t,p),p), \tag{1}$$

where the right-hand side $f : \Upsilon \times \Pi \to \mathbb{R}^N$ ($\Upsilon \subseteq \mathbb{R}^N$, $\Pi \subseteq \mathbb{R}^Q$) depends on parameters $p \in \Pi$. Furthermore, we assume that f is continuous. Consequently, the solution y of an initial value problem is smooth with respect to time and continuous with respect to the parameters.

We assume that it exists a unique equilibrium $\hat{y}(p)$ for each $p \in \Pi$, i.e., a stationary solution satisfying

$$f(\hat{y}(p),p) = 0 \qquad \text{for each } \; p \in \Pi. \tag{2}$$

Let the function $\hat{y} : \Pi \to \mathbb{R}^N$ be continuous.

For modelling uncertainties, we replace the parameters by independent random variables on some probability space

$$p : \Omega \to \Pi, \quad p = (p_1(\omega), \ldots, p_Q(\omega))^\top.$$

We apply traditional random distributions like Gaussian, uniform, beta or others. Hence a probability density function $\rho : \Pi \to \mathbb{R}$ is available. Assuming measurable functions, we define the spaces

$$L^k(\Pi,\rho) := \left\{ f : \Pi \to \mathbb{R} : \int_\Pi |f(p)|^k \rho(p) \; \mathrm{d}p < \infty \right\}.$$

Given a function $f \in L^1(\Pi,\rho)$, we denote the expected value by

$$\langle f(p) \rangle := \int_\Pi f(p)\rho(p) \; \mathrm{d}p. \tag{3}$$

This notation is employed also for vector-valued functions or matrix-valued functions $f : \Pi \to \mathbb{R}^{N_1 \times N_2}$ in each component separately. For $f, g \in L^2(\Pi, \rho)$, the bilinear form

$$\langle f(p)g(p) \rangle = \int_\Pi f(p)g(p)\rho(p) \, \mathrm{d}p \tag{4}$$

represents the inner product on the Hilbert space $L^2(\Pi, \rho)$. Let $\| \cdot \|_*$ be the norm of this Hilbert space induced by the bilinear form (4).

2.2 Stochastic Galerkin method

Given a solution $y : [t_0, t_1] \times \Pi \to \mathbb{R}^N$ of the ODEs (1), the corresponding expansion of the generalised polynomial chaos (gPC) reads, cf. [21],

$$y(t, p) = \sum_{i=0}^{\infty} v_i(t)\Phi_i(p), \tag{5}$$

where $(\Phi_i(p))_{i \in \mathbb{N}}$ represents a complete system of multivariate basis polynomials $\Phi_i : \Pi \to \mathbb{R}$. We assume orthonormal polynomials, i.e., $\langle \Phi_i(p)\Phi_j(p) \rangle = \delta_{ij}$ with the Kronecker-delta. The polynomials follow from the probability distribution of the random parameters, see [17]. For example, the uniform distribution and the Gaussian distribution imply the Legendre polynomials and the Hermite polynomials, respectively. The coefficient functions $v_i : [t_0, t_1] \to \mathbb{R}^N$ represent inner products (4) defined by

$$v_i(t) = \langle y(t, p)\Phi_i(p) \rangle \qquad \text{for each } \; t. \tag{6}$$

Each component of the gPC expansion (5) converges in the norm of the Hilbert space $L^2(\Pi, \rho)$ point-wise for t provided that $y_j(t, p) \in L^2(\Pi, \rho)$ holds for each $j = 1, \ldots, N$ and each $t \in [t_0, t_1]$. Furthermore, Parseval's equality shows that

$$\|y_j(t, p)\|_*^2 = \sum_{i=0}^{\infty} v_{i,j}(t)^2 \qquad \text{for each } \; t \tag{7}$$

and each component.

The unknown coefficients can be determined numerically by either a stochastic collocation using the probabilistic integrals (6) or the stochastic Galerkin method, see [4, 19, 20]. Considering the ODEs (1), the stochastic Galerkin method yields a larger coupled system of ODEs

$$\tilde{v}_l'(t) = \left\langle f\left(\sum_{i=0}^{M-1} \tilde{v}_i \Phi_i(p), p \right) \Phi_l(p) \right\rangle \qquad \text{for } \; l = 0, 1, \ldots, M-1 \tag{8}$$

satisfied by an approximation of the exact coefficients $v_0, \ldots, v_{M-1}$.

Now we focus on the equilibria $\hat{y} : \Pi \to \mathbb{R}^N$ defined by (2). Assuming finite second moments again, it follows the convergence of the gPC expansion

$$\hat{y}(p) = \sum_{i=0}^{\infty} \hat{v}_i \Phi_i(p). \tag{9}$$

Like in (6), the coefficients $\hat{v}_i \in \mathbb{R}^N$ satisfy the equation $\hat{v}_i = \langle \hat{y}(p)\Phi_i(p) \rangle$. For a truncated series of (9), we apply the notation

$$\hat{y}^M(p) := \sum_{i=0}^{M-1} \hat{v}_i \Phi_i(p)$$

for some integer $M \geq 1$. Using approximations $\tilde{v}_i$, it follows a truncated series $\tilde{y}^M$.

The equilibria of the system of ODEs (8) from the stochastic Galerkin method are given by the nonlinear system of algebraic equations

$$\left\langle f\left(\sum_{i=0}^{M-1} \tilde{v}_i \Phi_i(p), p\right) \Phi_l(p) \right\rangle = 0 \qquad \text{for } \; l = 0, 1, \ldots, M-1. \tag{10}$$

Thereby, the coefficients $\tilde{v}_i$ are an approximation of the exact coefficients $\hat{v}_i$ of $\hat{y}(p)$. The system (10) coincides with the result from an application of the stochastic Galerkin method to the nonlinear system (2).

We write the system (10) in the convenient form

$$F : \Theta_M \to \mathbb{R}^{MN}, \; \Theta_M \subseteq \mathbb{R}^{MN}, \quad F(V) = 0$$

for $V = (v_0, v_1, \ldots, v_{M-1})$ and $F = (F_0, F_1, \ldots, F_{M-1})$ with

$$F_l(V) := \left\langle f\left(\sum_{i=0}^{M-1} v_i \Phi_i(p), p\right) \Phi_l(p) \right\rangle. \tag{11}$$

The domain of dependence of F is given by

$$\Theta_M := \left\{ V \in \mathbb{R}^{MN} : \sum_{i=0}^{M-1} v_i \Phi_i(p) \in \Upsilon \;\text{ for all }\; p \in \Pi \right\}. \tag{12}$$

If Υ is a convex set, then Θ_M is also convex. If Υ is an open set, then Θ_M is open provided that Π is bounded.

In Sect. 3, we investigate if a unique solution $\tilde{V}$ of $F(V) = 0$ exists. Moreover, we analyse if the convergence

$$\lim_{M \to \infty} \left\| \tilde{y}_j^M(p) - \hat{y}_j(p) \right\|_* = 0 \qquad \text{for } \; j = 1, \ldots, N$$

is fulfilled in the norm of the Hilbert space $L^2(\Pi, \rho)$.

