Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 12/02

Tareq Amro, Anton Arnold and Matthias Ehrhardt

Optimized Far Field Boundary Conditions for Hyperbolic Systems

January 2012

http://www.math.uni-wuppertal.de

NUMERICAL MATHEMATICS: Theory, Methods and Applications
Numer. Math. Theor. Meth. Appl., Vol. **xx**, No. x, pp. 1-44 (200x)

Optimized Far Field Boundary Conditions for Hyperbolic Systems

Tareq Amro[1,*], Anton Arnold[2] and Matthias Ehrhardt[3]

[1] *Faculty of Science and Technology, Hebron University P.O. Box 40, Hebron, West Bank via Israel*
[2] *Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstr. 8, A-1040 Wien, Austria*
[3] *Lehrstuhl für Angewandte Mathematik und Numerische Analysis, Fachbereich C Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gaußstrasse 20, D-42119 Wuppertal, Germany.*

Abstract. In this paper we improve large-time *absorbing boundary conditions* (ABCs) for first order hyperbolic systems. These boundary conditions combine the properties of ABCs for transient solutions and the properties of transparent boundary conditions for steady state problems. Initially they were defined up to an arbitrary matrix factor and then we develop a general strategy to specify this matrix factor in an optimal way with respect to the absorption of outgoing waves. Well-posedness of the resulting initial boundary value problem is studied in detail, and convergence in time of the transient solution to the steady state is established. Afterwards we consider a Lax-Wendroff-type finite difference scheme to solve the resulting initial boundary value problem and apply the GKS-stability theory to prove the stability of this scheme. Numerical examples are presented, to discuss convergence to the correct steady state and to compare the numerical solutions for different choices of the scaling matrix, illustrating the usefulness of our approach.

AMS subject classifications: 65M06, 35L50

Key words: absorbing boundary conditions, hyperbolic system, well-posedness, GKS-stability, far field boundary conditions.

1. Introduction

For the computation of a numerical solution to a hyperbolic partial differential equation on an unbounded domain, it is common practice to perform the calculation on a truncated finite computational domain Ω. This issue raises the problem of choosing appropriate boundary conditions for the resulting artificial boundary Γ. Ideally, these boundary conditions should prevent any nonphysical reflection of outgoing waves and should be easy to

*Corresponding author. *Email addresses:* `Tareqa@hebron.edu` (Tareq Amro), `anton.arnold@tuwien.ac.at` (Anton Arnold), `ehrhardt@math.uni-wuppertal.de` (Matthias Ehrhardt)

implement. They should also present together with the governing equation a well-posed truncated problem which is a necessary requirement for the corresponding numerical approximation to be stable. Examples of hyperbolic equations include the Euler equations of gas dynamics, the shallow water equations, Maxwell's equations, and equations of magnetohydrodynamics. For these hyperbolic problems the correct boundary condition is that waves traveling across the boundary should not be reflected back. Boundary conditions with this property are often referred to as non-reflecting, transparent, artificial or absorbing boundary conditions (ABCs).

The theoretical basis for ABCs stems from a pioneering paper by Engquist and Majda [24] which discusses both ideal ABCs and a general method for constructing approximate forms. In addition, Kreiss [52] analyzes the well-posedness of the initial boundary value problems (IBVP) for hyperbolic systems. Many researchers have been active in this area in the last years, the readers are referred to [19, 27, 39, 40, 64, 65] for further details. However, their work has been mainly concerned with ABCs that are better suited for a transient solution than for a steady solution, and most of these boundary conditions may lead to steady solutions of poor accuracy.

In this paper, we are concerned with ABCs that lead to accurate steady solutions. In this context, Bayliss and Turkel [14] derived non-reflecting conditions for the Euler equations used for steady state calculations. These boundary conditions are obtained from expansions of the solution at large distances. Accurate boundary conditions for the steady Euler equations in a channel geometry were also studied by Giles [26].

We base our considerations on the paper of Engquist and Halpern [23]. They constructed a new class of boundary conditions that combine the properties of ABCs for transient solutions and the properties of transparent boundary conditions for steady state problems. These boundary conditions, which are called *far field boundary conditions* (FBCs), can be used in both the transient regime and when the solution approaches the steady state. In this sense, they can be applied when the evanescent and traveling waves are present in the time-dependent calculation or when a time-dependent formulation is used for computations until the steady state.

For hyperbolic systems, these FBCs are defined up to matrix factor in front of the steady terms [23]. This poses the following computational problem (which is one of the main subjects of this work): How to choose this factor in a way to accelerate the convergence to the steady state, and to improve the accuracy of the transient solution.

Since the problem has wavelike solutions, these FBCs must model the radiation of energy out of the computational domain Ω. An incorrect specification of these boundary conditions can cause spurious reflected waves to be generated at the artificial boundary Γ. These waves represent energy propagating back into Ω. Since they are not part of the desired solution, they can substantially reduce the accuracy of the computed solution. On the other hand, if the time-dependent equations are only an intermediate step toward computing the steady state, then a flow of energy into Ω can delay or destroy the convergence to the steady state. Conversely, the correct specification of FBCs can accelerate the convergence. Thus, an answer of the above question consists in minimizing the spurious reflections.

Our work consists of two parts: an analytic part and a numerical (discrete) part. In the first analytic part we study briefly in Section 2.1 the procedure of constructing FBCs for linear hyperbolic systems, and propose a general tool of scaling the included factors. In Section 2.2 the well-posedness and regularity of the resulting IBVP are studied. A general result of convergence in time to the steady state is established in Section 2.3.

In the numerical part of this article, in Section 3.1, we introduce a finite difference scheme to solve the resulting IBVP. In Section 3.2, we apply the well known stability theory due to Gustafsson, Kreiss, and Sundström (GKS-theory) [36] to prove the stability of the proposed scheme. Two numerical examples are presented in Section 3.3. In the first example, we discuss briefly a 2×2 model system and show the convergence of this system with first order FBCs to the correct steady state. In the second example, we compare the numerical approximations for different choices of the scaling matrices for a 3×3 system.

2. Optimized Far Field Boundary Conditions for Hyperbolic Systems

2.1. Derivation of far field boundary conditions

We consider in this section the derivation of a hierarchy of FBCs, t $x = 0$ and $x = 1$, for a strictly hyperbolic system of the form

$$u_t + \Lambda u_x + Cu = f(x), \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1}$$

with the initial function

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2.2}$$

Here C and Λ are constant $n \times n$ matrices, u is a vector with n components. $f(x)$ and $u_0(x)$ are assumed to be C^∞-smooth functions and have supports in $(0,1)$. The eigenvalues of Λ are distinct and different form zero, that is

$$\Lambda = \begin{pmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{pmatrix}, \tag{2.3}$$

with $\Lambda^+ = \text{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_j > 0$, $\Lambda^- = \text{diag}(\lambda_{m+1}, \dots, \lambda_n)$, $\lambda_j < 0$. We assume that

$$C_1 = \frac{C^\top + C}{2} \geq \delta I, \qquad \delta > 0, \tag{2.4}$$

which is a necessary condition to ensure the convergence of the whole space problem to the steady state as $t \to \infty$. We will further assume that $\Lambda^{-1} C$ has distinct eigenvalues.

Notation . Any $n \times n$-matrix X is partitioned as

$$X = \begin{pmatrix} X^{++} & X^{+-} \\ X^{-+} & X^{--} \end{pmatrix},$$

where X^{++}, X^{+-}, X^{-+} are $m \times m$, $m \times (n-m)$, $(n-m) \times m$-matrices, respectively.

Also, we set

$$X^+ := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} X \qquad \text{and} \qquad X^- := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} X.$$

Solutions of (2.1) are made up of n different modes, which propagate at different speeds. A crucial step for developing boundary conditions for (2.1) is to determine the direction of propagation of each mode, and distinguishing which modes are outgoing and which are incoming at the boundary.

If we take a Laplace transform in t, with the dual variable s

$$\tilde{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\, dt, \quad s \in \mathbb{C}, \quad \text{Re}\, s > 0,$$

the system (2.1) becomes

$$\tilde{u}_x + \Lambda^{-1}(sI + C)\tilde{u} = \Lambda^{-1}\tilde{f}.$$

Define $E(s) := \Lambda^{-1} + \frac{1}{s}\Lambda^{-1}C$, we may write

$$\tilde{u}_x + sE(s)\tilde{u} = \Lambda^{-1}\tilde{f}. \tag{2.5}$$

Next, we wish to separate $\tilde{u}$ into "rightgoing" and "leftgoing" modes. Each of these modes corresponds to an eigenvalue of $E(s)$.

Definition 2.1. *[16] The inertia of a matrix M is the ordered triple $\mathrm{i}(M) = (\mathrm{i}_+(M), \mathrm{i}_-(M), \mathrm{i}_0(M))$, where $\mathrm{i}_+(M), \mathrm{i}_-(M)$, and $\mathrm{i}_0(M)$ are the numbers of eigenvalues of M with, respectively, positive, negative, and zero real part, all counting multiplicity.*

Lemma 2.1. *[16] Let G, H be $n \times n$-matrices with H Hermitian and regular, suppose $HG + G^*H$ is a positive semi-definite and, $\mathrm{i}_0(G) = 0$. Then $\mathrm{i}(G) = \mathrm{i}(H)$.*

Lemma 2.2. *For $\text{Re}\, s > 0$, $E(s)$ has exactly m eigenvalues with positive real part and $(n-m)$ with negative real part; i.e., $\mathrm{i}(\Lambda) = \mathrm{i}(E)$.*

Proof. Apply Lemma 2.1 with $H := \Lambda$ and $G := \Lambda^{-1}(sI + C)$:

$$\Lambda(\Lambda^{-1}(sI + C)) + (\Lambda^{-1}(sI + C))^*\Lambda = 2\,\text{Re}\, sI + 2C_1 > \delta I > 0.$$

Also $i_0(\Lambda^{-1}(sI + C)) = 0$; otherwise $\Lambda^{-1}(sI + C)$ will have a purely imaginary eigenvalue, say $i\omega$, $\omega \in \mathbb{R}$. Let ϕ denote its eigenvector. Then

$$i\omega\phi = \Lambda^{-1}(sI + C)\phi \iff (i\omega\Lambda - C)\phi = s\phi,$$

which is impossible since

$$\begin{aligned} 0 &< 2\,\text{Re}\, s\left|\phi\right|^2 = \langle s\phi, \phi\rangle + \langle \phi, s\phi\rangle = \langle (i\omega\Lambda - C)\phi, \phi\rangle + \langle \phi, (i\omega\Lambda - C)\phi\rangle \\ &= -\left\langle \phi, (C + C^\top)\phi \right\rangle \le -2\delta\left|\phi\right|^2 < 0. \end{aligned}$$

□

From Lemma 2.2 and [35], there exists $\eta_0 > 0$ and a nonsingular transformation $T(s)$ such that for Re $s > \eta_0$,

$$XEX^{-1} = D = \begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix}, \tag{2.6}$$

where $D(s)$ is the matrix of eigenvalues of $E(s)$, arranged so that $D^+(s)$ is an $m \times m$ positive definite matrix, corresponding to rightgoing solutions, and $D^-(s)$ is an $(n-m) \times (n-m)$ negative definite matrix, corresponding to leftgoing solutions. Here, we drop the explicit s-dependence; henceforth all the matrices are functions of s unless otherwise noted.
With the characteristic variables $\tilde{v} := X\tilde{u}$ the system (2.5) can be written as

$$\tilde{v}_x + sD\tilde{v} = X\Lambda^{-1}\tilde{f},$$

and then partitioned into

$$\frac{d}{dx}\begin{pmatrix} \tilde{v}^+ \\ \tilde{v}^- \end{pmatrix} - s\begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix}\begin{pmatrix} \tilde{v}^+ \\ \tilde{v}^- \end{pmatrix} = X\Lambda^{-1}\tilde{f},$$

where $\tilde{v}^+$ and $\tilde{v}^-$ represent purely “rightgoing” and “leftgoing” modes respectively.

Now, we restrict the domain of x in (2.1) to $(0,1)$. The exact nonreflecting boundary conditions follow immediately. Since there are no incoming modes at a nonreflecting boundary, at the left boundary $x = 0$ there should be no rightgoing modes, so the exact non-reflecting boundary condition reads

$$\tilde{v}^+ = [X\tilde{u}]^+ = 0, \qquad x = 0. \tag{2.7a}$$

At the right boundary, there should be no leftgoing modes, so the exact non-reflecting boundary condition is

$$\tilde{v}^- = [X\tilde{u}]^- = 0, \qquad x = 1. \tag{2.7b}$$

Two difficulties arise in implementing the above boundary conditions. First, since the boundary condition is expressed in the Laplace transformed (x,s)-space, and the matrix $X(s)$ contains non-rational functions of s (e.g., square roots), when we transform back to the physical (x,t)-space, the boundary conditions will be nonlocal in time. From a computational perspective, we would prefer a local boundary condition, which may be obtained by approximating non-rational elements of X by rational functions of s.

The second difficulty arises when approximations are introduced: then the resulting IBVP may be ill-posed. The theory of well-posedness will be discussed in the next section.

For high temporal frequencies, i.e. $s \to \infty$, we have $E(s) \to \Lambda^{-1}$ and hence $X(s) \to I$. Following standard practice in [23] we shall hence make a high frequency expansion of X for Re $s > \eta_0$:

$$X(s) = I + \frac{1}{s}X_1 + \frac{1}{s^2}X_2 + \mathrm{O}\left(\frac{1}{|s|^3}\right). \tag{2.8}$$

The zero order ABCs are then

$$\tilde{u}^+ = 0, \qquad x = 0,$$
$$\tilde{u}^- = 0, \qquad x = 1.$$

More accurate conditions are obtained by using higher order approximations. First and second order ABCs are respectively

$$\left[(I + \frac{1}{s}X_1)\tilde{u}\right]^+ = 0, \qquad x = 0,$$
$$\left[(I + \frac{1}{s}X_1)\tilde{u}\right]^- = 0, \qquad x = 1,$$

$$\left[(I + \frac{1}{s}X_1 + \frac{1}{s^2}X_2)\tilde{u}\right]^+ = 0, \qquad x = 0,$$
$$\left[(I + \frac{1}{s}X_1 + \frac{1}{s^2}X_2)\tilde{u}\right]^- = 0, \qquad x = 1,$$

which is transformed to the physical space by the formal substitution $s \to \frac{\partial}{\partial t}$.

Hence, the first and second order ABCs are respectively [24]

$$\left[(\frac{\partial}{\partial t} + X_1)u\right]^+ = 0, \qquad x = 0, \tag{2.9a}$$
$$\left[(\frac{\partial}{\partial t} + X_1)u\right]^- = 0, \qquad x = 1, \tag{2.9b}$$

$$\left[(\frac{\partial^2}{\partial t^2} + X_1\frac{\partial}{\partial t} + X_2)u\right]^+ = 0, \qquad x = 0, \tag{2.10a}$$
$$\left[(\frac{\partial^2}{\partial t^2} + X_1\frac{\partial}{\partial t} + X_2)u\right]^- = 0, \qquad x = 1. \tag{2.10b}$$

For large $\mathrm{Re}\, s > \eta_0$, the term $\frac{1}{s}\Lambda^{-1}C$ in $E(s)$ is a small perturbation of Λ^{-1}

With a high frequency expansion D is written as

$$D(s) = \Lambda^{-1} + \frac{1}{s}D_1 + \frac{1}{s^2}D_2 + \mathrm{O}\left(\frac{1}{|s|^3}\right),$$

where $D_j(s),\ j = 1, 2, \dots$ are diagonal. If we write (2.6) as $XE = DX$, then the $\mathrm{O}\left(|s|^{-1}\right)$-equation reads

$$X_1\Lambda^{-1} + \Lambda^{-1}C = \Lambda^{-1}X_1 + D_1.$$

Solving for X_1 and D_1 gives

$$D_1 = \mathrm{diag}\left(\frac{c_{11}}{\lambda_1}, \ldots, \frac{c_{nn}}{\lambda_n}\right),$$

and

$$(X_1)_{jk} = \begin{cases} 0, & j = k, \\ \dfrac{\lambda_k c_{jk}}{\lambda_k - \lambda_j}, & j \neq k, \end{cases}$$

where c_{jk} is the $(j,k)^{th}$ entry of C. The second order expansion of (2.6) yields

$$X_2\Lambda^{-1} + X_1\Lambda^{-1}C = \Lambda^{-1}X_2 + D_1X_1 + D_2.$$

Solving for X_2 and D_2 yields

$$D_2 = \mathrm{diag}\left(\sum_{k\neq 1}\frac{c_{1k}c_{k1}}{\lambda_k - \lambda_1}, \ldots, \sum_{k\neq n}\frac{c_{nk}c_{kn}}{\lambda_k - \lambda_n}\right),$$

and

$$(X_2)_{jk} = \begin{cases} 0, & j = k, \\ \dfrac{1}{\lambda_j - \lambda_k}\left[c_{jj}c_{jk}\dfrac{\lambda_k^2}{\lambda_k - \lambda_j} - \sum_{l\neq j} c_{jl}c_{lk}\dfrac{\lambda_j\lambda_k}{\lambda_l - \lambda_j}\right], & j \neq k. \end{cases}$$

Let us now turn to the stationary problem corresponding to (2.1) :

$$u_x + \Lambda^{-1}Cu = \Lambda^{-1}f(x), \qquad x \in \mathbb{R}, \tag{2.11}$$

$$u(x) \to 0, \qquad x \to \pm\infty. \tag{2.12}$$

The following lemma is similar to Lemma 2.2 but for the case $s = 0$.

Lemma 2.3. $\mathrm{i}(\Lambda) = \mathrm{i}(\Lambda^{-1}C)$.

Proof. Apply Lemma 2.1 with $H := \Lambda$ and $G := \Lambda^{-1}C$

$$\Lambda(\Lambda^{-1}C) + (\Lambda^{-1}C)^\top\Lambda = 2C_1 > 0.$$

Assume that $\Lambda^{-1}C$ has the purely imaginary eigenvalue $i\omega$. Then

$$i\omega\phi = \Lambda^{-1}C\phi \iff i\omega\Lambda\phi = C\phi.$$

But, on the other hand

$$\begin{aligned} 0 &= \langle i\omega\Lambda\phi, \phi\rangle + \langle\phi, i\omega\Lambda\phi\rangle \\ &= \langle C\phi, \phi\rangle + \langle\phi, C\phi\rangle = \left\langle\phi, (C + C^\top)\phi\right\rangle \geq 2\delta\left|\phi\right|^2 > 0, \end{aligned}$$

and hence $i_0(\Lambda^{-1}C) = 0$, which completes the proof. □

Using Lemma 2.3 and that $\Lambda^{-1}C$ has distinct eigenvalues, we diagonalize the system (2.11)

$$w_x + Rw = S\Lambda^{-1}f(x), \tag{2.13}$$

where w is given by $w := Su$,

$$S\Lambda^{-1}CS^{-1} = R = \begin{pmatrix} R^+ & 0 \\ 0 & R^- \end{pmatrix}, \tag{2.14}$$

$R^+ = \mathrm{diag}(r_1, \dots, r_m)$, Re $r_j > 0$, $R^- = \mathrm{diag}(r_{m+1}, \dots, r_n)$, Re $r_j < 0$

The following boundary conditions for the steady problem on the bounded domain $(0,1)$ are satisfied by the steady solution on the unbounded domain

$$(Su)^+ = 0, \qquad x = 0, \tag{2.15a}$$

$$(Su)^- = 0, \qquad x = 1. \tag{2.15b}$$

This holds since the general solution of (2.13) outside the support of f reads

$$w(x) = \begin{pmatrix} w^+(0)e^{-R^+x} \\ w^-(0)e^{-R^-x} \end{pmatrix}, \quad x \le 0, \qquad w(x) = \begin{pmatrix} w^+(1)e^{R^+(1-x)} \\ w^-(1)e^{R^-(1-x)} \end{pmatrix}, \quad x \ge 1,$$

where $w = (w^+, w^-)^\top$ is partitioned in the same way as u. For the decay condition (2.12) to be valid, it is necessary that

$$w^+(0) = w^-(1) = 0.$$

(2.15) is unique up to a multiplication by regular matrices V^+ and V^-, respectively

$$(VSu)^+ = 0, \qquad x = 0, \tag{2.16a}$$

$$(VSu)^- = 0, \qquad x = 1. \tag{2.16b}$$

Engquist and Halpern [23] defined a family of first order FBCs from a combination of the first order ABCs (2.9) and the transparent steady boundary conditions (2.16):

$$\left[(\frac{\partial}{\partial t} + VS)u \right]^+ = 0, \qquad x = 0, \tag{2.17a}$$

$$\left[(\frac{\partial}{\partial t} + VS)u \right]^- = 0, \qquad x = 1, \tag{2.17b}$$

which is defined up to a matrix factor, $V = \mathrm{diag}(V^+, V^-)$, in front of S. Higher order boundary conditions can formally be derived analogously

$$\left[(\frac{\partial^2}{\partial t^2} + X_1\frac{\partial}{\partial t} + VS)u \right]^+ = 0, \qquad x = 0, \tag{2.18a}$$

$$\left[(\frac{\partial^2}{\partial t^2} + X_1\frac{\partial}{\partial t} + VS)u \right]^- = 0, \qquad x = 1. \tag{2.18b}$$

The solution of the IVP (2.1)) on $(0,1)$ with the boundary conditions (2.17) or (2.18), for arbitrary regular V, converges for long time to the steady state, see [23] and Section 4. But since spurious reflections pollute the computed solution, a good choice of V^+ and V^- that annihilate the spurious reflections up to higher order can accelerate this convergence for long time computations and gives higher accuracy for short time computations.
To clarify that, we transform the first order left boundary condition (2.17a) into Laplace space, and use the notation

$$\left[(I+\frac{1}{s}VS)\tilde{u}\right]^+ = \left(I^+ + \frac{1}{s}V^+S^{++} \ , \ \frac{1}{s}V^+S^{+-}\right)\tilde{u} = 0. \tag{2.19}$$

In terms of the characteristic variables, $\tilde{u} = X^{-1}\tilde{v}$, where

$$X^{-1}(s) = I - \frac{1}{s}X_1 - \frac{1}{s^2}(X_2 - X_1^2) + \mathrm{O}\left(\frac{1}{|s|^3}\right).$$

(2.19) then becomes

$$\begin{aligned}
&\left(I^+ + \frac{1}{s}V^+S^{++} \ , \ \frac{1}{s}V^+S^{+-}\right)\begin{pmatrix} I^+ - \frac{1}{s}X_1^{++} & -\frac{1}{s}X_1^{+-} \\ -\frac{1}{s}X_1^{-+} & I^- - \frac{1}{s}X_1^{--}\end{pmatrix}\tilde{v} + \mathrm{O}\left(\frac{1}{|s|^2}\right) \\
&\qquad = \left[I^+ + \frac{1}{s}(V^+S^{++} - X_1^{++})\right]\tilde{v}^+ + \frac{1}{s}\left[V^+S^{+-} - X_1^{+-}\right]\tilde{v}^- + \mathrm{O}\left(\frac{1}{|s|^2}\right) = 0.
\end{aligned}$$

Neglecting $\mathrm{O}(|s|^{-2})$-terms, we may solve for the incoming (rightgoing) modes in terms of outgoing ones as long as $\left[I^+ + \frac{1}{s}(V^+S^{++} - X_1^{++})\right]$ is nonsingular (this holds true at least for $|s|$ large)

$$\tilde{v}^+(0) = -\left[sI^+ + (V^+S^{++} - X_1^{++})\right]^{-1}\left[V^+S^{+-} - X_1^{+-}\right]\tilde{v}^-(0) =: R_c^+\tilde{v}^-(0),$$

where R_c^+ is the matrix of reflection coefficients.

Similarly the right boundary condition (2.17b) may be written in terms of the characteristic variables as

$$\begin{aligned}
&\left(\frac{1}{s}V^-S^{-+} \ , \ I^- + \frac{1}{s}V^-S^{--}\right)\begin{pmatrix} I^+ - \frac{1}{s}X_1^{++} & -\frac{1}{s}X_1^{+-} \\ -\frac{1}{s}X_1^{-+} & I^- - \frac{1}{s}X_1^{--}\end{pmatrix}\tilde{v} \\
&\qquad = \frac{1}{s}\left[V^-S^{-+} - X_1^{-+}\right]\tilde{v}^+ + \left[I^- + \frac{1}{s}(V^-S^{--} - X_1^{--})\right]\tilde{v}^- + \mathrm{O}\left(\frac{1}{|s|^2}\right) = 0.
\end{aligned}$$

Neglecting $\mathrm{O}(|s|^{-2})$-terms and solving for the incoming (leftgoing) modes in terms of outgoing ones as long as $\left[I^- + \frac{1}{s}(V^-S^{--} - X_1^{--})\right]$ is nonsingular, gives

$$\tilde{v}^-(1) = -\left[sI^- + (V^-S^{--} - X_1^{--})\right]^{-1}\left[V^-S^{-+} - X_1^{-+}\right]\tilde{v}^+(1) =: R_c^-\tilde{v}^+(1),$$

where R_c^- is the matrix of reflection coefficients at the right boundary.

For the pair of boundary conditions to be absorbing up to order $O(|s|^{-2})$, the matrices R_c^+ and R_c^- must be identically zero, that is $V^+S^{+-} - X_1^{+-}$ and $V^-S^{-+} - X_1^{-+}$ must be zero. So the optimal choices of V^+ and V^- are then given as solutions of

$$V^+S^{+-} = X_1^{+-}, \tag{2.20a}$$

$$V^-S^{-+} = X_1^{-+}. \tag{2.20b}$$

If $(S^{+-})^{-1}$ exists, then $V^+ = X_1^{+-}(S^{+-})^{-1}$ and the first order FBC at $x=0$ reads

$$u_t^+ + X_1^{+-}(S^{+-})^{-1}S^{++}u^+ + X_1^{+-}u^- = 0, \tag{2.21a}$$

which is different from the first order ABC (2.9a) only by the middle term.
Similarly if $(S^{-+})^{-1}$ exists, then $V^- = X_1^{-+}(S^{-+})^{-1}$ and the first order FBC at $x=1$ is

$$u_t^- + X_1^{-+}u^+ + X_1^{-+}(S^{-+})^{-1}S^{--}u^- = 0. \tag{2.21b}$$

We shall denote these FBCs as

$$\left[(\frac{\partial}{\partial t} + \hat{X}_1)u\right]^+ = 0, \qquad x = 0, \tag{2.22a}$$

$$\left[(\frac{\partial}{\partial t} + \hat{X}_1)u\right]^- = 0, \qquad x = 1, \tag{2.22b}$$

where

$$\hat{X}_1 = \begin{pmatrix} X_1^{+-}(S^{+-})^{-1}S^{++} & X_1^{+-} \\ X_1^{-+} & X_1^{-+}(S^{-+})^{-1}S^{--} \end{pmatrix}.$$

For the second order case, we write (2.18a) in terms of the characteristic variables:

$$\begin{aligned}
&\left(I^+ + \frac{1}{s}X_1^{++} + \frac{1}{s^2}V^+S^{++} \;,\; \frac{1}{s}X_1^{+-} + \frac{1}{s^2}V^+S^{+-}\right)\tilde{u} \\
&= \left(I^+ + \frac{1}{s}X_1^{++} + \frac{1}{s^2}V^+S^{++} \;,\; \frac{1}{s}X_1^{+-} + \frac{1}{s^2}V^+S^{+-}\right) \\
&\quad \begin{pmatrix} I^+ - \frac{1}{s}X_1^{++} - \frac{1}{s^2}[X_2 - X_1^2]^{++} & -\frac{1}{s}X_1^{+-} - \frac{1}{s^2}[X_2 - X_1^2]^{+-} \\ -\frac{1}{s}X_1^{-+} - \frac{1}{s^2}[X_2 - X_1^2]^{-+} & I^- - \frac{1}{s}X_1^{--} - \frac{1}{s^2}[X_2 - X_1^2]^{--} \end{pmatrix} \tilde{v} + O\left(\frac{1}{|s|^3}\right) \\
&= \left[I^+ + \frac{1}{s^2}(V^+S^{++} - X_2^{++})\right]\tilde{v}^+ + \frac{1}{s^2}\left[V^+S^{+-} - X_2^{+-}\right]\tilde{v}^- + O\left(\frac{1}{|s|^3}\right).
\end{aligned}$$

The optimal choice of V^+ is to annihilate the coefficient of the outgoing mode up to order $O(|s|^{-2})$. Similar computations at the right boundary condition give the analogous equation for V^-. Finally V^+ and V^- are chosen as solutions of

$$V^+S^{+-} = X_2^{+-}, \tag{2.23a}$$

$$V^-S^{-+} = X_2^{-+}. \tag{2.23b}$$

Again, if $(S^{+-})^{-1}$ and $(S^{-+})^{-1}$ exist, then the second order FBCs can be written as

$$\left[(\frac{\partial^2}{\partial t^2} + X_1 \frac{\partial}{\partial t} + \hat{X}_2) u \right]^+ = 0, \qquad x = 0, \tag{2.24a}$$

$$\left[(\frac{\partial^2}{\partial t^2} + X_1 \frac{\partial}{\partial t} + \hat{X}_2) u \right]^- = 0, \qquad x = 1, \tag{2.24b}$$

where

$$\hat{X}_2 = \begin{pmatrix} X_2^{+-}(S^{+-})^{-1}S^{++} & X_2^{+-} \\ X_2^{-+} & X_2^{-+}(S^{-+})^{-1}S^{--} \end{pmatrix}.$$

In the case S^{+-} and/or S^{-+} are not invertible, generalized solutions of (2.20),(2.23) have to be considered.

General cases: Considering the systems (2.20), let V^{*+} and V^{*-} denote generalized solutions of (2.20a) and (2.20b) respectively. Then we have two cases:

- $m \geq n - m$, then equation (2.20a) can be written as

  $$(S^{+-})^\top (V^+)^\top = (X_1^{+-})^\top.$$

  Let $v_1^{(i)}$ and $b_1^{(i)}$ be the i^{th} columns of $(V^+)^\top$ and $(X_1^{+-})^\top$, respectively. Then this system is equivalent to the m underdetermined systems

  $$(S^{+-})^T v_1^{(i)} = b_1^{(i)}, \quad i = 1, \ldots, m.$$

  the solution $(v_1^*)^{(i)} \in \mathbb{R}^m$ (in the least-squares sense, that is minimizing the Euclidean norm of residuals $\left\| (S^{+-})^\top v_1^{(i)} - b_1^{(i)} \right\|^2, \quad i = 1, \ldots, m$) is given by

  $$(v_1^*)^{(i)} = S^{+-}((S^{+-})^\top S^{+-})^{-1} b_1^{(i)}, \quad i = 1, \ldots, m.$$

  The solution exists and is unique if S^{+-} has full rank. If S^{+-} does not have full rank, then the solution is not unique, since in this case if $(v_1^*)^{(i)}$ is a solution then the vector $(v_1^*)^{(i)} + z$ with $z \in Ker(S^{+-})$, is a solution too. A further constraint is introduced to enforce uniqueness of the solution. Typically, one requires that $(v_1^*)^{(i)}$ has minimal Euclidean norm.
  On the other side, Equation (2.20b) is equivalent to $n - m$ overdetermined systems

  $$(S^{-+})^\top v_2^{(i)} = b_2^{(i)}, \quad i = 1, \ldots, n - m,$$

  where $v_2^{(i)}$ and $b_2^{(i)}$ are the i^{th} columns of $(V^-)^\top$ and $(X_1^{-+})^\top$ respectively. The general solution is given by

  $$(v_2^*)^{(i)} = (S^{-+}(S^{-+})^\top)^{-1} S^{-+} b_2^{(i)}, \quad i = 1, \ldots, n - m.$$

- The case $m < n-m$, is similar but with $n-m$ underdetermined and m overdetermined systems.