A critical property is related to the domain of dependence of the function f. The functions $\hat{y}(p)$ and $\hat{y}^M(p)$ are both well-defined for each $p \in \Pi$ and are continuous. However, the limit of $\hat{y}^M$ for increasing M is given in the sense of $L^2(\Pi, \rho)$. Hence a property like $\|\tilde{y}^M(p) - \hat{y}(p)\|_* < \varepsilon$ does not impose a bound on the difference $\tilde{y}^M(p) - \hat{y}(p)$ point-wise for each p. Although $\hat{y}(p) \in \Upsilon$ holds for all $p \in \Pi$, we cannot guarantee this property for $\tilde{y}^M(p)$ in general. Thus we add this property as a requirement. Nevertheless, no problems appear if $\Upsilon = \mathbb{R}^N$ holds.

2.3 Jacobian matrix in Galerkin method

For our analysis, we require the Jacobian matrix $\mathrm{D}F$ of the function F defined by (11). We assume that f is continuously differentiable with respect to y and thus the Jacobian matrix $\mathrm{D}f$ exists and is continuous. It holds

$$\frac{\partial F_l}{\partial v_k} = \left\langle \mathrm{D}f \left(\sum_{i=0}^{M-1} v_i \Phi_i(p), p \right) \Phi_l(p) \Phi_k(p) \right\rangle \quad \text{for} \quad l, k = 0, 1, \ldots, M-1 \tag{13}$$

provided that the differentiation and the probabilistic integration can be interchanged. If Π is compact, then this interchange is allowed due to the smoothness of f. If Π is not compact, additional integrability conditions have to be satisfied to guarantee the interchange. It follows that the Jacobian matrix $\mathrm{D}f$, which consists of the minors (13), is continuous with respect to the parameters as well as the arguments $v_0, \ldots, v_{M-1}$.

We can write the complete Jacobian matrix in the form

$$\mathrm{D}F(V) = \left\langle S(p) \otimes \mathrm{D}f \left(\sum_{i=0}^{M-1} v_i \Phi_i(p), p \right) \right\rangle \tag{14}$$

using the Kronecker product and the symmetric matrix $S := (\Phi_l \Phi_k) \in \mathbb{R}^{M \times M}$.

3 Analysis of convergence

We investigate the convergence of the stochastic Galerkin method introduced in Sect. 2.2.

3.1 Theorem of Newton-Kantorovich

We apply the famous Theorem of Newton-Kantorovich to show the convergence of the stochastic Galerkin method under certain assumptions. The complete theorem and its proof can be found in [9], for example. The following theorem quotes the statements of Newton-Kantorovich from [16] with respect to the existence of a root only, whereas statements on a corresponding Newton iteration are omitted. Thereby, an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^N$ is used.

Theorem 1 *Let $g : \Xi \to \mathbb{R}^N$, $\Xi \subseteq \mathbb{R}^N$ be continuously differentiable on a convex subset $\Xi_0 \subseteq \Xi$ and*

$$\|\mathrm{D}g(y) - \mathrm{D}g(z)\| \le \gamma \|y - z\| \quad \textit{for all } \; y, z \in \Xi_0$$

with a constant $\gamma > 0$. For some $y_0 \in \Xi_0$ satisfying $\det(\mathrm{D}g(y_0)) \neq 0$, assume $\|\mathrm{D}g(y_0)^{-1} g(y_0)\| \le \alpha$ and $\|\mathrm{D}g(y_0)^{-1}\| \le \beta$. Consider the quantities

$$\delta := \alpha\beta\gamma, \quad r_{1/2} := \frac{1 \mp \sqrt{1 - 2\delta}}{\delta}\,\alpha$$

and sets $S_r(y_0) := \{y \in \mathbb{R}^N : \|y - y_0\| < r\}$. If $\delta \le \frac{1}{2}$ and $\overline{S_{r_1}(y_0)} \subset \Xi_0$, then a root of g exists in $\overline{S_{r_1}(y_0)}$. Moreover, this root is unique in $\Xi_0 \cap S_{r_2}(y_0)$.

We apply this theorem to the functions $F : \Theta_M \to \mathbb{R}^{MN}$ defined by (11). The assumptions of the theorem should be satisfied for sufficiently large M. Our idea is to specify the unknown data $\hat{V}$ of the exact equilibria $\hat{y}(p)$ as starting values in Theorem 1. In the following, we discuss the Lipschitz condition, the residuals $F(\hat{V})$ and the inverse of the Jacobian matrices $\mathrm{D}F(\hat{V})$. We consider the Euclidean norm $\|\cdot\|_2$, since it agrees to the L^2-norm of the probability space due to (7). Moreover, the Euclidean norm induces the spectral matrix norm, which is employed for the stability analysis in Sect. 4.

3.2 Lipschitz-continuity of Jacobian matrix

We assume that the function f is globally Lipschitz-continuous, i.e.,

$$\|f(y, p) - f(z, p)\|_2 \le \gamma_f \cdot \|y - z\|_2 \quad \text{for all } \; y, z \in \Upsilon, \; p \in \Pi \tag{15}$$

with a constant $\gamma_f > 0$.

For the Jacobian matrix $\mathrm{D}f$, we assume the Lipschitz condition

$$\|\mathrm{D}f(y, p) - \mathrm{D}f(z, p)\|_2 \le \gamma_0 \|y - z\|_2 \quad \text{for all } \; y, z \in \Upsilon, \; p \in \Pi \tag{16}$$

with a constant $\gamma_0 > 0$.

The Jacobian matrix $\mathrm{D}F$ of the coupled system (10) is given by the minors (13). Since F depends on the choice of M, we assume Lipschitz conditions of the form

$$\|\mathrm{D}F(V) - \mathrm{D}F(W)\|_2 \leq \gamma_M \|V - W\|_2 \quad \text{for all } \ V, W \in \Theta_M, \tag{17}$$

where the constants $\gamma_M > 0$ are chosen as small as possible.

Lemma 1 *If the Jacobian matrices* $\mathrm{D}f$ *satisfy the Lipschitz condition (16), then the Jacobian matrices* $\mathrm{D}F$ *fulfill the Lipschitz-continuity (17) with constants*

$$\gamma_M \leq \gamma_0 M^{\frac{3}{2}} N \left(\max_{i=0,1,\ldots,M-1} \sqrt{\langle \Phi_i(p)^4 \rangle} \right)$$

for each integer M*.*

Proof:

Let

$$y^M(p) := \sum_{i=0}^{M-1} v_i \Phi_i(p) \quad \text{and} \quad z^M(p) := \sum_{i=0}^{M-1} w_i \Phi_i(p).$$

Employing the notation (14), it holds

$$\mathrm{D}F(V) - \mathrm{D}F(W) = \left\langle S(p) \otimes \left[\mathrm{D}f(y^M(p), p) - \mathrm{D}f(z^M(p), p) \right] \right\rangle .$$

Basic calculations and the Cauchy-Schwarz inequality yield the estimate

$$\begin{aligned} \|\mathrm{D}F(V) - \mathrm{D}F(W)\|_2 \quad &\leq \quad \sqrt{MN} \|\mathrm{D}F(V) - \mathrm{D}F(W)\|_\infty \\ \leq \quad M^{\frac{3}{2}} N^{\frac{1}{2}} &\sqrt{\max_{i,j=0,1,\ldots,M-1} \langle \Phi_i(p)^2 \Phi_j(p)^2 \rangle} \sqrt{\langle \|\mathrm{D}f(y^M(p), p) - \mathrm{D}f(z^M(p), p)\|_\infty^2 \rangle}. \end{aligned}$$