This generalization applies to the case of (2.23).

2.2. Well-posedness of the IBVP

In this section, following the book of Kreiss and Lorenz [53], we discuss the well-posedness of the IVP

$$u_t + \Lambda(x,t)u_x + C(x,t)u = f(x,t), \qquad 0 < x < 1, \ \ t \geq 0, \tag{2.25a}$$

$$u(x,0) = u_0(x), \qquad 0 < x < 1, \tag{2.25b}$$

together with boundary conditions of the form

$$\left[(\frac{\partial}{\partial t} + B(t))u \right]^+ = 0, \qquad x = 0, \tag{2.26a}$$

$$\left[(\frac{\partial}{\partial t} + B(t))u \right]^- = 0, \qquad x = 1. \tag{2.26b}$$

$B(t)$ is partitioned in the same way as in notation :

$$B(t) = \begin{pmatrix} S_0(t) & K_0(t) \\ \\ S_1(t) & K_1(t) \end{pmatrix}.$$

We assume that S_0, S_1, K_0, and K_1 are uniformly bounded for all $t \geq 0$. Clearly the first order ABCs (2.9), and the first order FBCs (2.22) are special cases of (2.26).

$\Lambda(x,t), C(x,t) \in \mathbb{R}^{n\times n}$ and $f(x,t), u_0(x) \in \mathbb{R}^n$ are assumed to be C^∞-smooth functions with respect to all variables. Moreover, $f(x,t), u_0(x)$ are assumed to vanish in a neighborhood of the corners $(x,t) = (0,0)$, $(x,t) = (1,0)$. Using this compatibility assumption, we write the boundary conditions (2.26) in the integral form:

$$u^+(0,t) = -\int_0^t S_0(\tau)u^+(0,\tau)d\tau - \int_0^t K_0(\tau)u^-(0,\tau)d\tau, \tag{2.27a}$$

$$u^-(1,t) = -\int_0^t S_1(\tau)u^-(1,\tau)d\tau - \int_0^t K_1(\tau)u^+(1,\tau)d\tau. \tag{2.27b}$$

Roughly speaking, the initial value problem (2.25) with boundary conditions (2.27) is called well-posed if for all smooth compatible data u_0 and f there is a unique smooth solution u, and in every finite interval $0 \leq t \leq T$ the solution can be estimated in terms of the data.

We define the outflow and inflow norms respectively by

$$\|u(t)\|_+^2 := \sum_{\lambda_j(1,t)>0} \lambda_j(1,t)|u_j(1,t)|^2 - \sum_{\lambda_j(0,t)<0} \lambda_j(0,t)|u_j(0,t)|^2,$$

and

$$\|u(t)\|_-^2 := \sum_{\lambda_j(0,t)>0} \lambda_j(0,t)|u_j(0,t)|^2 - \sum_{\lambda_j(1,t)<0} \lambda_j(1,t)|u_j(1,t)|^2.$$

Lemma 2.4. *Assume that the boundary is not characteristic and that $\Lambda(x,t)$, $\Lambda_x(x,t), C(x,t), B(t)$ are uniformly bounded for all $0 \le x \le 1$ and $0 \le t \le T$. Then, for every finite time interval $0 \le t \le T$ there is a constant C_T such that, if u solves the IBVP (2.25),(2.27) for $0 \le t \le T$, then*

$$\|u(.,t)\|_2^2 + \int_0^t \left(\|u(\tau)\|_-^2 + \|u(\tau)\|_+^2\right) d\tau \ \le C_T \left(\|u_0\|_2^2 + \int_0^t \|f(.,\tau)\|_2^2 d\tau\right),$$

C_T is independent of f and u_0.

Proof.

$$\begin{aligned} \frac{d}{dt}\|u(.,t)\|_2^2 &= (u,u_t) + (u_t,u) \\ &\le -(u,\Lambda u_x) - (\Lambda u_x, u) + c_1\left\{\|u(.,t)\|_2^2 + \|f(.,t)\|_2^2\right\}. \end{aligned}$$

Integration by parts gives

$$(u,\Lambda u_x) + (u_x, \Lambda u) = \langle u, \Lambda u\rangle\big|_0^1 - (u, \Lambda_x u),$$

moreover,

$$\langle u, \Lambda u\rangle\big|_0^1 = \|u(t)\|_+^2 - \|u(t)\|_-^2.$$

Since Λ_x is uniformly bounded, we have

$$\frac{d}{dt}\|u(.,t)\|_2^2 \le \|u(t)\|_-^2 - \|u(t)\|_+^2 + c_2\left\{\|u(.,t)\|_2^2 + \|f(.,t)\|_2^2\right\}. \tag{2.28}$$

Choose

$$\rho_1 := \max_{j=1,\dots,n}\left(|\lambda_j(1,t)|, |\lambda_j(0,t)|\right), \quad 0 \le t \le T,$$

then

$$\|u(t)\|_-^2 \le \rho_1\left(|u^+(0,t)|^2 + |u^-(1,t)|^2\right). \tag{2.29}$$

The Cauchy-Schwarz inequality for the boundary condition (2.27a) yields

$$\begin{aligned} |u^+(0,t)|^2 &\le 2\left|\int_0^t S_0(\tau)u^+(0,\tau)d\tau\right|^2 + 2\left|\int_0^t K_0(\tau)u^-(0,\tau)d\tau\right|^2 \\ &\le 2t\left(s_0^2\int_0^t \left|u^+(0,\tau)\right|^2 d\tau + k_0^2\int_0^t \left|u^-(0,\tau)\right|^2 d\tau\right), \end{aligned}$$

where $\left|S_0(t)\right| \le s_0$ and $|K_0(t)| \le k_0$ for $0 \le t \le T$.
In a similar way, the boundary condition (2.27b) can be estimated as

$$\left|u^-(1,t)\right|^2 \le 2t\left(s_1^2\int_0^t \left|u^-(1,\tau)\right|^2 d\tau + k_1^2\int_0^t \left|u^+(1,\tau)\right|^2 d\tau\right).$$

With $\hat{k} = \max(k_1, k_0)$ and $\hat{s} = \max(s_0, s_1)$, (2.29) becomes

$$\|u(t)\|_-^2 \leq 2\rho_1 t \left(\hat{s}^2 \int_0^t \left(\left|u^+(0,\tau)\right|^2 + \left|u^-(1,\tau)\right|^2 \right) d\tau \right.$$
$$\left. + \hat{k}^2 \int_0^t \left(\left|u^-(0,\tau)\right|^2 + \left|u^+(1,\tau)\right|^2 \right) d\tau \right). \tag{2.30}$$

Consider

$$z(t) := \|u(t)\|_2^2 + \int_0^t \left(\|u(\tau)\|_-^2 + \|u(\tau)\|_+^2 \right) d\tau.$$

Using (2.28), (2.30) yields

$$\begin{aligned} z'(t) &\leq 2\|u(t)\|_-^2 + c_2 \left\{ \|u(.,t)\|_2^2 + \|f(.,t)\|_2^2 \right\} \\ &\leq 4\rho_1\rho_2 t \left(\hat{s}^2 \int_0^t \|u(\tau)\|_-^2 d\tau + \hat{k}^2 \int_0^t \|u(\tau)\|_+^2 d\tau \right) + c_2 \left\{ \|u(.,t)\|_2^2 + \|f(.,t)\|_2^2 \right\} \\ &\leq \alpha \left(\|u(.,t)\|_2^2 + \int_0^t \|u(\tau)\|_-^2 + \|u(\tau)\|_+^2 d\tau \right) + c_2 \|f(.,t)\|_2^2, \end{aligned}$$

where $\alpha := 4\rho_1\rho_2 T(s^2 + k^2) + c_2 > 0$, and

$$\rho_2 := \max_{j=1,\ldots,n} \left(|\lambda_j^{-1}(1,t)|, |\lambda_j^{-1}(0,t)| \right), \ \ 0 \leq t \leq T.$$

ρ_2 is used for the estimates:
$|u^+(1,t)|^2 + |u^-(0,t)|^2 \leq \rho_2 \|u(t)\|_+^2$ and $|u^-(0,t)|^2 + |u^+(1,t)|^2 \leq \rho_2 \|u(t)\|_-^2$.
The Gronwall inequality gives the result

$$z(t) \leq C_T \left(\|u_0\|_2^2 + \int_0^t \|f(.,\tau)\|_2^2 d\tau \right),$$

where $C_T := (c_2 + 1)e^{\alpha T}$. □

Remark 2.1. *The last proof, as well as the proof of Theorem 7.6.4 [53], can be done under weaker regularity assumptions, namely $f \in L^2((0,T), L^2((0,1);\mathbb{R}^n))$ and $u_0 \in L^2((0,1);\mathbb{R}^n)$. First derived for the classical solution, the result for mild solutions then follows from a density argument.*

Theorem 2.1. *Under the assumptions of Lemma 2.4 and $u_0 \in L^2((0,1);\mathbb{R}^n)$ and $f \in L^2((0,T), L^2((0,1);\mathbb{R}^n))$ the IBVP (2.25),(2.27) has a unique mild solution in $L^2((0,T), L^2((0,1);\mathbb{R}^n))$.*

Proof. A fixed point method will be used to show the existence and uniqueness of the solution. For $g = \left(g^+, g^-\right) \in L^2((0,T);\mathbb{R}^n)$ solve the equation:

$$y_t + \Lambda(x,t)y_x + C(x,t)y = f(x,t), \qquad 0 < x < 1,\ 0 \le t \le T, \tag{2.31a}$$

$$y(x,0) = u_0(x), \qquad 0 \le x \le 1, \tag{2.31b}$$

$$y^+(0,t) = g^+(t), \tag{2.31c}$$

$$y^-(1,t) = g^-(t). \tag{2.31d}$$

Theorem 7.6.4 [53] and Remark 2.1 guarantee the existence of unique solution $y \in C([0,T], L^2((0,1);\mathbb{R}^n))$, and $y(0,.), y(1,.) \in L^2((0,T);\mathbb{R}^n)$. Define $Fg = ((Fg)^+, (Fg)^-)$ as

$$(Fg)^+(t) := -\int_0^t S_0(\tau)y^+(0,\tau)d\tau - \int_0^t K_0(\tau)y^-(0,\tau)d\tau, \qquad t \ge 0,$$

$$(Fg)^-(t) := -\int_0^t S_1(\tau)y^-(1,\tau)d\tau - \int_0^t K_1(\tau)y^+(1,\tau)d\tau, \qquad t \ge 0.$$

The first to show is that F maps $L^2((0,T);\mathbb{R}^n)$ into itself:
The solution of (2.31)) can be estimated [53], for $0 \le t \le T$ as

$$\begin{aligned} &\|y(.,t)\|_2^2 + \int_0^t \left(|y(0,\tau)|^2 + |y(1,\tau)|^2\right) d\tau \\ &\le K_T \left[\|u_0\|_2^2 + \int_0^t \left(\|f(.,\tau)\|_2^2 + |g(\tau)|^2\right) d\tau \right]. \end{aligned} \tag{2.32}$$

A similar computation as in the last proof shows that with $\alpha_1 = k_0^2 + s_0^2 + s_1^2 + k_1^2$

$$|Fg(t)|^2 \le 2\alpha_1 t \int_0^t \left(|y(0,\tau)|^2 + |y(1,\tau)|^2\right) d\tau.$$

Integration by parts, using (2.32) and that $g \in L^2((0,T);\mathbb{R}^n)$, gives

$$\begin{aligned} \|Fg\|_2^2 &\le \alpha_1 \left(T^2 \int_0^T \left(|y(0,t)|^2 + |y(1,t)|^2\right) dt - t^2 \left(|y(0,t)|^2 + |y(1,t)|^2\right) \right) \\ &\le Constant. \end{aligned}$$

The second to show is that F is contractive at least on a subinterval $(0,T_1)$ of $(0,T)$:
Given two inflow data: $g_1, g_2 \in L^2((0,T);\mathbb{R}^n)$, the difference between the corresponding outflow data can be estimated by (2.32), for $0 \le t \le T$ as

$$\begin{aligned} &\|(y_1-y_2)(.,t)\|_2^2 + \int_0^t \left(|(y_1-y_2)(0,\tau)|^2 + |(y_1-y_2)(1,\tau)|^2\right) d\tau \\ &\qquad \le K_T \int_0^t |(g_1-g_2)(\tau)|^2 d\tau \end{aligned} \tag{2.33}$$

Now,

$$|Fg_1^+(t) - Fg_2^+(t)|^2 \leq 2t(s_0^2 + k_0^2) \int_0^t |(y_1 - y_2)(0, \tau)|^2 d\tau,$$

$$|Fg_1^-(t) - Fg_2^-(t)|^2 \leq 2t(s_1^2 + k_1^2) \int_0^t |(y_1 - y_2)(1, \tau)|^2 d\tau.$$

Using (2.33) we get

$$|Fg_1(t) - Fg_2(t)|^2 \leq 2t\alpha_1 \int_0^t \Big(|(y_1 - y_2)(0, \tau)|^2 + |(y_1 - y_2)(1, \tau)|^2\Big)\, d\tau.$$

Integration by parts gives

$$\begin{aligned} \|Fg_1 - Fg_2\|_2^2 \leq \alpha_1 \{T^2 & \int_0^T \Big(|(y_1 - y_2)(0, t)|^2 + |(y_1 - y_2)(1, t)|^2\Big)\, dt \\ & - t^2 \Big(|(y_1 - y_2)(0, t)|^2 + |(y_1 - y_2)(1, t)|^2\Big)\} \\ & \leq \alpha_1 K_T T^2 \|g_1 - g_2\|_2^2. \end{aligned}$$

F is contraction for $T_1 < 1/\sqrt{\alpha_1 K_T}$. The contractivity of F depends only on $\alpha_1 K_T$ (but not on the initial condition or the inhomogeneity term). So we can apply the iteration first on a subinterval $(0, T_1)$ of $(0, T)$, then $(T_1, 2T_1)$ and so on. The local solution can be continued in t to reach T. □

This theorem shows that our problem is strongly well-posed in the sense of Kreiss [45].

2.3. Convergence to the steady state

We now consider the convergence of the IVP (2.1)-(2.2) with first order FBCs (2.17) as $t \to \infty$ to the solution of the corresponding steady problem. The non-singular matrix V in (2.17) is used to accelerate the convergence to the steady state and it does not effect the convergence itself. For convenience we take V^+ and V^- as identity matrices. Then, we have

$$u_t + \Lambda u_x + Cu = f(x), \qquad 0 < x < 1, \quad t > 0, \tag{2.34a}$$

$$u(x, 0) = u_0(x), \qquad 0 < x < 1, \tag{2.34b}$$

$$\left[(\frac{\partial}{\partial t} + S)u\right]^+ = 0, \qquad x = 0, \tag{2.34c}$$

$$\left[(\frac{\partial}{\partial t} + S)u\right]^- = 0, \qquad x = 1. \tag{2.34d}$$

The corresponding steady state problem reads

$$\Lambda u_x^* + Cu^* = f(x), \qquad 0 < x < 1, \tag{2.35a}$$

with transparent boundary conditions

$$\big(Su^*\big)^+ = 0, \qquad x = 0, \tag{2.35b}$$
$$\big(Su^*\big)^- = 0, \qquad x = 1. \tag{2.35c}$$

Define:

$$q(x,t) := u(x,t) - u^*(x),$$

then q satisfies

$$q_t + \Lambda q_x + Cq = 0, \qquad 0 < x < 1,\ t \geq 0, \tag{2.36a}$$
$$q(x,0) = u_0(x) - u^*(x) =: q_0(x), \qquad 0 < x < 1, \tag{2.36b}$$
$$\left[(\frac{\partial}{\partial t} + S)q\right]^+ = 0, \qquad x = 0, \tag{2.36c}$$
$$\left[(\frac{\partial}{\partial t} + S)q\right]^- = 0, \qquad x = 1. \tag{2.36d}$$

By taking the Laplace transform (with dual variable s) of (2.36) we obtain the ordinary differential equation

$$s\tilde{q} + \Lambda\tilde{q}_x + C\tilde{q} = q_0(x), \qquad 0 < x < 1, \tag{2.37a}$$
$$s\tilde{q}^+ + (S\tilde{q})^+ = q_0^+(0), \qquad x = 0, \tag{2.37b}$$
$$s\tilde{q}^- + (S\tilde{q})^- = q_0^-(1), \qquad x = 1. \tag{2.37c}$$

The following lemma, which relates the asymptotic behavior of the original function with the limit of the image function, will be used to prove Theorem 2.2, the main theorem of this section.

Lemma 2.5. *[21] Suppose that $b(t)$ belongs to a Banach space with norm $\|.\|$ and $\tilde{b}(s)$ is its Laplace transform. Then we have*

$$\lim_{t\to\infty} b(t) = \lim_{s\to 0^+} s\tilde{b}(s),$$

provided that $\lim_{t\to\infty} b(t)$ exists.

Theorem 2.2. *If the solution $q(x,t)$ of (2.36) converges in $L^2(0,1)$ as $t \to \infty$, then it converges to zero in $L^2(0,1)$.*

Proof. To make use of the previous Lemma we need to show that $\tilde{q}(x,s)$ defined in (2.37) is a natural extension of the Laplace transform near $s = 0$. That is, $\tilde{q}(x,s)$ is an analytic function of s in the neighborhood of 0. Suppose that

$$Z := H^1((0,1);\mathbb{R}^n) \times \mathbb{R}^n, \qquad Y := L^2((0,1);\mathbb{R}^n) \times \mathbb{R}^n.$$

Define the operator L as

$$L : Y \supset Z \supset H^1(0,1)^n =: D(L) \ni \tilde{q} \longmapsto \begin{pmatrix} (\Lambda\partial_x + C)\tilde{q} \\ (S\tilde{q})^+(0) \\ (S\tilde{q})^-(1) \end{pmatrix} \in Y.$$

Accordingly, (2.37) can be written as

$$(L + sI) \begin{pmatrix} \tilde{q} \\ \tilde{q}^+(0) \\ \tilde{q}^-(1) \end{pmatrix} = \begin{pmatrix} q_0(x) \\ q_0^+(0) \\ q_0^-(1) \end{pmatrix}, \qquad \tilde{q}^+(0) \in \mathbb{R}^m, \qquad \tilde{q}^-(1) \in \mathbb{R}^{n-m}. \tag{2.38}$$

Focusing on the operator L, we notice that it has some useful properties:
First, L is invertible. Consider

$$q_1 = \begin{pmatrix} f(x) \\ \alpha^+ \\ \alpha^- \end{pmatrix} \in Y, \qquad \alpha^+ \in \mathbb{R}^m, \qquad \alpha^- \in \mathbb{R}^{n-m},$$

and search for $\tilde{q} \in D(L)$, such that

$$\begin{aligned} &\Lambda\tilde{q}_x + C\tilde{q} = f(x), \qquad 0 < x < 1, \\ &(S\tilde{q})^+(0) = \alpha^+, \\ &(S\tilde{q})^-(1) = \alpha^-. \end{aligned}$$

This inhomogeneous boundary value problem is equivalent to

$$\begin{aligned} &\breve{w}_x + R\breve{w} = h(x), \qquad 0 < x < 1, \\ &\breve{w}^+(0) = 0, \breve{w}^-(1) = 0, \end{aligned} \tag{2.39}$$

where $\breve{w} = S\tilde{q} - \begin{pmatrix} \alpha^+ \\ \alpha^- \end{pmatrix}$, $h(x) = S\Lambda^{-1} f(x) - (I + R)\begin{pmatrix} \alpha^+ \\ \alpha^- \end{pmatrix}$, and S is defined as in (2.14).
The last equation (2.39) has the unique solution

$$\breve{w}^+(x) = \int_0^x e^{-R^+(x-y)} h^+(y)dy, \qquad 0 < x < 1,$$

$$\breve{w}^-(x) = -\int_x^1 e^{-R^-(y-x)} h^-(y)dy, \qquad 0 < x < 1.$$

Second, L^{-1} is a bounded operator from Y to $D(L)$. Multiplying (2.37a) by $\tilde{q}$ and integrating

$$(\Lambda\tilde{q}_x, \tilde{q}) + (\tilde{q}, \Lambda\tilde{q}_x) + (C\tilde{q}, \tilde{q}) + (\tilde{q}, C\tilde{q}) = 2(f, \tilde{q}). \tag{2.40}$$

Integration by parts gives

$$(\tilde{q}, \Lambda\tilde{q}_x) + (\tilde{q}_x, \Lambda\tilde{q}) = \langle \tilde{q}, \Lambda\tilde{q} \rangle \big|_0^1,$$

using the properties of C

$$(C\tilde{q},\tilde{q})+(\tilde{q},C\tilde{q})=(\tilde{q},(C+C^{\top})\tilde{q})\geq\ 2\delta\|\tilde{q}\|_2^2>0,\quad \delta>0,$$

and that

$$2(f,\tilde{q})\leq\frac{2}{\delta}\|f\|_2^2+\frac{\delta}{2}\|\tilde{q}\|_2^2.$$

Thus, (2.40) gives

$$\langle\tilde{q},\Lambda\tilde{q}\rangle\,\Big|_0^1+\frac{3\delta}{2}\|\tilde{q}\|_2^2\leq\frac{2}{\delta}\|f\|_2^2. \tag{2.41}$$

Choose

$$\lambda_M:=\max_{j=1,\ldots,n}\left\{|\lambda_j|\right\},$$

then

$$\begin{aligned}\langle\tilde{q},\Lambda\tilde{q}\rangle\,\Big|_0^1 &= \sum_{j=1}^{m}\lambda_j(|\tilde{q}_j(1)|^2-|\tilde{q}_j(0)|^2)+\sum_{j=m+1}^{n}\lambda_j(|\tilde{q}_j(1)|^2-|\tilde{q}_j(0)|^2)\\ &\geq -\sum_{j=1}^{m}\lambda_j|\tilde{q}_j(0)|^2+\sum_{j=m+1}^{n}\lambda_j|\tilde{q}_j(1)|^2\\ &\geq -n\lambda_M(|\alpha^+|^2+|\alpha^-|^2).\end{aligned}$$

Substituting the result in (2.41), we get

$$\|\tilde{q}\|_2^2\leq C(\|f\|_2^2+|\alpha^+|^2+|\alpha^-|^2). \tag{2.42}$$

But

$$\tilde{q}_x=-\Lambda^{-1}C\tilde{q}+\Lambda^{-1}f,$$

then

$$\|\tilde{q}_x\|_2^2\leq C_1(\|\tilde{q}\|_2^2+\|f\|_2^2). \tag{2.43}$$

Using (2.42) and (2.43) we get

$$\|\tilde{q}\|_{D(L)}^2\leq C(\|f\|_2^2+|\alpha^+|^2+|\alpha^-|^2)=C\|q_1\|_Y. \tag{2.44}$$

Now, define $L_p^{-1}:=P\circ L^{-1}:Y\longmapsto Y$, where $P:Z\longmapsto Y$ is the identity compact operator. Since $L^{-1}:Y\longmapsto D(L)$ is bounded, then L_p^{-1} is compact.
Choose s small enough such that

$$|s|<\|L_p^{-1}\|^{-1}, \tag{2.45}$$

then $(I+sL_p^{-1})$ is an invertible operator from Y to Y. Moreover, the resolvent function $(I+sL_p^{-1})^{-1}$ is an analytic function of s with norm that satisfies

$$\|(\frac{I}{s}+L_p^{-1})^{-1}\|_Y\leq\frac{1}{\frac{1}{s}-\|L_p^{-1}\|_Y},$$

and so in particular

$$\|(\frac{I}{s}+L_p^{-1})^{-1}\|_Y \to 0 \quad as\ s \to 0. \tag{2.46}$$

Consider the first line of equation (2.38)

$$(L+sI)\tilde{q} = q_0, \tag{2.47}$$

and let s be as in (2.45). We apply L_p^{-1} to both sides of (2.47), then

$$(I+sL_p^{-1})\tilde{q} = L_p^{-1}q_0.$$

Since $(I+sL_p^{-1})$ is invertible, we obtain

$$\tilde{q} = (I+sL_p^{-1})^{-1}L_p^{-1}q_0,$$

which is analytic in s and bounded. The multiplication of the last equation by s yields

$$s\tilde{q} = (\frac{I}{s}+L_p^{-1})^{-1}L_p^{-1}q_0.$$

Using (2.46) and that L_p^{-1} is bounded, gives that $\|s\tilde{q}(x,s)\|_2 \to 0 \ \ as \ \ s \to 0$.
From Lemma 2.5 we obtain the result. □

3. Numerical Approximation

Consider the first order hyperbolic system in characteristic form

$$u_t + \Lambda u_x + Cu = f(x,t), \tag{3.1a}$$

in the stripe $0 < x < 1$, $t > 0$. Here, Λ and C are constant $n \times n$ matrices and Λ is partitioned as in (2.3). The solution of (3.1a) is uniquely determined if we prescribe initial values for $t = 0$:

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1, \tag{3.1b}$$

and boundary conditions at $x = 0,1$:

$$u_t^+ + (Su)^+ = 0, \qquad x = 0, \tag{3.1c}$$

$$u_t^- + (Su)^- = 0, \qquad x = 1. \tag{3.1d}$$

Here, S is defined as in (2.14). While (3.1c)–(3.1d) represent general FBCs. Furthermore, the support of f and u_0 are assumed to be in $(0,1)$.

We want to solve the above problem by a finite difference approximation. For that reason, we introduce a mesh size $h := \Delta x$, a time step $k := \Delta t$, and discretize the $(x,t)-$stripe $[0,1] \times \mathbb{R}_0^+$ using the mesh points

$$x_j = jh, \qquad j = 0,1,2,\dots,J t_l \qquad\qquad = lk, \qquad l = 0,1,2,\dots$$

We assume that $r := k/h$ is constant and use the notation $u_j^l \in \mathbb{R}^n$ to approximate the exact solution u at (x_j, t_l). According to the partition of u, we set

$$u_j^l = \begin{pmatrix} (u^+)_j^l \\ (u^-)_j^l \end{pmatrix}, \qquad \left(u^+\right)_j^l \in \mathbb{R}^m, \quad \left(u^-\right)_j^l \in \mathbb{R}^{n-m}.$$

in the rest of this part, the derived ABCs are discretized adequately and we show that the resulting difference scheme for the IBVP in 1D is stable in the sense of Gustafsson, Kreiss and Sundström [36]. As well as results from numerical experiments are presented.

3.1. Numerical scheme

Lax-Wendroff scheme (LW-scheme) based on the expansion

$$u(x,t+k) = u(x,t) + ku_t(x,t) + \frac{k^2}{2}u_{tt}(x,t) + O(k^3), \tag{3.2}$$

where u_{tt} can be determined using (3.1a) as follows

$$\begin{aligned} u_{tt} &= (-\Lambda u_x - Cu + f)_t \\ &= -\Lambda u_{tx} - Cu_t + f_t \\ &= \Lambda(\Lambda u_x + Cu - f)_x + C(\Lambda u_x + Cu - f) + f_t \\ &= \Lambda^2 u_{xx} + (\Lambda C + C\Lambda)u_x + C^2 u + f_t - \Lambda f_x - Cf. \end{aligned}$$

Substituting the above equation into (3.2) yields

$$\begin{aligned} u(x,t+k) &= u(x,t) - k(\Lambda u_x(x,t) + Cu(x,t) - f(x,t)) \\ &\quad + \frac{k^2}{2}(\Lambda^2 u_{xx}(x,t) + (\Lambda C + C\Lambda)u_x(x,t) + C^2 u(x,t) \\ &\quad + f_t(x,t) - \Lambda f_x(x,t) - Cf(x,t)) + O(k^3). \end{aligned}$$

The LW-scheme uses centered differences to approximate the spatial derivatives of u. Furthermore, the derivatives of f will be appropriately discertized. The resulting scheme will be then

$$\begin{aligned} u_j^{l+1} = u_j^l &- \frac{1}{2}r\Lambda(u_{j+1}^l - u_{j-1}^l) - kCu_j^l + \frac{1}{2}(r\Lambda)^2(u_{j+1}^l - 2u_j^l + u_{j-1}^l) \\ &+ \frac{1}{4}rk(\Lambda C + C\Lambda)(u_{j+1}^l - u_{j-1}^l) + \frac{1}{2}(kC)^2 u_j^l + \frac{1}{2}k(f_j^{l+1} + f_j^l) \\ &- \frac{1}{4}rk\Lambda(f_{j+1}^l - f_{j-1}^l) - \frac{1}{2}k^2 Cf_j^l, \qquad l = 0,1,\dots,\ j = 1,\dots,J-1. \end{aligned} \tag{3.3a}$$

To solve (3.3a) uniquely, we provide initial values

$$u_j^0 = u_0(x_j), \qquad j = 0,1,2,\dots,J, \tag{3.3b}$$

and specify at each time level $t_l = lk$, $l = 1,2,\dots$, boundary values u_0^{l+1}, u_J^{l+1}. These boundary values split into two groups: The first group, which we refer to as inflow boundary conditions is

$$\left(u^+\right)_0^{l+1}, \quad \left(u^-\right)_J^{l+1}.$$

The second group is

$$\left(u^-\right)_0^{l+1}, \quad \left(u^+\right)_J^{l+1},$$

which we refer to as the outflow boundary conditions.
The inflow values are determined by the discretization of the boundary conditions (3.1c)–(3.1d), while the outflow values are obtained by introducing numerical boundary conditions. In this work we will consider two types of numerical boundary conditions, the first type is upwinding in which u at the boundaries satisfy the homogeneous version of the system (3.1a), and the second type is first order extrapolation.

Definition 3.1. *The general horizontal extrapolation of order q for the outflow data u^- at $x = 0$ is*

$$\left(E_+ - I\right)^{q+1} \left(u^-\right)_0^{l+1} = 0, \quad q = 0,1,\dots,$$

and that of u^+ at $x = 1$

$$\left(I - E_+^{-1}\right)^{q+1} \left(u^+\right)_J^{l+1} = 0, \quad q = 0,1,\dots,$$

where $E_+ u_j := u_{j+1}$.

Using the boundary condition (3.1c), we write

$$D_+^t \left(u^+\right)_0^l + \left((Su)^+\right)_0^l = 0,$$

which gives

$$\left(u^+\right)_0^{l+1} = \left(u^+\right)_0^l - k\left((Su)^+\right)_0^l. \tag{3.4}$$

Since f is compactly supported in $(0,1)$, the outflow part of (3.1a) at $x = 0$ satisfies

$$u_t^- + \Lambda^- u_x^- + (Cu)^- = 0,$$

which is discretized as

$$D_+^t \left(u^-\right)_0^l + \Lambda^- D_+^x \left(u^-\right)_0^l + \left((Cu)^-\right)_0^l = 0.$$

Hence

$$\left(u^-\right)_0^{l+1} = \left(I + r\Lambda^-\right)\left(u^-\right)_0^l - k\left((Cu)^-\right)_0^l - r\Lambda^- \left(u^-\right)_1^l. \tag{3.5a}$$

An alternative numerical boundary condition is the first order extrapolation

$$\left(u^-\right)_0^{l+1} = 2\left(u^-\right)_1^{l+1} - \left(u^-\right)_2^{l+1}. \tag{3.5b}$$

The discretization of the right boundary conditions is treated in a similar manner.