Using the Lipschitz-continuity (16) and Parseval's equality (7), we obtain

$$\begin{aligned} \langle \|\mathrm{D}f(y^M(p), p) - \mathrm{D}f(z^M(p), p)\|_\infty^2 \rangle \quad &\leq \quad {\gamma_0}^2 N \langle \|y^M(p) - z^M(p)\|_2^2 \rangle \\ &= \quad {\gamma_0}^2 N \sum_{i=0}^{M-1} \sum_{j=1}^{N} (v_{i,j} - w_{i,j})^2 \\ &= \quad {\gamma_0}^2 N \|V - W\|_2^2. \end{aligned}$$

Furthermore, the Cauchy-Schwarz inequality implies

$$\langle \Phi_i(p)^2 \Phi_j(p)^2 \rangle \leq \sqrt{\langle \Phi_i(p)^4 \rangle} \cdot \sqrt{\langle \Phi_j(p)^4 \rangle} \leq \max_{i=0,1,\ldots,M-1} \langle \Phi_i(p)^4 \rangle$$

for all $0 \leq i, j \leq M - 1$. □

If the sequence $(\langle \Phi_i(p)^4 \rangle)_{i \in \mathbb{N}}$ grows just polynomially, then the coefficients γ_M are bounded by some polynomial in M due to Lemma 1. This property is satisfied for uniformly distributed random parameters, for example.

3.3 Residual

We employ the Lipschitz condition (15) of the right-hand side f to impose a bound on the residuals F.

Lemma 2 *Let f satisfy the global Lipschitz-continuity (15). For the gPC coefficients $\hat{V} \in \mathbb{R}^{MN}$ of the exact equilibria $\hat{y}(p)$, it holds*

$$\|F(\hat{V})\|_2 \leq \gamma_f \sqrt{M} \sqrt{\langle \|\hat{y}^M(p) - \hat{y}(p)\|_2^2 \rangle}$$

for each M.

Due to (2), it holds $f_j(\hat{y}(p), p) = 0$ for each component $j = 1, \ldots, N$. The Lipschitz-continuity (15), the Cauchy-Schwarz inequality and the orthonormality of the basis polynomials imply the estimate

$$\begin{aligned} \|F_l(\hat{V})\|_2^2 &= \sum_{j=1}^{N} \langle f_j(\hat{y}^M(p), p) \Phi_l(p) \rangle^2 \\ &= \sum_{j=1}^{N} \langle \left[f_j(\hat{y}^M(p), p) - f_j(\hat{y}(p), p) \right] \Phi_l(p) \rangle^2 \\ &\leq \sum_{j=1}^{N} \left\langle \left[f_j(\hat{y}^M(p), p) - f_j(\hat{y}(p), p) \right]^2 \right\rangle \cdot \langle \Phi_l(p)^2 \rangle \\ &= \left\langle \left\| f(\hat{y}^M(p), p) - f(\hat{y}(p), p) \right\|_2^2 \right\rangle \\ &\leq \gamma_f^2 \left\langle \left\| \hat{y}^M(p) - \hat{y}(p) \right\|_2^2 \right\rangle . \end{aligned}$$

It follows

$$\|F(\hat{V})\|_2 = \sqrt{\sum_{l=0}^{M-1} \|F_l(\hat{V})\|_2^2} \leq \sqrt{M} \gamma_f \sqrt{\langle \|\hat{y}^M(p) - \hat{y}(p)\|_2^2 \rangle}$$

as stated above. □

A linear convergence of the gPC expansion of the equilibria $\hat{y}$ is already sufficient for the fundamental property

$$\lim_{M \to \infty} \|F(\hat{V})\|_2 = 0$$

by Lemma 2.

3.4 Inverse of Jacobian matrix

The Jacobian matrices $\mathrm{D}F \in \mathbb{R}^{MN \times MN}$ depend on $V \in \mathbb{R}^{MN}$. We define the matrices $J^{(M)} \in \mathbb{R}^{MN \times MN}$ by the minors

$$J_{lk}^{(M)} := \langle \mathrm{D}f\,(\hat{y}(p), p)\, \Phi_l(p)\Phi_k(p) \rangle \quad \text{for } \; l, k = 0, 1, \ldots, M-1 \tag{18}$$

using the exact equilibria $\hat{y}(p)$. Thus the random-dependent Jacobian matrix $\mathrm{D}f(\hat{y}(p), p)$ is involved. We define two conditions corresponding to the spectrum and the numerical range, which will also be used in Sect. 4.

Condition 1 *It exists a constant $\nu \in \mathbb{R}$ with $\nu \leq 0$ such that $\mathrm{Re}(\lambda) \leq \nu$ holds for all eigenvalues $\lambda \in \mathbb{C}$ of $\mathrm{D}f(\hat{y}(p), p)$ and for all $p \in \Pi$.*

Condition 2 *It exists a constant $\nu \in \mathbb{R}$ with $\nu \leq 0$ such that $\mu(\mathrm{D}f(\hat{y}(p), p)) \leq \nu$ holds for all $p \in \Pi$ using the logarithmic matrix norm μ corresponding to the spectral norm $\| \cdot \|_2$.*

For the definition and more details on the logarithmic matrix norm μ, see [6], for example. Condition 2 is equivalent to the property

$$\sup\{\mathrm{Re}(z^* \mathrm{D}f(\hat{y}(p), p) z) : z \in \mathbb{C}^N, z^* z = 1\} \leq \nu$$

for each $p \in \Pi$, which involves the numerical range of the Jacobian matrix. Thus Condition 2 is sufficient for Condition 1, since the numerical range of a matrix includes the spectrum. If the Jacobian matrices $\mathrm{D}f(\hat{y}(p), p)$ are normal for each $p \in \Pi$, then the two conditions are equivalent.

Condition 2 gives information on the spectrum of the matrices $J^{(M)}$ in view of results from [15].

Lemma 3 *If the Jacobian matrices $\mathrm{D}f(\hat{y}(p), p)$ satisfy Condition 2 for some $\nu < 0$, then the matrices $J^{(M)}$ are regular for each $M \geq 1$.*

Proof:

Condition 2 guarantees $\mathrm{Re}(z^* \mathrm{D}f(\hat{y}(p), p) z) \leq \nu$ for all $z^* z = 1$ and all $p \in \Pi$ with some $\nu < 0$. The half-plane $\{\eta \in \mathbb{C} : \mathrm{Re}(\eta) \leq \nu\} \subset \mathbb{C}$ is closed and convex. If $\lambda \in \mathbb{C}$ is an eigenvalue of $J^{(M)}$ for arbitrary M, then it follows $\mathrm{Re}(\lambda) \leq \nu$ by Theorem 2 in [15]. Thus all eigenvalues of $J^{(M)}$ are non-zero. □

In case of normal matrices, we obtain a uniform bound on the spectral norm of the inverse matrices.

Lemma 4 *Let the matrices* $\mathrm{D}f(\hat{y}(p), p)$ *be normal for all* $p \in \Pi$ *and the matrices* $J^{(M)}$ *be normal for all* $M \geq 1$*. If Condition 1 or, equivalently, Condition 2 is satisfied for some* $\nu < 0$*, then the matrices* $J^{(M)}$ *are regular and*

$$\left\| \left(J^{(M)}\right)^{-1} \right\|_2 \leq \frac{1}{|\nu|} \qquad \text{for all } M \geq 1. \tag{19}$$

Proof:

Lemma 3 implies that all eigenvalues of $J^{(M)}$ are non-zero. A complex number λ is an eigenvalue of $J^{(M)}$ if and only if $\frac{1}{\lambda}$ is an eigenvalue of $(J^{(M)})^{-1}$. For a normal matrix, the spectral norm $\|\cdot\|_2$ is equal to the spectral radius. Since $(J^{(M)})^{-1}$ is a normal matrix, it follows

$$\left\| \left(J^{(M)}\right)^{-1} \right\|_2 = \max \left\{ \left| \frac{1}{\lambda} \right| : \lambda \text{ eigenvalue of } J^{(M)} \right\} \leq \frac{1}{|\nu|}$$

due to $|\lambda| \geq |\mathrm{Re}(\lambda)| \geq |\nu|$ for all eigenvalues of $J^{(M)}$ uniformly in M. □

Note that the normality of the matrices $\mathrm{D}f(\hat{y}(p), p)$ does not imply the normality of the matrices $J^{(M)}$ in general. Nevertheless, symmetric matrices $\mathrm{D}f(\hat{y}(p), p)$ for all p yield symmetric matrices $J^{(M)}$ for each M.

Finally, we require an estimate for the inverse matrices of $\mathrm{D}F$ corresponding to the coupled system from the stochastic Galerkin method.

Lemma 5 *Assume that the Lipschitz condition (16) holds with constant* γ_0*. Let* $\hat{V} \in \mathbb{R}^{MN}$ *be the gPC coefficients corresponding to the equilibria* $\hat{y}(p)$*. It follows*

$$\left\| \mathrm{D}F(\hat{V}) - J^{(M)} \right\|_2 \leq \gamma_0 M^{\frac{3}{2}} N \left(\max_{i=0,1,\ldots,M-1} \sqrt{\langle \Phi_i(p)^4 \rangle} \right) \sqrt{\langle \|\hat{y}^M(p) - \hat{y}(p)\|_2^2 \rangle}$$

for each $M \geq 1$*.*

Proof:

Let $\hat{y}^M(p)$ be the truncated gPC expansion of the equilibria $\hat{y}(p)$ again. The minors of the matrices yield

$$\frac{\partial F_l}{\partial v_k}(\hat{V}) - J_{lk}^{(M)} = \left\langle \left[\mathrm{D}f\left(\hat{y}^M(p), p\right) - \mathrm{D}f\left(\hat{y}(p), p\right) \right] \Phi_l(p) \Phi_k(p) \right\rangle .$$

It follows using the notation (14)

$$\mathrm{D}F(\hat{V}) - J^{(M)} = \left\langle S(p) \otimes \left[\mathrm{D}f(y^M(p), p) - \mathrm{D}f(y(p), p) \right] \right\rangle .$$

By repeating the steps in the proof of Lemma 1, we obtain the desired estimate in the spectral norm. $\square$

For a sufficiently fast convergence of the gPC expansion of $\hat{y}$, it follows

$$\lim_{M\to\infty} \left\| \mathrm{D}F(\hat{V}) - J^{(M)} \right\|_2 = 0. \tag{20}$$

The Jacobian matrices $\mathrm{D}F$ inherit the regularity of the matrices $J^{(M)}$ for sufficiently large M provided that a uniform bound (19) holds. This property allows us to use the data $\hat{V}$ as starting values in the Theorem of Newton-Kantorovich. However, we require also some uniform bound on the corresponding norms $\|(\mathrm{D}F(\hat{V}))^{-1}\|_2$.

Lemma 6 *Let the matrices $J^{(M)}$ be regular and*

$$\left\| \left(J^{(M)}\right)^{-1} \right\|_2 \leq C \qquad \text{for all } M \geq 1$$

with a constant $C > 0$. If the convergence (20) is given in the norm $\|\cdot\|_2$, then the matrices $\mathrm{D}F(\hat{V})$ are regular for all $M \geq M'$ with some integer $M' \geq 1$ and

$$\left\| \left(\mathrm{D}F(\hat{V})\right)^{-1} \right\|_2 \leq 2C \qquad \text{for all } M \geq M'.$$

Proof:

Due to (20), it exists an $M' \geq 1$ such that $\|\mathrm{D}F(\hat{V}) - J^{(M)}\|_2 \leq \frac{1}{2C}$ for all $M \geq M'$. We obtain $\|(J^{(M)})^{-1}(\mathrm{D}F(\hat{V}) - J^{(M)})\|_2 \leq \frac{1}{2}$ for all $M \geq M'$. It follows

$$\left\| (\mathrm{D}F(\hat{V}))^{-1} - (J^{(M)})^{-1} \right\|_2 \leq 2 \left\| (J^{(M)})^{-1} \right\|_2^2 \cdot \left\| \mathrm{D}F(\hat{V}) - J^{(M)} \right\|_2$$

for $M \geq M'$. We achieve the statement via

$$\left\| (\mathrm{D}F(\hat{V}))^{-1} \right\|_2 \leq \left\| (\mathrm{D}F(\hat{V}))^{-1} - (J^{(M)})^{-1} \right\|_2 + \left\| (J^{(M)})^{-1} \right\|_2,$$

since both terms on the right-hand side are bounded uniformly. $\square$

3.5 Convergence of stochastic Galerkin method

Now we are able to prove a theorem on the convergence of the stochastic Galerkin method. Let the continuity and smoothness of the involved functions hold as introduced in the previous sections. Moreover, we impose the following properties.

Assumptions:

- domain of dependence Υ of f is convex,
- equilibria $\hat{y}(p)$ from (2) have finite second moments,
- truncated gPC expansions of equilibria satisfy $\hat{y}^M(p) \in \Upsilon$ for each $p \in \Pi$ and all $M \geq M'$.
- $\mathrm{D}F$ fulfills Lipschitz conditions (17) with constants γ_M for all $M \geq M'$.

Now we are able to show the main result.

Theorem 2 *Let the above assumptions be satisfied. For $\hat{V} \in \mathbb{R}^{MN}$ including the first M coefficients of the gPC expansion of $\hat{y}(p)$, assume $\det(\mathrm{D}F(\hat{V})) \neq 0$ for all $M \geq M'$ and define*

$$\alpha_M := \|F(\hat{V})\|_2, \quad \beta_M := \|\mathrm{D}F(\hat{V})^{-1}\|_2, \quad \delta_M := \alpha_M \beta_M^2 \gamma_M.$$

If it holds

$$\lim_{M\to\infty} \alpha_M \beta_M = 0 \quad \textit{and} \quad \lim_{M\to\infty} \delta_M = 0 \tag{21}$$

as well as

$$\left\{ V \in \mathbb{R}^{MN} : \|V - \hat{V}\|_2 \leq \frac{1 - \sqrt{1 - 2\delta_M}}{\beta_M \gamma_M} \right\} \subset \Theta_M \quad \textit{for all} \;\; M \geq M', \tag{22}$$

then a locally unique solution $\tilde{V}$ of the stochastic Galerkin method exists for each $M \geq M'$ with some sufficiently large integer M'. Furthermore, the stochastic Galerkin method is convergent and the error is bounded by

$$\|\tilde{V} - \hat{V}\|_2 \leq \alpha_M \beta_M$$

asymptotically.