3.2. Stability of the finite difference scheme

In solving linear hyperbolic partial differential equations numerically by means of finite difference approximations, a principal difficulty both theoretically and in practice is the question of stability. For the "Cauchy problem" on the unbounded domain $(-\infty,\infty)$, a fairly complete stability theory based on the Fourier analysis has been worked out during the last few decades by von Neumann, Lax, Kreiss, and others [57,59,62]. For the "initial boundary value problem" on a domain such as $[0,\infty)$ or $[0,1]$, however, Fourier analysis cannot be applied in a straightforward way, and progress has been slower and technically more complex. Important contributions in this area were made by S. Osher [56] and by H.-O. Kreis [51], and are based on various kinds of normal mode analysis that extend the Fourier methods. A comprehensive theory of this type was presented in an influential paper by Gustafsson, Kreiss, and Sundström (briefly: GKS) [36]. The complicated algebraic conditions of the GKS-theory were simplified in following work of Goldberg and Tadmor [37]. In this section we apply the GKS-theory to show the stability of the difference approximation (3.3)-(3.5a)(or (3.5b)) and the corresponding right boundary discretization. We intend to provide both sufficient and necessary conditions for the stability of this discrete IBVP. It appears that the IBVP does not have the standard form presented in the GKS-theory and thus, this stability theory is not directly applicable.

The discrete IBVP under consideration is given with two boundaries. According to the Theorem 3.1 below, which is valid for any of the GKS stability definitions, it is sufficient to consider the problem on the positive plane $x \geq 0$, i.e., on the index range $j \geq 0$.

Theorem 3.1. *[[36], Thm. 5.4] Consider the difference approximation for* $t \geq 0$ *and* $0 \leq x \leq 1$ *and assume that the corresponding left and right quarter-plane problems (which we get by removing one boundary to infinity) are stable, then the original problem is also stable.*

The idea behind the theorem is that the basic difference scheme (3.3) and each of the boundary conditions separated into the two quarter plane problems that are relatively nice to handle. Therefore, we will consider only the stability of the right quarter plane problem, while the left quarter one is analogue.

To fit our approximation into the form discussed in [36], we write (3.3) as

$$u_j^{l+1} = Qu_j^l + kb_j^l, \tag{3.6a}$$

$$u_j^0 = u_0(x_j), \qquad j = 0,1,2,\ldots, \tag{3.6b}$$

where

$$Q = \sum_{\sigma=-1}^{1} \Lambda_\sigma E_+^\sigma, \qquad E_+ u_j = u_{j+1},$$

$$\Lambda_0 = I - kC - (r\Lambda)^2 + \frac{1}{2}(kC)^2,$$
$$\Lambda_{\pm 1} = \mp\frac{1}{2}r\Lambda + \frac{1}{2}(r\Lambda)^2 \pm \frac{1}{4}rk(\Lambda C + C\Lambda),$$
$$b_j^l = \frac{1}{2}\left(f_j^{l+1} + f_j^l\right) - \frac{1}{4}r\Lambda\left(f_{j+1}^l - f_{j-1}^l\right) - \frac{1}{2}kCf_j^l.$$

The boundary values are written as

$$u_0^{l+1} \;=\; B_{0,0}u_0^l + B_{1,0}u_1^l + B_{1,1}u_1^{l+1} + B_{2,1}u_2^{l+1}, \qquad (3.7)$$

where the above matrices are determined by the numerical boundary conditions under consideration. For the upwinding case (3.4)-(3.5a), we have

$$B_{0,0} = \begin{pmatrix} I^+ & 0 \\ 0 & I + r\Lambda^- \end{pmatrix} - k\begin{pmatrix} S^{++} & S^{+-} \\ C^{-+} & C^{--} \end{pmatrix},$$

$$B_{1,0} = \begin{pmatrix} 0 & 0 \\ 0 & -r\Lambda^- \end{pmatrix}, \qquad B_{1,1} = B_{2,1} = 0. \qquad (3.8a)$$

However, if extrapolation (3.4)-(3.5b) is used, then (3.7) is given by

$$B_{0,0} = \begin{pmatrix} I^+ & 0 \\ 0 & 0 \end{pmatrix} - k\begin{pmatrix} S^{++} & S^{+-} \\ 0 & 0 \end{pmatrix},$$

$$B_{1,1} = \begin{pmatrix} 0 & 0 \\ 0 & -2I^- \end{pmatrix}, \qquad B_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & -I^- \end{pmatrix}, \qquad B_{1,0} = 0. \qquad (3.8b)$$

There are different ways to define stability of finite difference schemes. GKS [36] discussed some possible definitions of which we choose the one that allows us to make use of the available results.

Let $l^2(x)$ denote the space of all grid functions $u_j = u(x_j),\ x_j = jh,\ j = 0,1,\dots,$ with $\sum_{j=0}^{\infty} \left|u_j\right|^2 < \infty$ and define the scalar product and norm by

$$(u,v)_h = \sum_{j=0}^{\infty} hu_j^* v_j, \qquad \|u\|_h^2 = (u,u)_h.$$

We define $l^2(t)$ and $l^2(x,t)$ in the corresponding way and denote by

$$(u,v)_k = \sum_{l=0}^{\infty} ku^*(t_l)v(t_l), \qquad \|u\|_k^2 = (u,u)_k,$$

$$(u,v)_{h,k} = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} hku_j^*(t_l)v_j(t_l), \qquad \|u\|_{h,k}^2 = (u,u)_{h,k},$$

the corresponding norms and scalar products.

Definition 3.2. *[[36], Def. 3.3] Assume that the initial function is zero. The difference scheme (3.6)-(3.8a)(or(3.8b)) is stable, if there exist constants* $c_0 > 0, \alpha_0 \geq 0$ *such that, for all* $t = t_l = lk$, *all* $\alpha > \alpha_0$, *and all* h, *an estimate*

$$\left(\frac{\alpha-\alpha_0}{\alpha k+1}\right)\left\|e^{-\alpha t}u_0\right\|_k^2+\left(\frac{\alpha-\alpha_0}{\alpha k+1}\right)^2\left\|e^{-\alpha t}u\right\|_{h,k}^2 \leq c_0\left\|e^{-\alpha(t+k)}b\right\|_{h,k}^2$$

holds.

While here the vector b_j^l of the basic scheme (3.6a) is a general combination of f and its derivatives, in [36] we have $b_j^l = f_j^l$. However, Goldberg *et al.* [37] showed that this generalization does not affect the results of [36] and they raised the question of stability in the sense of Definition 3.2.

The definition of stability for the difference scheme for the left quarter plane problem is the same, except that the norm is taken over the grid on $(-\infty, 1]$ and u_0 is replaced by u_J.

In the following, we shall reduce the above stability question to that of the principal part of the scalar outflow approximations, i.e., the part obtained by eliminating the terms of order k, k^2, and all inhomogeneity vectors. This result is based on Theorem 4.3 of [36], which provides a necessary and sufficient determinantal stability criterion given entirely in terms of the principal part of the approximations. The mere existence of such a criterion implies that for the stability purposes we may consider a basic scheme of (3.6)-(3.8a)(or(3.8b)) of the form

$$u_j^{l+1} = \tilde{Q}u_j^l, \qquad \tilde{Q} = \sum_{\sigma=-1}^{1} \tilde{\Lambda}_\sigma E_+^\sigma, \qquad E_+u_j = u_{j+1}, \tag{3.9}$$

where

$$\begin{aligned}\tilde{\Lambda}_0 &= I - (r\Lambda)^2, \\ \tilde{\Lambda}_{\pm 1} &= \mp\frac{1}{2}r\Lambda + \frac{1}{2}(r\Lambda)^2,\end{aligned}$$

and the boundary conditions

$$\left(u^+\right)_0^{l+1} = \left(u^+\right)_0^l, \tag{3.10}$$

$$\left(u^-\right)_0^{l+1} = \left(I + r\Lambda^-\right)\left(u^-\right)_0^l - r\Lambda^-\left(u^-\right)_1^l, \tag{3.11a}$$

$$\left(u^-\right)_0^{l+1} = 2\left(u^-\right)_1^{l+1} - \left(u^-\right)_2^{l+1}. \tag{3.11b}$$

The scheme (3.9) is now consistent with

$$u_t + \Lambda u_x = 0.$$

We split the basic scheme and the boundary values into inflow and outflow parts respectively

$$\begin{aligned}\left(u^-\right)_j^{l+1} &= \left(u^-\right)_j^l - \frac{r\Lambda^-}{2}\left(\left(u^-\right)_{j+1}^l - \left(u^-\right)_{j-1}^l\right) \\ &+ \frac{\left(r\Lambda^-\right)^2}{2}\left(\left(u^-\right)_{j+1}^l - 2\left(u^-\right)_j^l + \left(u^-\right)_{j-1}^l\right), + k\left(d^-\right)_j^l,\end{aligned} \tag{3.12}$$

$$\left(u^-\right)_0^{l+1} = \left(I + r\Lambda^-\right)\left(u^-\right)_0^l - r\Lambda^-\left(u^-\right)_1^l, \tag{3.13a}$$

$$\left(u^-\right)_0^{l+1} = 2\left(u^-\right)_1^{l+1} - \left(u^-\right)_2^{l+1}, \tag{3.13b}$$

and

$$\begin{aligned}\left(u^+\right)_j^{l+1} &= \left(u^+\right)_j^l - \frac{r\Lambda^+}{2}\left(\left(u^+\right)_{j+1}^l - \left(u^+\right)_{j-1}^l\right) \\ &+ \frac{\left(r\Lambda^+\right)^2}{2}\left(\left(u^+\right)_{j+1}^l - 2\left(u^+\right)_j^l + \left(u^+\right)_{j-1}^l\right), + k\left(d^-\right)_j^l,\end{aligned} \tag{3.14}$$

$$\left(u^+\right)_0^{l+1} = \left(u^+\right)_0^l. \tag{3.15}$$

Obviously, (3.6)-(3.8a)(or (3.8b)) is stable if and only if both parts are stable. Before we proceed, we include the following assumptions that are necessary for the result contained in this section.

Assumption 3.1. *1. The associated initial value scheme is stable.*

2. The difference scheme is either dissipative or nondissipative.

A necessary condition for the stability of the initial value scheme is to satisfy the CFL. For the LW-scheme, this gives

$$\max_{\nu=1,\dots,n} \left|\lambda_\nu r\right| \le 1, \tag{3.16}$$

Definition 3.3. *[62] The difference scheme (3.9) is dissipative of order* $2s$ *if there exists* $c > 0$ *such that the eigenvalues* $\mu_\nu(\xi)$ *of the amplification matrix of* $\tilde{Q}$ *satisfies the following estimate*

$$\left|\mu_\nu(\theta)\right|^2 \le 1 - c\,|\theta|^{2s}, \qquad |\theta| \le \pi.$$

This condition is equivalent to (see [59])

$$\left|\mu_\nu(\theta)\right|^2 \le 1 - \acute{c}\sin^{2s}(\theta/2), \qquad \acute{c} > 0.$$

The amplification matrix of $\tilde{Q}$ reads λ_ν denotes the eigenvalue of Λ, has the form

$$I - ir\Lambda\sin\theta - (r\Lambda)^2(1-\cos\theta),$$

with eigenvalues

$$\mu_\nu(\theta) = 1 - ir\lambda_\nu \sin\theta - 2r^2\lambda_\nu^2 \sin^2(\theta/2), \qquad \nu = 1,\dots,n.$$

This gives

$$\begin{aligned} \left|\mu_\nu(\theta)\right|^2 &= [1-2(r\lambda_\nu)^2\sin^2(\theta/2)]^2 + (r\lambda_\nu)^2\sin^2\theta \\ &= 1-2(r\lambda_\nu)^2[4\sin^2(\theta/2)+4(r\lambda_\nu)^2\sin^4(\theta/2)+\sin^2\theta] \\ &= 1-4(r\lambda_\nu)^2[1-(r\lambda_\nu)^2]\sin^4(\theta/2), \qquad \nu=1,\dots,n. \end{aligned} \tag{3.17}$$

Thus, the difference scheme (3.9) is dissipative of order 4 if r is chosen to satisfy

$$0 < \left|\lambda_\nu r\right| \le 1, \quad \nu = 1,\dots,n.$$

Since Λ is regular, Assumption 3.1 is fulfilled if the CFL-condition (3.16) is satisfied.
We split the outflow approximation (3.12)-(3.13a)(or (3.13b)) into $n-m$ scalar components, each of the form

$$\begin{aligned} v_j^{l+1} &= v_j^l - \frac{\kappa}{2}(v_{j+1}^l - v_{j-1}^l) + \frac{\kappa^2}{2}(v_{j+1}^l - 2v_j^l + v_{j-1}^l) \\ &= \frac{1}{2}(\kappa^2+\kappa)v_{j-1}^l + (1-\kappa^2)v_j^l + \frac{1}{2}(\kappa^2-\kappa)v_{j+1}^l \end{aligned} \tag{3.18}$$

where $\kappa := r\lambda_\nu$, for fixed $\lambda_\nu \in \Lambda^-$, and

$$v_0^{l+1} = v_0^l - \kappa(v_1^l - v_0^l). \tag{3.19a}$$

or

$$v_0^{l+1} = 2v_1^{l+1} - 2v_2^{l+1}. \tag{3.19b}$$

The scheme (3.6)-(3.8a)(or (3.8b)) is stable if and only if (3.12)-(3.13a)(or (3.13b)) and (3.14)-(3.15) are stable, and the latter are stable if and only if their scalar components are. Lemma 2.3 of [37] shows that the scalar components of the inflow approximation (3.14)-(3.15) are stable (for $0 < \kappa \le 1$). So we conclude the main result of this section

Lemma 3.1. *The approximation (3.6)-(3.8a)(or (3.8b)) is stable if and only if the scalar outflow components (3.18)-(3.19a)(or (3.19b)) are stable.*

To discuss the stability of (3.18)-(3.19a)(or (3.19b)) we use the discrete Laplace transform, which is one of the few approaches available for analyzing the stability of difference schemes for initial boundary value problems. This approach is used to transform out the temporal differences (time derivatives) and consider the scheme in transform space as a difference scheme in j.

Definition 3.4. *The discrete Laplace transform of $u = \left\{u^l\right\}$ is the function $\tilde{u} := \mathscr{L}\left(\left\{u^l\right\}\right)$ defined by*

$$\tilde{u}(z) := \sum_{l=0}^{\infty} e^{-zl}u^l$$

where $z \in \mathbb{C}$, $\operatorname{Re} z > 0$ and $\operatorname{Im} z \in [-\pi,\pi]$. the function $\tilde{u}(z)$, $z \in \mathbb{Z}$, is called the Z-transformation of the sequence $\{u^n\}$, where $n \in \mathbb{N}_0$.

We take the discrete Laplace transform of equation (3.18)-(3.19a)(or (3.19b)) and obtain the resolvent equation

$$z\tilde{v}_j = \frac{1}{2}(\kappa^2+\kappa)\tilde{v}_{j-1} + (1-\kappa^2)\tilde{v}_j + \frac{1}{2}(\kappa^2-\kappa)\tilde{v}_{j+1}, \tag{3.20}$$

and the transformed boundary conditions

$$z\tilde{v}_0 = \tilde{v}_0 - \kappa(\tilde{v}_1 - \tilde{v}_0), \tag{3.21a}$$

$$z(\tilde{v}_0 - 2\tilde{v}_1 + \tilde{v}_2) = 0. \tag{3.21b}$$

Definition 3.5. *The complex number z, $|z| > 1$, is an eigenvalue of equations (3.20)-(3.21a)(or (3.21b)) if*

1. *there exists a vector $\tilde{v} = [\tilde{v}_0 \ \tilde{v}_1 \ \dots]^T$ such that $(z, \tilde{v})$ satisfies equations (3.20)-(3.21a)(or (3.21b)), and*
2. $\|\tilde{v}\|_h < \infty$.