Proof:

The sets Θ_M from (12) are convex for each M, since Υ is assumed to be convex. The function F is smooth on Θ_M for each M. Note that

$$\|\mathrm{D}F(\hat{V})^{-1} F(\hat{V})\|_2 \leq \alpha_M \beta_M.$$

Due to the condition (21), it holds

$$\alpha_M \beta_M^2 \gamma_M \leq \tfrac{1}{2} \quad \text{for all} \;\; M \geq M'$$

with some integer $M' \geq 1$.

We apply Theorem 1 of Newton-Kantorovich with the starting values $y_0 := \hat{V}$ and the constants $\alpha := \alpha_M \beta_M$, $\beta := \beta_M$ and $\gamma := \gamma_M$. It follows the existence of a solution $\tilde{V}$ in the neighbourhood (22) of the data $\hat{V}$. In particular, it holds

$$\|\tilde{V} - \hat{V}\|_2 \leq \frac{1 - \sqrt{1 - 2\alpha_M \beta_M^2 \gamma_M}}{\beta_M \gamma_M} \doteq \alpha_M \beta_M$$

asymptotically for large M, which can be verified by a Taylor expansion of the square root. Thus the convergence of the method is shown under those assumptions. $\square$

In general, we do not expect the constants β_M or γ_M to converge to zero. An optimal case appears if these constants are uniformly bounded. In contrast, the constants α_M of the residuals converge to zero due to Lemma 2 provided that f fulfills the Lipschitz condition (15).

We also obtain the convergence of the stochastic Galerkin technique with respect to the mean square error of its approximation.

Corollary 1 *Let the assumptions of Theorem 2 be satisfied. Then the approximations $\tilde{y}^M(p)$ of the stochastic Galerkin method converge in the norm of $L^2(\Pi, \rho)$, i.e.,*

$$\lim_{M \to \infty} \|\hat{y}_j(p) - \tilde{y}_j^M(p)\|_* = 0$$

for each $j = 1, \ldots, N$.

Proof:

The mean square error of the approximation can be estimated by

$$\|\hat{y}_j(p) - \tilde{y}_j^M(p)\|_* \leq \|\hat{y}_j(p) - \hat{y}_j^M(p)\|_* + \|\hat{y}_j^M(p) - \tilde{y}_j^M(p)\|_*$$

in each component $j = 1, \ldots, N$. The first term tends to zero, since the gPC expansion of the exact equilibria converges in $L^2(\Pi, \rho)$. We estimate for the second term

$$\|\hat{y}_j^M(p) - \tilde{y}_j^M(p)\|_* \leq \|\hat{V} - \tilde{V}\|_2$$

due to Parseval's equality (7). Thus the convergence of the coefficients as given by Theorem 2 implies that the second term also tends to zero. $\square$

4 Analysis of Stability

In this section, we assume just the existence of a solution of the coupled system (10) of the stochastic Galerkin method. Note that a solution of the nonlinear system (10) represents an equilibrium of the system of ODEs (8) and vice versa. We analyse the stability of the equilibria of the ODEs (8) with respect to the stability of the equilibria of the original ODEs (1).

The stability of the equilibria $\hat{y}(p)$ is determined by the eigenvalues of the Jacobian matrix $\mathrm{D}f(\hat{y}(p), p)$. An equilibrium is stable if and only if the real part of each eigenvalue is non-positive. An equilibrium is asymptotically stable if and only if the real part of each eigenvalue is negative.

4.1 Criterion from numerical range

We use Condition 1 and Condition 2 from Sect. 3.4 again, which correspond to the spectrum and the numerical range, respectively. Condition 1 for $\nu = 0$ is equivalent to the stability of each equilibrium of the original system of ODEs (1). Condition 1 for $\nu < 0$ is sufficient for the asymptotical stability of each equilibrium. For compact domains Π, Condition 1 for some $\nu < 0$ is also necessary.

We consider Condition 2 corresponding to the numerical range of the Jacobian matrices, which is sufficient but not necessary for Condition 1. It follows a positive result for the stability of the equilibria.

Theorem 3 *If Condition 2 holds with $\nu \leq 0$, then the equilibria of the system of ODEs (8) from the stochastic Galerkin method are stable. If Condition 2 holds with $\nu < 0$, then the equilibria of (8) are asymptotically stable.*

Proof:

The statement follows as in the proof of Lemma 3. Condition 2 allows for the conclusion that each eigenvalue $\eta \in \mathbb{C}$ of the Jacobian matrix for an equilibrium of (8) satisfies $\mathrm{Re}(\eta) \leq \nu$ by Theorem 2 in [15]. Thus the cases $\nu \leq 0$ and $\nu < 0$ imply stability and asymptotical stability, respectively. $\square$

We note a restriction on the regularity of the matrices. If the equilibria of the original system (1) are stable, then the Jacobian matrix $\mathrm{D}F$ of the coupled system (10) can be singular even if the Jacobian matrices $\mathrm{D}f$ are regular for all $p \in \Pi$. A corresponding example can be constructed as in Sect. 3.2 of [13] using a probability distribution of p such that the eigenvalues are located on the imaginary axis only.

4.2 Criterion from spectrum

Now we discuss the weaker Condition 1 corresponding to the spectrum of the Jacobian matrices. Condition 1 for $\nu \leq 0$ is sufficient for the stability of the equilibria of the coupled system (8) in case of $N = 1$, since the Jacobian matrix becomes just a scalar. However, Condition 1 is not sufficient for the stability of the equilibria of (8) in case of $N \geq 2$ even if $\nu < 0$ holds. We construct a corresponding counterexample already in the linear case for $N = 2$. Let

$$y'(t,p) = A(p)y(t,p) \quad \text{with} \quad A(p) = \begin{pmatrix} -41 & 40p \\ \frac{30}{p} & -31 \end{pmatrix}$$

assuming $p \neq 0$. The eigenvalues of $A(p)$ are $\lambda_1 = -1$ and $\lambda_2 = -71$ independent of p. Note that the matrix is non-normal for $p^2 \neq \frac{3}{4}$. If $p \in [1,3]$ is uniformly distributed, then the larger matrix $B := (\langle A(p)\Phi_i(p)\Phi_j(p)\rangle) \in \mathbb{R}^{2M \times 2M}$ exhibits a real positive eigenvalue $\eta > 0.3$ for each $M = 1, \ldots, 20$, for example. Thereby, the eigenvalues are computed numerically.

Likewise, we achieve linear counterexamples in each case, where a matrix B exhibits an eigenvalue η with $\mathrm{Re}(\eta) > \nu$ for a constant $\nu < 0$ in Condition 1. If $\mathrm{Re}(\eta) < 0$ appears, then a constant shift of the spectra can be used to achieve a positive real part.

Nevertheless, we obtain a sufficient criterion in the case of normal matrices.

Theorem 4 *Let the matrices $\mathrm{D}f(\hat{y}(p), p)$ be normal for each $p \in \Pi$. If Condition 1 holds with $\nu \leq 0$, then the equilibria of the system of ODEs (8) from the stochastic Galerkin method are stable. If Condition 1 holds with $\nu < 0$, then the equilibria of (8) are asymptotically stable.*

The proof of Theorem 4 follows directly from the equivalence of Condition 1 and Condition 2 in case of normal Jacobian matrices corresponding to the right-hand sides of the ODEs (1). In contrast to Lemma 4, we do not need the normality of the Jacobian matrices of the larger system of ODEs (8) in Theorem 4.

5 Illustrative Examples

We investigate the equilibria for some test examples either theoretically or using numerical simulations.

5.1 Stationary solutions of heat equation

We consider the heat equation in one space dimension

$$\frac{\partial u}{\partial t} = \lambda(p)\frac{\partial^2 u}{\partial x^2} + s(x,p) \tag{23}$$

for $x \in [0,1]$ assuming homogeneous Dirichlet boundary conditions. The heat conduction $\lambda > 0$ and the source term s are allowed to depend on random parameters p. We apply a method of lines to solve (23), see [5]. Thereby, the spatial derivative is replaced by a symmetric difference formula of second order with equidistant step size $h = \frac{1}{N+1}$. It follows the linear system of ODEs

$$y'(t) = \lambda(p)A_N y(t) + w(p) \tag{24}$$

for the approximations $y_j(t) \doteq u(jh,t)$ including $w_j(p) := s(jh,p)$ with $j = 1,\ldots,N$. The tridiagonal matrices A_N depend on the step size h. Nevertheless, the matrices A_N are symmetric and negative definite for all N.

Stationary solutions of the partial differential equation (23) are identified by the property $\frac{\partial u}{\partial t} \equiv 0$. Likewise, the equilibria of the approximative ODEs (24) read

$$\hat{y}(p) = -\frac{1}{\lambda(p)}A_N^{-1}w(p) \tag{25}$$

assuming $\lambda(p) > 0$ for all p. Thus the equilibria inherit the smoothness of the functions λ and s with respect to the parameters. Although the systems of ODEs (24) are linear, the dependence of the equilibria (25) on the parameters is nonlinear in general.

The stochastic Galerkin method yields a linear system

$$\sum_{i=0}^{M-1} \langle \lambda(p)\Phi_i(p)\Phi_l(p)\rangle A_N \tilde{v}_i = \langle w(p)\Phi_l(p)\rangle \quad \text{for } l = 0,1,\ldots,M-1$$

satisfied by the approximations $\tilde{v}_0,\ldots,\tilde{v}_{M-1}$. This larger linear system involves the matrix

$$B := L_M \otimes A_N \quad \text{with} \quad L_M := (\langle \lambda(p)\Phi_i(p)\Phi_l(p)\rangle) \in \mathbb{R}^{M\times M}.$$

Thus L_M and B are also symmetric. Assuming $\lambda(p) > 0$ for all p again, the symmetric matrices L_M are positive definite. It follows that B is always negative definite. Hence the equilibria of the ODE system (24) as well as the equilibria of the coupled system from the stochastic Galerkin method are always stable due to the symmetry by Theorem 4.

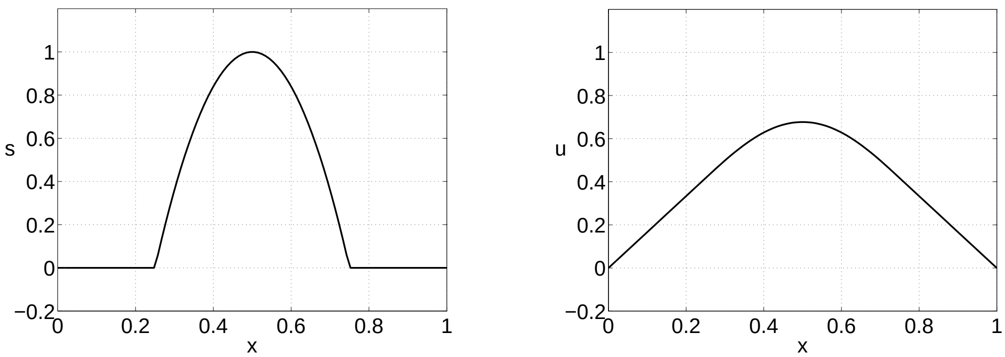

Figure 1: Source term $s(x)$ (left) and a corresponding stationary solution $u(t,x) \equiv \hat{u}(x)$ (right) for heat equation with constant heat conduction $\lambda = 0.1$.

We arrange the deterministic source term

$$s(x) := \begin{cases} 1 - 16x^2 & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 0 & \text{otherwise,} \end{cases}$$

see Figure 1 (left). Although the source term has a compact support, corresponding stationary solutions are non-zero outside the boundaries, see Figure 1 (right).

Now we replace the heat conductance by a uniformly distributed random variable $\lambda(p) = p \in [0.1, 0.2]$. Consequently, the polynomial chaos applies the Legendre polynomials. Since the equilibria (25) are analytic functions with respect to the heat conductance, the convergence of the exact gPC expansion is exponentially fast, see [21]. Thus the assumptions guaranteeing the convergence of the stochastic Galerkin method by Theorem 2 are satisfied.

We use $N = 100$ and $M = 4$ now. Fig. 2 illustrates the resulting approximations of the expected values and the variances for the random equilibria. The other coefficient functions of the gPC are depicted in Fig. 3. Furthermore, we compute the mean square errors of the approximations from the stochastic Galerkin method with respect to the exact random equilibria (25), i.e., the error in the L^2-norm. Fig. 4 shows the errors for different polynomial degrees $M - 1$, where the maximum error of the components $j = 1, \ldots, 100$ is computed for each M. It follows that the stochastic Galerkin method inherits the exponential rate of convergence from the exact gPC expansion.

Now we investigate the spectra of the involved matrices for different numbers N of the spatial discretisation as well as different polynomial degrees given by M. Since each matrix is symmetric, all eigenvalues are real. Fig. 5 illustrates the smallest and largest eigenvalues for different polynomial degrees and fixed grid

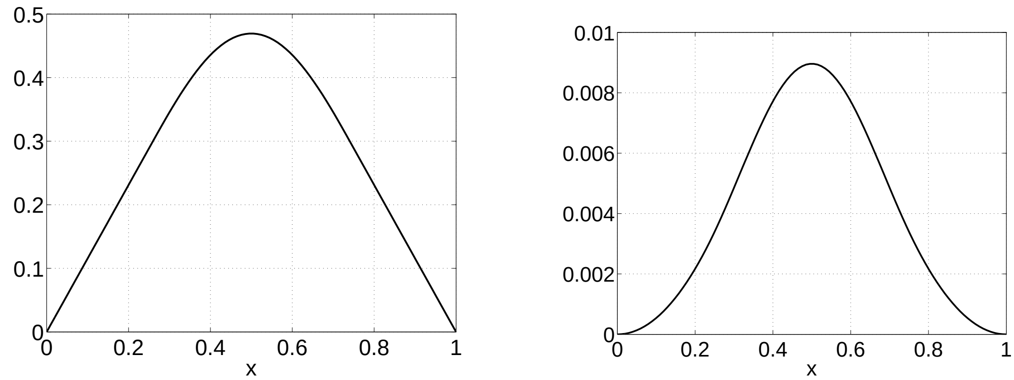

Figure 2: Expected values (left) and variances (right) of random equilibria for heat equation.