Definition 3.6. *The complex number z is a generalized eigenvalue of equations (3.20)-(3.21a)(or (3.21b)) if*

1. *there exists a vector $\tilde{v} = [\tilde{v}_0 \ \tilde{v}_1 \ \dots]^T$ such that $(z, \tilde{v})$ satisfies equations (3.20)-(3.21a)(or (3.21b)),*
2. $|z| = 1$, *and*
3. $\tilde{v}_k$ *satisfies*

$$\tilde{v}_k(z) = \lim_{\omega \to z, |\omega|>1} \tilde{v}_k(\omega),$$

where $(\omega, \tilde{v}(\omega))$ is a solution to equation (3.20).

The result from [35] is given in the following proposition.

Proposition 3.1. *The difference scheme (3.18)-(3.19a)(or (3.19b)) is stable if and only if the eigenvalue problem (3.20)-(3.21a)(or (3.21b)) has no eigenvalues and no generalized eigenvalues.*

Theorem 3.2. *The approximation (3.18) in combination with one of the boundary conditions (3.19a) or (3.19b) is stable for $-1 \le \kappa < 0$.*

To prove this theorem we apply Proposition 3.1 and the first part of the following lemma, which describes the root of the characteristic equation of (3.20)

$$zk = k + \frac{\kappa}{2}(k^2-1) + \frac{\kappa^2}{2}(k-1)^2. \tag{3.22}$$

Lemma 3.2. *[51] There exists a $\delta > 0$, such that for the roots k_1, k_2 of (3.22) the following estimates hold*

1. If $\kappa < 0$, then

$$\begin{array}{rcll} |k_1| & \leq & 1-\delta, & for\ |z| \geq 1, \\ |k_2| & > & 1, & for\ |z| \geq 1,\ z \neq 1, \\ k_2 & = & 1, & for\ z = 1. \end{array}$$

2. If $\kappa > 0$, then

$$\begin{array}{rcll} |k_1| & < & 1, & for\ |z| \geq 1,\ z \neq 1, \\ k_1 & = & 1, & for\ z = 1, \\ |k_2| & \geq & 1+\delta, & for\ |z| \geq 1. \end{array}$$

Proof. [of Theorem 3.2] To solve the difference equation (3.20)-(3.21b) for $|z| > 1$, we note that the general solution of (3.20) belonging to $l^2(x)$ has the form $\tilde{v}_j = k_1^j \varphi_1$, where k_1 is the (smaller) root of the characteristic equation (3.22). We insert this solution into the condition (3.21b) and obtain $\varphi_1(k_1-1)^2 = 0$. But, according to the previous lemma, $|k_1 - 1| \geq \delta$. Hence, equations (3.20)-(3.21b) have no eigenvalues.
To determine whether $z = 1$ is a generalized eigenvalue of (3.20)-(3.21b), we substitute $z = 1$ into equation (3.22) and obtain

$$k = \frac{\kappa^2 \pm |\kappa|}{\kappa^2 - \kappa} = \begin{cases} 1 & =: k_2 \\ \dfrac{\kappa^2+\kappa}{\kappa^2-\kappa} & =: k_1. \end{cases}$$

For this scheme, k_1 is not relevant, since $|k_1| = \left|\dfrac{\kappa^2+\kappa}{\kappa^2-\kappa}\right| < 1$ (for $-1 \leq \kappa < 0$), and hence k_1 will not satisfy equation (3.21b).
We notice that for $|z| > 1$, k_2 will satisfy $|k_2| > 1$. This is the case because k_1 is clearly inside the circle $|z| = 1$, so k_2 must be outside that circle. Since $|z| = 1$ is associated with k_2, the solution at $|z| = 1$, does not satisfy condition 3 of Definition 3.6. Thus, $z = 1$ is not a generalized eigenvalue, and the difference scheme (3.18)-(3.19b) is stable.
We emphasize that we have already assumed that the difference scheme is stable as an initial value problem scheme. Hence, the stability proved here will be conditional stability with condition $-1 \leq \kappa < 0$.
Considering the case (3.19a), we substitute $\tilde{v}_j = k_1^j \varphi_1$, $|k_1| \leq 1-\delta$, into boundary condition (3.19a)

$$\varphi_1(z - 1 + \kappa k_1 - \kappa) = 0.$$

For $-1 \le \kappa < 0$ (the stability condition for the Cauchy problem) and $|z| \ge 1$, we have

$$\begin{aligned} \left|z-1+\kappa k_1-\kappa\right| &> \left|1+k_1\frac{\kappa}{z-1-\kappa}\right| \\ &\ge 1-k_1\left|\frac{\kappa}{1+\kappa-z}\right| \\ &\ge 1+(\delta-1)\left|\frac{\kappa}{1+\kappa-z}\right| \ge \delta. \end{aligned}$$

It follows that (3.20)-(3.21a) has no eigenvalues.
Analogously to the computations used in the first part, we show that $z=1$ is not a generalized eigenvalue of (3.20)-(3.21a) . □

3.3. Numerical tests

In the following numerical experiments we compare the performance of the ABCs and the FBCs, as well as the numerical approximation with FBCs for different scaling matrices.

3.3.1. Example 1

We consider the linear hyperbolic system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 & 1 \\ \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \quad x \in \mathbb{R}, \tag{3.23a}$$

$$u(x,0)=u^0(x), \qquad v(x,0)=v^0(x), \tag{3.23b}$$

where u^0, v^0, f and g have compact support in $(0,1)$.
The corresponding steady equation on $\mathbb{R}$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 & 1 \\ -\frac{3}{4} & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \quad x \in \mathbb{R}, \tag{3.24a}$$

with the decay conditions

$$u, v \to 0, \qquad x \to \pm\infty. \tag{3.24b}$$

The zero and first order ABCs for the restriction to the interval $0 \le x \le 1$ are, respectively

$$u = 0, \qquad x = 0, \tag{3.25a}$$

$$v = 0, \qquad x = 1, \tag{3.25b}$$

and

$$u_t + v/2 = 0, \qquad x = 0, \tag{3.26a}$$

$$v_t + 3u/8 = 0, \qquad x = 1. \tag{3.26b}$$

The matrix

$$S = \begin{pmatrix} a & 2a/3 \\ b & 2b \end{pmatrix}, \qquad 0 \neq a, b \in \mathbb{R},$$

transforms the steady state problem (3.24a) to the diagonal form. Diagonalize $\Lambda^{-1}C$

$$S\Lambda^{-1}CS^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{3.27}$$

For the decay condition to be valid we need

$$u + 2v/3 = 0, \qquad x = 0, \tag{3.28a}$$

$$u + 2v = 0, \qquad x = 1. \tag{3.28b}$$

The first order FBCs combine ABCs (3.26) and the steady boundary conditions (3.28), in analogue to (2.17),

$$u_t + a(u + 2v/3) = 0, \qquad x = 0, \tag{3.29a}$$

$$v_t + b(u + 2v) = 0, \qquad x = 1. \tag{3.29b}$$

The finite difference scheme introduced in the first section is used in the following numerical tests.

(i) Consider (3.23) with zero initial condition, $f = 0$, and

$$g(x) = \begin{cases} \cos^2(\pi(x - 0.5)/0.9), & x \in (0.05, 0.95), \\ 0, & \text{else,} \end{cases}$$

together with each of the boundary conditions 3.26 and (3.29). The convergence as $t \to \infty$ of the solution of the resulting IBVP to the solution of the steady unbounded problem has been tested numerically ($h = 0.0005$, $r = k/h = 0.9$). The steady state solution (3.24) is given in Figure 1 and the convergence to this solution is described in Figures 2 and 3.
Figure 2 shows, with different choices of a and b, that the solution of (3.23) with the new boundary conditions (3.29) converges in $(0, 1)$ to the solution of the steady unbounded problem. Figure 3 shows that this is not true for the first order boundary conditions (3.26).

(ii) Using equation (2.20), the optimal choices of a and b are

$$a = \frac{\lambda_2 c_{12}}{\lambda_2 - \lambda_1} \frac{1}{s_{12}} = \frac{3}{4}, \qquad b = \frac{\lambda_1 c_{21}}{\lambda_1 - \lambda_2} \frac{1}{s_{21}} = \frac{3}{8},$$

and the FBCs (3.29a) become

$$u_t + \frac{3}{4}(u + 2v/3) = 0, \qquad x = 0, \tag{3.30a}$$

$$v_t + \frac{3}{8}(u + 2v) = 0, \qquad x = 1. \tag{3.30b}$$

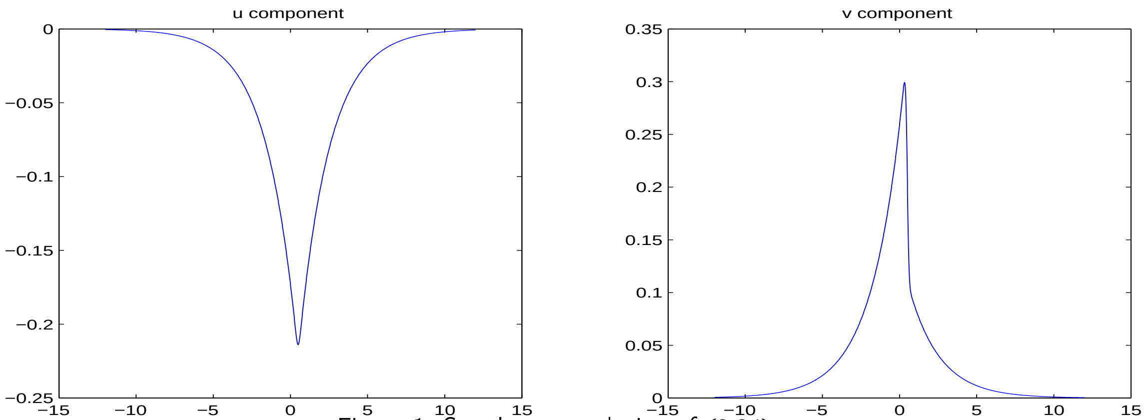

Figure 1: Steady state solution of (3.24).

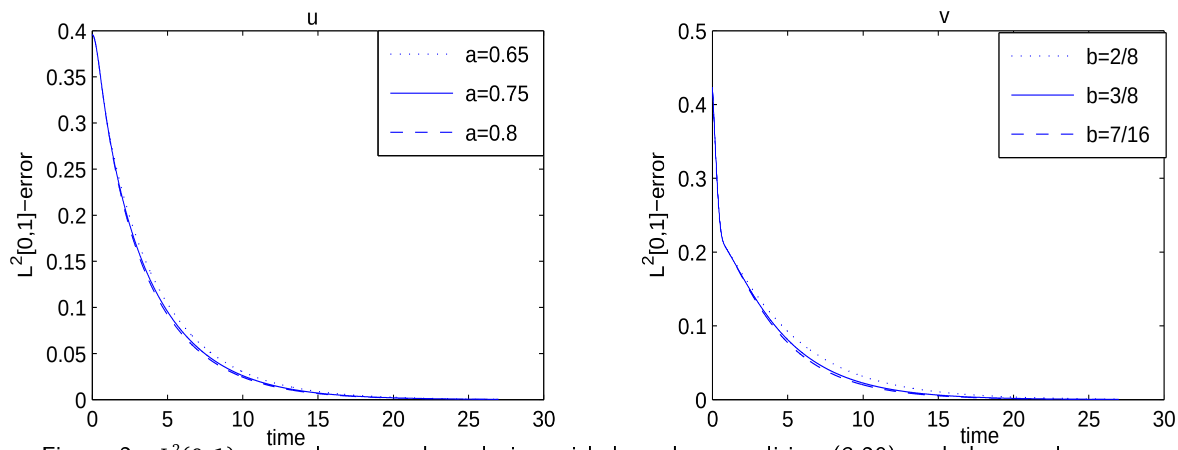

Figure 2: $L^2(0,1)$−error between the solution with boundary condition (3.29) and the steady state solution.

In this test we show that this choice, among other arbitrary choices, improve the approximation for short time computations. The convergence to steady state is tested, for arbitrary non-zero constants a and b, in part (i). Therefore, for short time comparison, it is reasonable to consider $f(x) = g(x) = 0$.

Since an asymptotic approximation is used to localize the exact nonlocal boundary conditions, we consider a highly-oscillatory initial data (2.7).

$$u(x,0) = v(x,0) = \begin{cases} \cos^2(2\pi(x-0.5))\sin(2\pi p x), & x \in (0.25, 0.75), \\ 0, & \text{else}, \end{cases} \tag{3.31}$$

The cases $p = 10, 20$ are plotted in Figure 4.
The error between the whole space exact solution and the solution with the boundary condition (3.29) for different values of a and b has been tested. The absolute errors of the inflow data (u at $x = 0$ and v at $x = 1$) and the $L^2(0,1)$-error are considered. The step size is rather small ($h = 0.0005$) in order to estimate the errors due to the

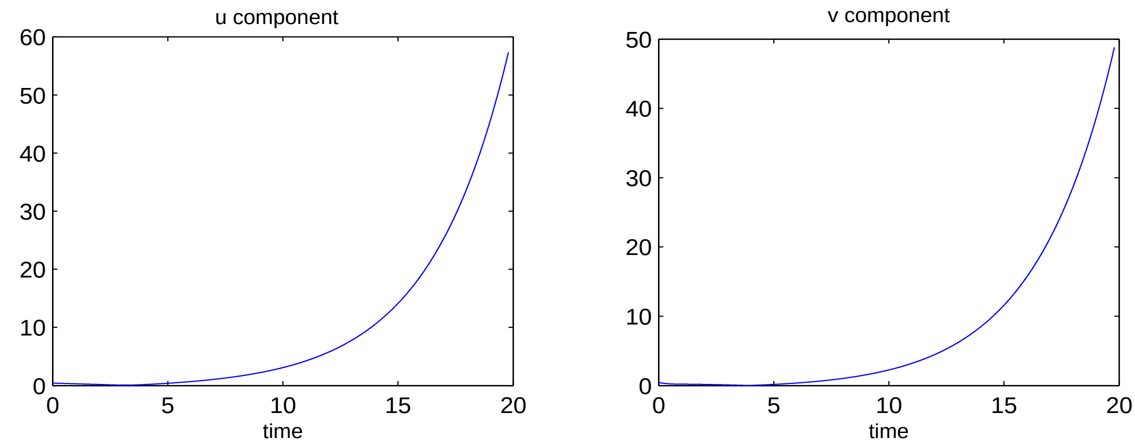

Figure 3: $L^2(0,1)-$error between the solution with boundary condition (3.26) and the steady state solution.

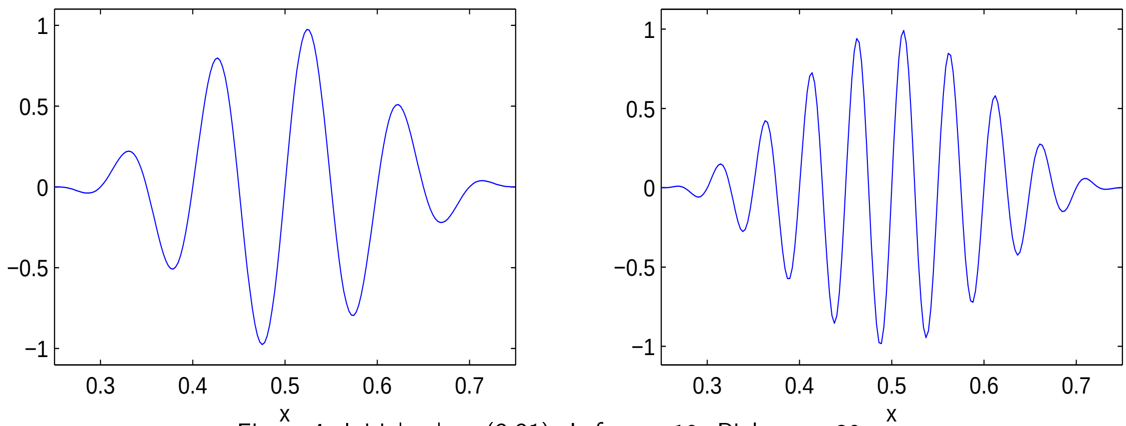

Figure 4: Initial values (3.31). Left: $p=10$. Right: $p=20$.

boundary conditions and not the discretization errors. In case $p = 10$, Figures 5-6 clearly show that our choices of a and b give the minimum error. The same result holds for the case $p = 20$.

(iii) In this example, we test the dependence of the boundary condition on the initial frequency. We consider the system (3.23), boundary conditions (3.30), and the initial data (3.31). In the case $p = 10$, Figure 7 compares the absolute error of inflow data for different refinements of the space step size h. Figures 8-9 show the same comparison but for the case $p = 20$.
Tables 1-2 list the maximal absolute errors at the inflow data, the tables show that as h is getting smaller the error is reduced slower. As a result the last row of the two tables are good approximations of the errors due to the boundary conditions (3.30). The maximal absolute error for the case $p = 10$ of u at $x = 0$ and v at $x = 1$ are $2.5702 \cdot 10^{-5}$ and $1.9215 \cdot 10^{-5}$, respectively. In the case of $p = 20$, they

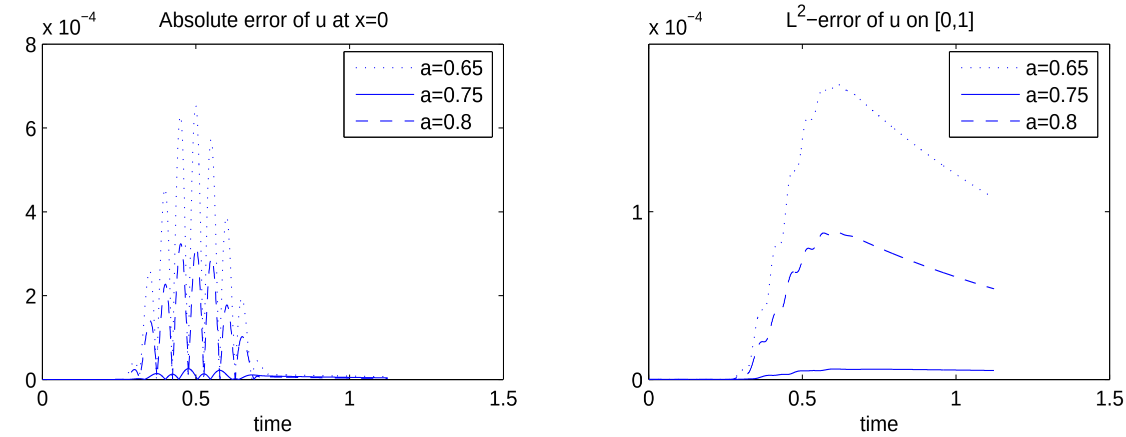

Figure 5: Comparison of the error between the exact solution u and the solution with boundary conditions (3.29) for different values of a. ($a = 0.75$ is the theoretical prediction for optimal value)

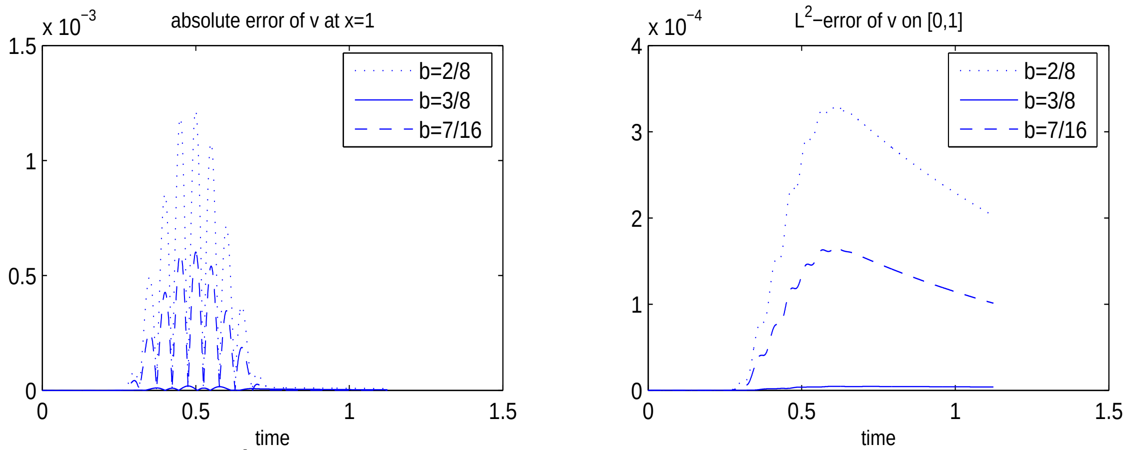

Figure 6: Comparison of the error between the exact solution v and the solution with boundary conditions (3.29) for different value of b. ($b = 3/8$ is the theoretical prediction for optimal value)

are reduced to $5.5009 \cdot 10^{-6}$ and $4.1164 \cdot 10^{-6}$, respectively. This gives numerical evidence that with highly oscillating initial data the errors become smaller, which agrees with the approximation of the nonlocal exact boundary condition (2.7) with asymptotic expansion.

h	u at $x = 0$, $(\cdot 10^{-5})$	v at $x = 1$, $(\cdot 10^{-5})$
0.001	2.5997	1.9439
0.0005	2.5713	1.9227
0.00025	2.5702	1.9215

Table 1: Maximal absolute error, $p = 10$.

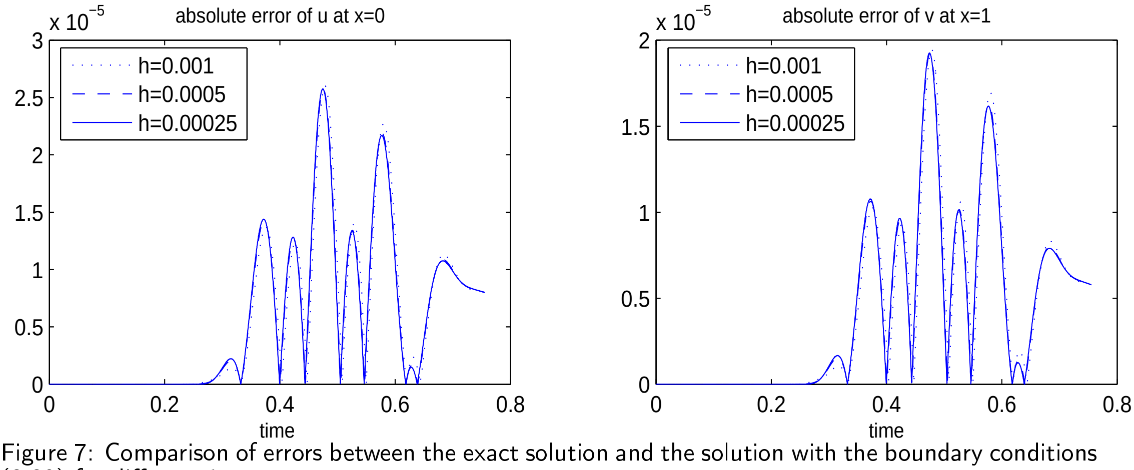

Figure 7: Comparison of errors between the exact solution and the solution with the boundary conditions (3.30) for different h, $p = 10$.

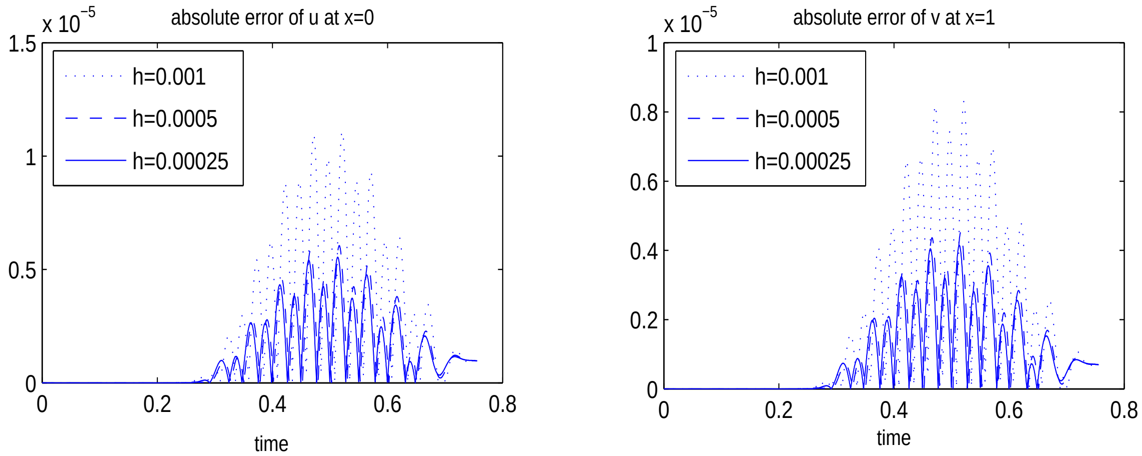

Figure 8: Comparison of errors between the exact solution and the solution with the boundary conditions (3.30) for different h, $p = 20$.

h	u at $x = 0$, $(\cdot 10^{-6})$	v at $x = 1$, $(\cdot 10^{-6})$
0.001	11.079	8.2997
0.0005	6.0525	4.5299
0.00025	5.5394	4.1452
0.0001	5.5009	4.1164

Table 2: Maximal absolute error, $p = 20$.

3.3.2. Example 2

Consider (3.1a) in $\mathbb{R}$ with

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & -0.8 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.2 & 0 & 1 \\ 0 & 0.4 & 2 \\ -1 & -2 & 1 \end{pmatrix}, \tag{3.32}$$

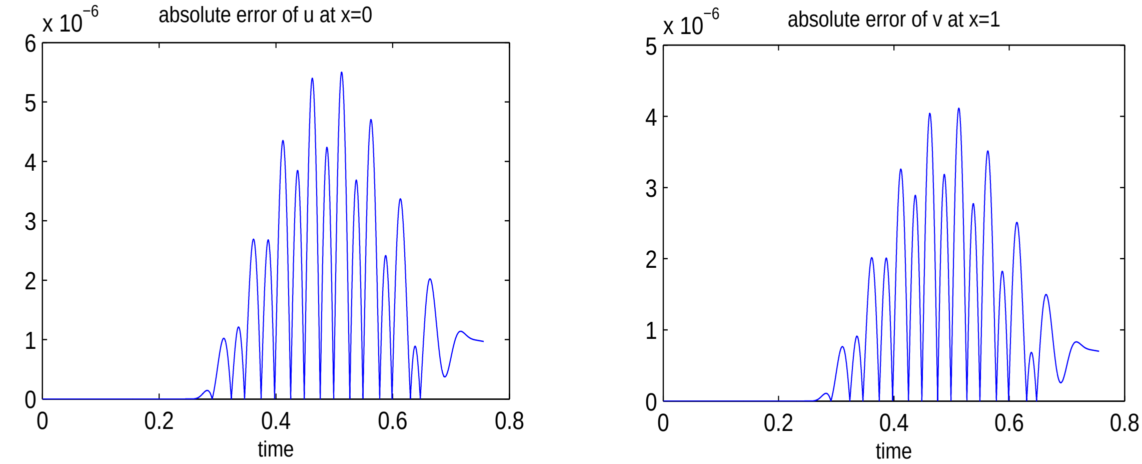

Figure 9: Comparison of errors between the exact solution and the solution with the boundary conditions (3.30) for $h = 0.0001$ and $p = 20$.

and $f(x,t) = (f_1(x), f_1(x), f_1(x))^T$, where

$$f_1(x) = \begin{cases} 10\exp(-100(2x-1)^2), & x \in (0.25, 0.75), \\ 0, & \text{else}. \end{cases} \tag{3.33}$$

The initial function is given by (cf. Figure 10)

$$u^0(x) = \begin{cases} \cos(\pi(x-0.5)/0.9), & x \in (0.05, 0.95), \\ 0, & \text{else}. \end{cases} \tag{3.34}$$

In the first part of this example we compare the performance of the FBCs and ABCs for

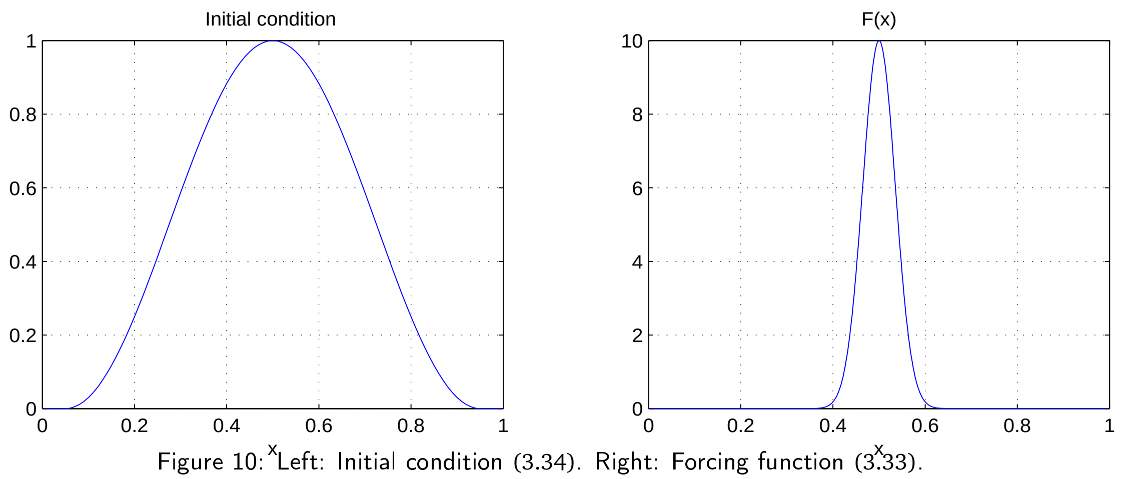

Figure 10: Left: Initial condition (3.34). Right: Forcing function (3.33).

long time computations. The first order ABCs for the restriction to the interval $0 \le x \le 1$

are

$$u_{1,t} + 0.444u_{3,t} = 0, \qquad x = 0, \tag{3.35a}$$
$$u_{2,t} + 1.6u_{3,t} = 0, \qquad x = 0, \tag{3.35b}$$
$$u_{3,t} - 0.5556u_{1,t} - 0.4u_{2,t} = 0, \qquad x = 1, \tag{3.35c}$$

while the first order FBCs read

$$u_t^+ + V^+(S^{++}u^+ + S^{+-}u^-) = 0, \qquad x = 0, \tag{3.36a}$$
$$u_t^- + V^-(S^{-+}u^+ + S^{--}u^-) = 0, \qquad x = 1. \tag{3.36b}$$

The matrix S, which diagonalizes $\Lambda^{-1}C$, is given by

$$S = \begin{pmatrix} -0.2287 & -0.6791 & -1.0083 \\ 1.0522 & -0.0952 & 0.0656 \\ -0.2690 & -0.4004 & 1.1275 \end{pmatrix}$$

The scaling matrices

$$V^+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad V^- = e$$

are chosen as a general solution of $V^+S^{+-} = X_1^{+-}$ and $V^-S^{-+} = X_1^{-+}$, respectively. Following the procedure presented in Section 1.2, we take

$$V^{*+} = \begin{pmatrix} -0.4389 & 0.0285 \\ -1.5801 & 0.1028 \end{pmatrix}, \qquad V^{*-} = 1.3306.$$

The step sizes are chosen small in order to see the errors due to different boundary conditions and not the discretization errors ($h = 0.0005$, $k = 0.0004$). It is clear that the CFL-condition, $\max_{j=1,2,3}|r\lambda_j| < 1$, is satisfied.

The steady state solution is given in Figure 11-Right and the convergence to this solution as $t \to \infty$ in $(0,1)$ is described in Figure 12. In Figures 13-15 the solutions with FBCs for different choices of the scaling matrices are compared to the exact solution over $(-\infty,\infty)$. The plots show that the FBCs with the proposed optimal choices of V^+, V^- give the best approximate solutions in the transient phase to the exact solution in the unbounded domain.