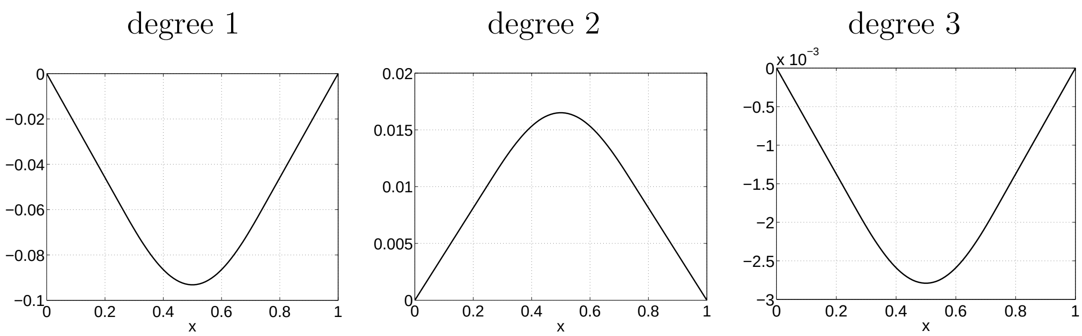

Figure 3: Coefficients of random equilibria within polynomial chaos for heat equation.

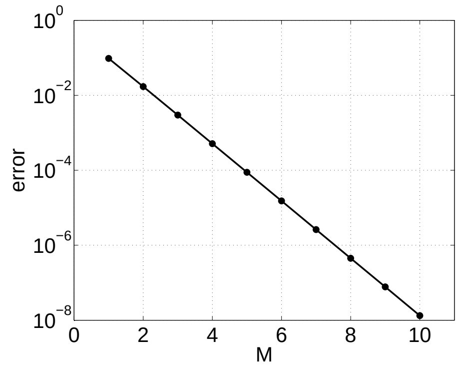

Figure 4: Mean square errors of the approximations from the stochastic Galerkin method applied to heat equation (semi-logarithmic scale).

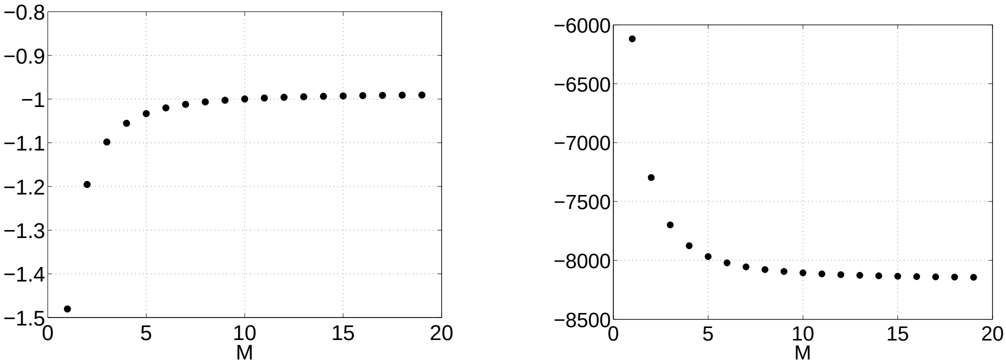

Figure 5: Smallest (left) and largest (right) eigenvalues of the matrices in the stochastic Galerkin method for $N = 100$ and varying M.

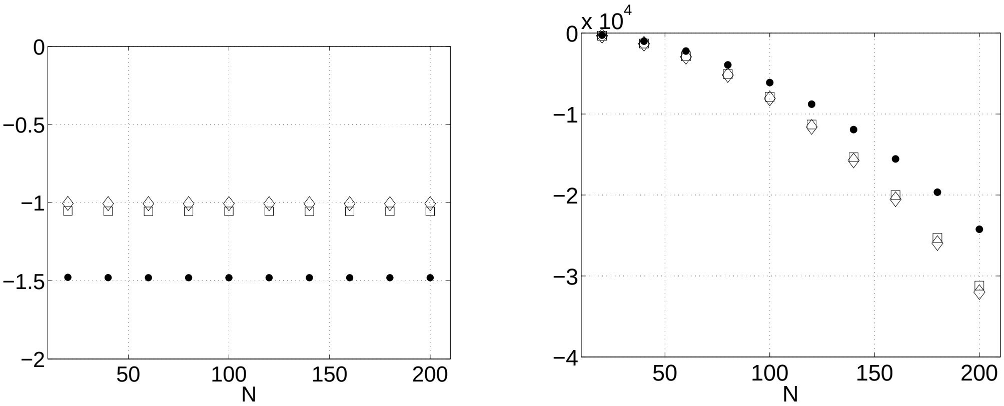

Figure 6: Smallest (left) and largest (right) eigenvalues of the matrices in the stochastic Galerkin method for varying N and $M = 4$ (squares), $M = 8$ (diamonds) as well as deterministic matrices $\langle\lambda\rangle A_N$ (circles).

$N = 100$. We observe that these bounds of the spectrum approach constant values for increasing M. The results are in agreement to the theoretical investigations of Sect. 4. Alternatively, the smallest and largest eigenvalues in case of $M = 4$ and $M = 8$ are shown for different N in Fig. 6. For comparison, the corresponding values are depicted for the deterministic matrices $\langle \lambda \rangle A_N$. We recognise that the order of magnitude of the eigenvalues is similar in each case. Thus the stochastic Galerkin method also inherits the quantitative stability properties of the original systems.

5.2 Duffing oscillator

The Duffing oscillator represents a frequently used example in stability analysis, see [14]. We apply the Duffing oscillator in the form

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= y_1(t) - p_1 y_1(t)^3 - p_2 y_2(t) \end{aligned} \tag{26}$$

with parameters p_1 for the nonlinear term and p_2 for the damping constant. Assuming $p_1 > 0$, it follows exactly three equilibria

$$(0,0), \quad (\tfrac{1}{\sqrt{p_1}}, 0), \quad (-\tfrac{1}{\sqrt{p_1}}, 0)$$

which are all independent of p_2. The equilibrium $(0, 0)$ is unstable. Fig. 7 depicts two solutions of initial value problems of the Duffing oscillator (26) using the parameters $p_1 = p_2 = 1$. We observe that the solutions tend to the two stable equilibria in the form of damped oscillations.

For the equilibria $(\pm 1/\sqrt{p_1}, 0)$, the corresponding Jacobian matrix

$$\mathrm{D}f(\hat{y}(p), p) = \hat{J}(p_2) := \begin{pmatrix} 0 & 1 \\ -2 & -p_2 \end{pmatrix}$$

is independent of p_1 now. It follows that the equilibria are asymptotically stable for $p_2 > 0$, stable for $p_2 = 0$ and unstable for $p_2 < 0$. Consequently, Condition 1 is satisfied for non-negative ranges of p_2, whereas Condition 2 is violated in case of $p_2 = 1$, for example, due to the logarithmic matrix norm $\mu(\hat{J}(1)) = \frac{1}{2}[-1+\sqrt{2}]$.

We consider beta-distributed random variables now. The corresponding probability density function reads

$$\rho(p) := C(\alpha, \beta)(1-p)^\alpha (1+p)^\beta \quad \text{for} \quad p \in [-1, 1] \tag{27}$$

with constants $\alpha, \beta \geq 0$ and a constant $C(\alpha, \beta) > 0$ for standardisation. We choose the two random parameters in (26) with independent beta distributions

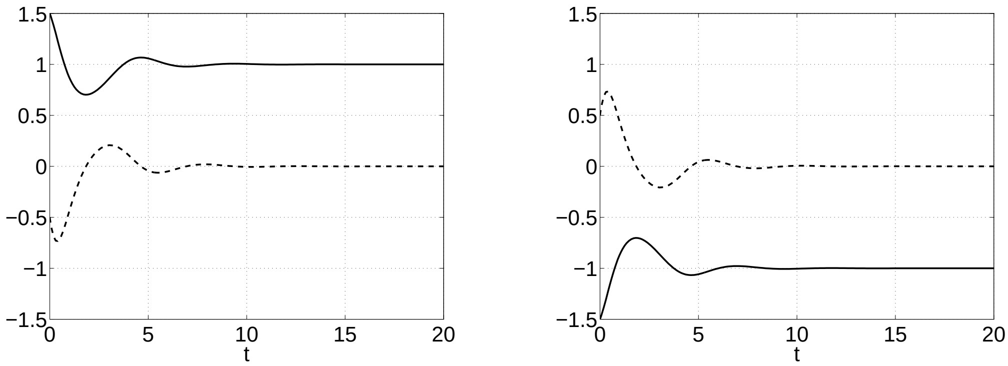

Figure 7: Solutions of initial value problems of Duffing oscillator (y_1: solid line, y_2: dashed line).

via $p_i := 1 + \theta_i \tilde{p}_i$ using constants $\theta_i > 0$, where $\tilde{p}_i$ is a random variable with density (27). The corresponding polynomial chaos induces the Jacobi polynomials.

We investigate the equilibrium $(1/\sqrt{p_1}, 0)$ now. We select $\alpha = \beta = 1$ in (27) for both random parameters and $\theta_1 = \theta_2 = \frac{1}{2}$. Since the second component of the equilibrium is zero, the results are presented just for the first component. The stochastic Galerkin method resolves the right-hand side of (26). We include all polynomials up to degree five, which implies $M = 21$ basis functions. Fig. 8 (left) shows the absolute values of the computed gPC coefficients. Coefficients corresponding to polynomials with dependence on p_1 only yield a significant contribution, whereas the other coefficients are reproduced correctly to zero except for roundoff errors. The mean square error of the stochastic Galerkin method is demonstrated for different polynomial degrees by Fig. 8 (right). Again we observe an exponential convergence due to the high smoothness of the equilibria.

Finally, we discuss the eigenvalues of the Jacobian matrix for the equilibria $(1/\sqrt{p_1}, 0)$ in the stochastic Galerkin method. Assuming $\theta_1 = \theta_2 = \frac{1}{2}$, Condition 1 holds due to $p_2 > 0$ for all realisations of the random parameter, whereas Condition 2 is violated. Fig. 9 (left) depicts the eigenvalues of the larger Jacobian matrix for $M = 21$. We recognise that asymptotical stability is given for the equilibria of the coupled system. This simulation demonstrates that Condition 2 is sufficient but not necessary for the stability of the equilibria of the system from the stochastic Galerkin method.

Alternatively, we consider the case of $\theta_1 = \frac{1}{2}$ and $\theta_2 = \frac{3}{2}$. Now even Condition 1 is violated, since negative realisations $p_2 < 0$ appear. Although the probability of a negative value of p_2 is relatively small, the stability is lost in the stochastic Galerkin method as can be seen in Fig. 9 (right).

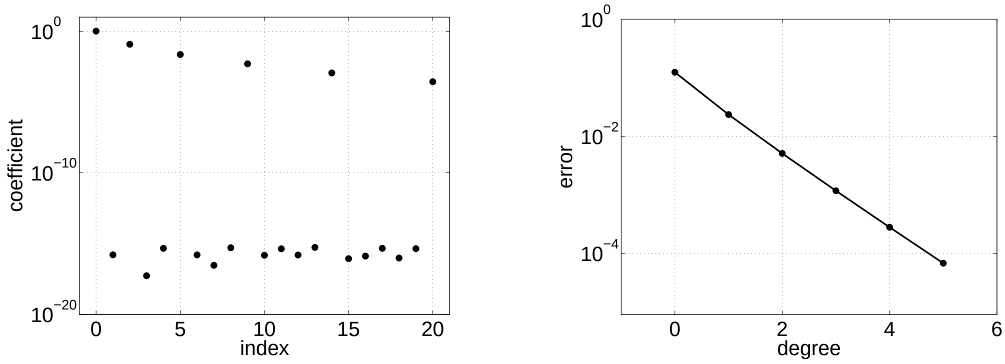

Figure 8: Absolute values of gPC coefficients for simulation with all polynomials up to degree 5 (left) and mean square error of approximations for different polynomial degrees (right) for component y_1 (both in semi-logarithmic scale).

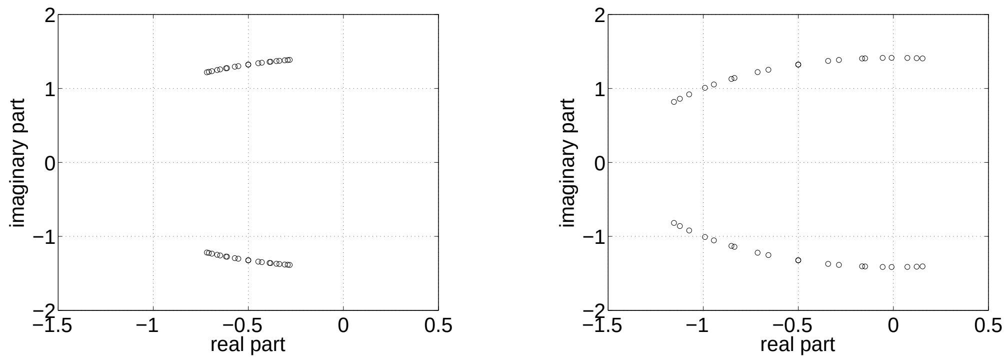

Figure 9: Eigenvalues of Jacobian matrix in stochastic Galerkin method using all polynomials up to degree 5 for $p_2 \in [\frac{1}{2}, \frac{3}{2}]$ (left) and for $p_2 \in [-\frac{1}{2}, \frac{5}{2}]$ (right) corresponding to Duffing oscillator.

6 Conclusions

The expansions of the polynomial chaos have been applied to equilibria of autonomous systems of ordinary differential equations including random parameters. The stochastic Galerkin method yielded an approximation of the random-dependent equilibria by a larger coupled system. We proved the convergence of the stochastic Galerkin technique under certain assumptions, which allow for the application of the Theorem of Newton-Kantorovich. The assumptions are satisfied in case of symmetric Jacobian matrices together with a sufficiently fast convergence of the polynomial chaos expansions, for example. Furthermore, the equilibria of the coupled system inherit the stability of the equilibria corresponding to the original systems provided that a specific condition holds for the numerical ranges of the underlying Jacobian matrices. We presented numerical simulations of illustrative examples, which demonstrate the convergence and the stability properties. The concept of our proof of convergence may be useful also for conclusions on convergence of the stochastic Galerkin method in other fields of application.

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