Tables 3-5 list the maximal absolute errors at the inflow data (u_1, u_2 at $x = 0$ and u_3 at $x = 1$). As well as the $L^2(0,1)-$error between exact solution and the solution with the boundary condition (3.29) for different values of a, b, c, d, and e.

The numerical results give quantitative evidence that the FBCs are useful for both short and long times.

a,	b	abs. error at $x=0$	$L^2(0,1)$−error
−0.2,	0	0.0804	0.0548
−0.8,	0.2	0.0829	0.0506
a^*,	b^*	0.0318	0.0293

Table 3: Maximum errors due to the first order FBCs of u_1 for different choices of a and b, see also Figure 13.

c,	d	abs. error at $x=0$	$L^2(0,1)$−error
−1,	0	0.2208	0.0664
−2.5,	0.5	0.1312	0.0430
c^*,	d^*	0.1233	0.0408

Table 4: Maximum errors of u_2 due to the first order FBCs for different choices of c and d, see also Figure 14.

e	abs. error at $x=1$	$L^2(0,1)$−error
1	0.1080	0.0421
2.5	0.0943	0.0468
e^*	0.0875	0.0344

Table 5: Maximum errors due to the first order FBCs of u_3 for different choices of e, see also Figure 15.

Acknowledgments

The first author was partially supported by a DAAD scholarship.

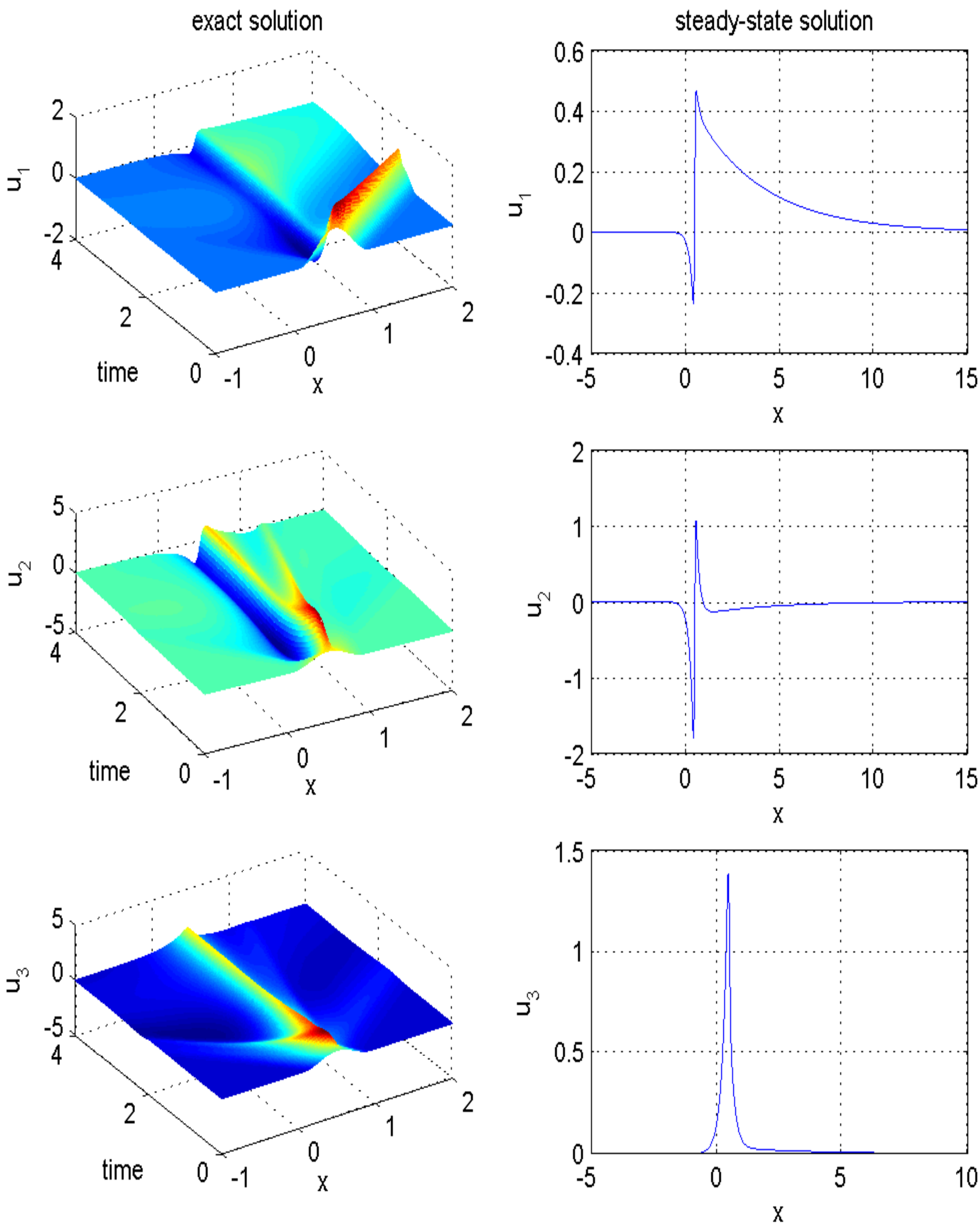

Figure 11: Left: Exact solution. Right: Steady state solution.

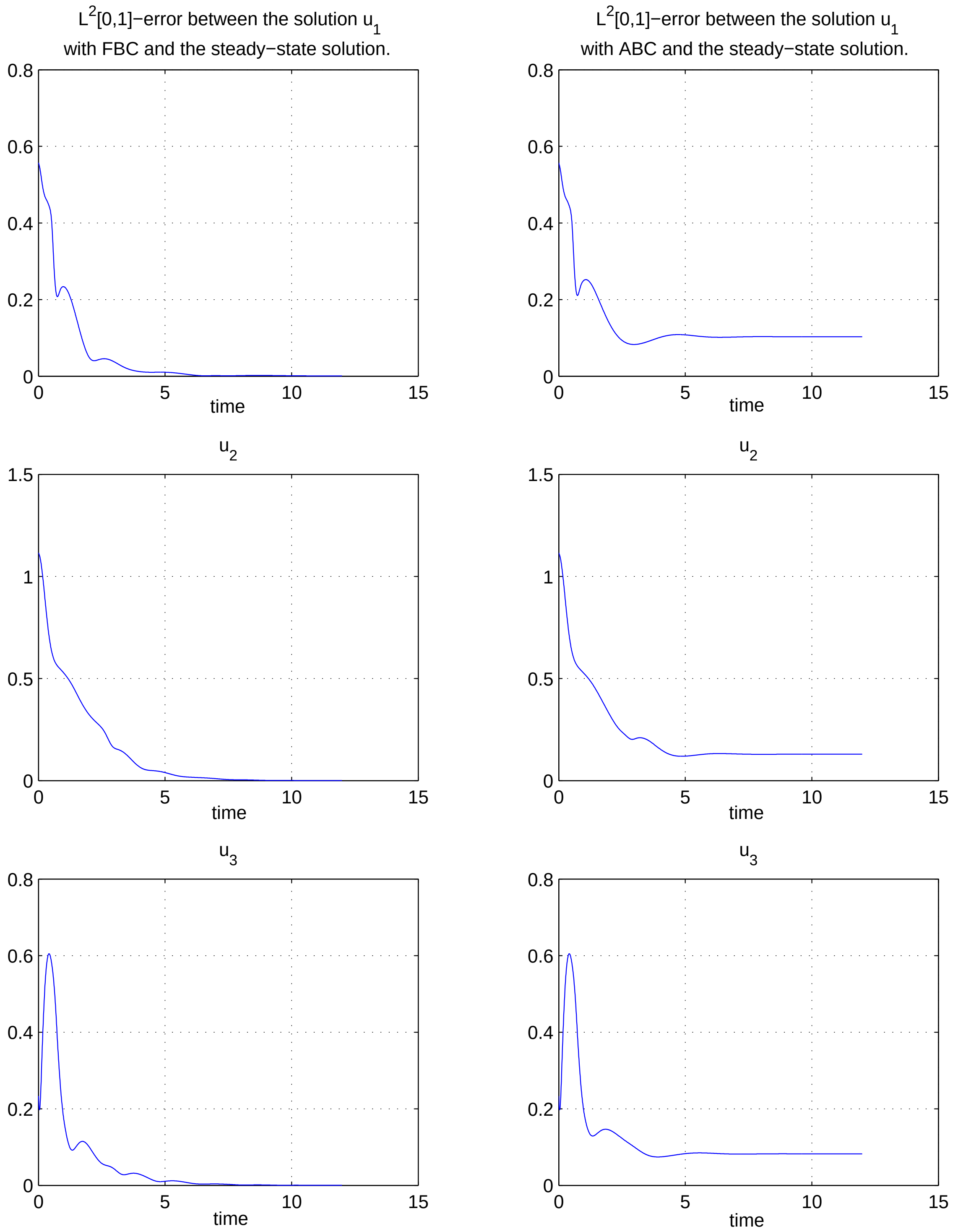

Figure 12: Convergence to the steady state solution as $t \to \infty$ in $(0,1)$.

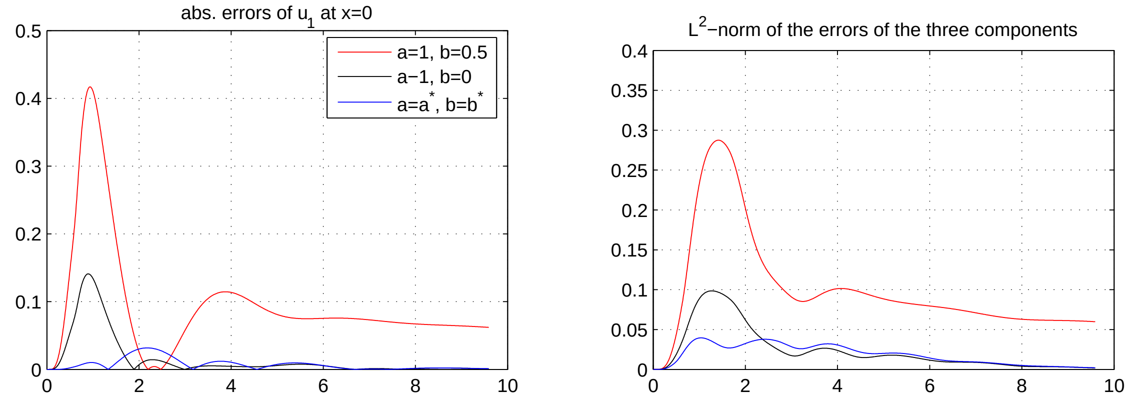

Figure 13: Comparison of the errors between the exact solution of u_1 and the solution with FBCs for different choices of a and b.

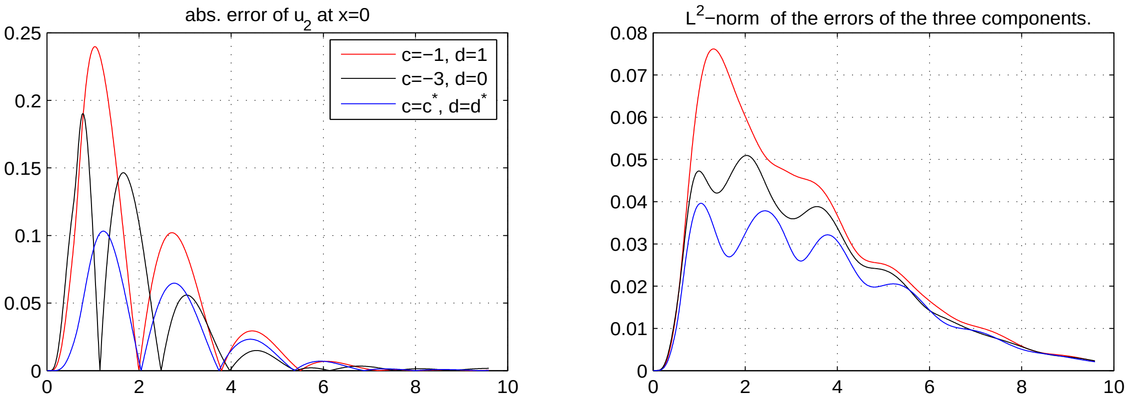

Figure 14: Comparison of the errors between the exact solution of u_2 and the solution with FBCs for different choices of c and d.

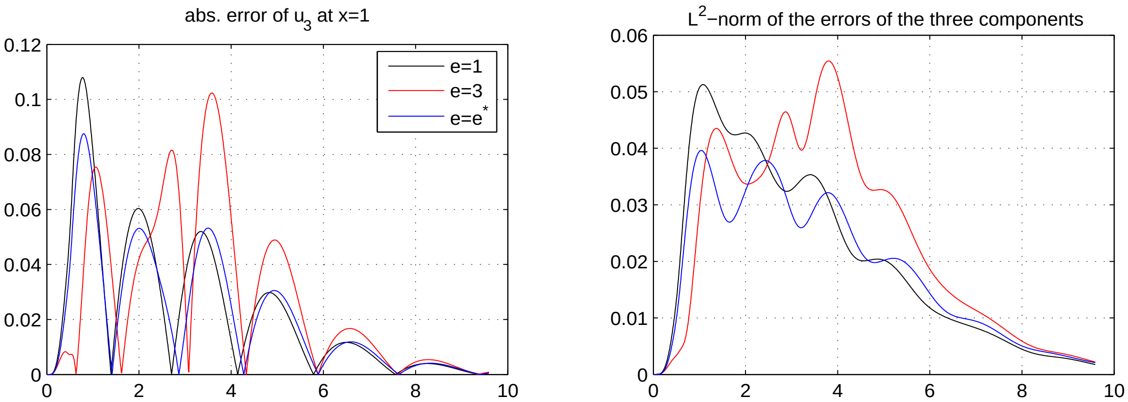

Figure 15: Comparison of the errors between the exact solution and the solution with FBCs for different choices of e.

References

[1] S. ABARBANEL AND D. GOTTLIEB, *On the construction and analysis of absorbing layers in CEM*, Appl. Numer. Math., 27 (1998), pp. 331–340.

[2] S. ABARBANEL, D. GOTTLIEB, AND J.S. HESTHAVEN, *Well-posed perfectly matched layers for advective acoustics*, J. Comput. Phys., 154 (1999), pp. 266–283.

[3] B. ALPERT, L. GREENGARD AND T. HAGSTROM, *Rapid evaluation of nonreflecting boundary kernels for time-domain wave propagation*, SIAM J. Numer. Anal., 37 (2000), pp. 1138–1164.

[4] B. ALPERT, L. GREENGARD AND T. HAGSTROM, *Nonreflecting boundary conditions for the time-dependent wave equation*, J. Comput. Phys., 180 (2002), pp. 270–296.

[5] T. AMRO, *Far Field Boundary Conditions and Perfectly Matched Layer for some Wave Propagation Problems*, Dissertation, Westfälische Wilhelms-Universität Münster, 2007.

[6] C.D. ANGELOPOULOS, *Nonreflecting boundary conditions for hyperbolic conservation laws*, Bull. Greek Math. Soc., 34 (1992), pp. 161–176.

[7] X. ANTOINE, H. BARUCQ AND A. BENDALI, *Bayliss-Turkel-like Radiation Conditions on Surfaces of Arbitrary Shapes*, J. Math. Anal. Appl., 229 (1999), pp. 184–211.

[8] X. ANTOINE, A. ARNOLD, C. BESSE, M. EHRHARDT, AND A. SCHÄDLE, *A Review of Transparent and Artificial Boundary Conditions Techniques for Linear and Nonlinear Schrödinger Equations*, Commun. Comput. Phys., 4 (2008), pp. 729–796.

[9] D. APPELÖ AND G. KREISS, *Evaluation of a Well-posed Perfectly Matched Layer for Computational Acoustic*, Proceedings of the HYP2002 conference, Pasadena, USA, 2002

[10] D. APPELÖ, T. HAGSTROM AND G. KREISS, *Perfectly matched layers for hyperbolic systems: general formulation, well-posedness and stability*, SIAM Appl. Math., 67 (2006), pp. 1–23.

[11] A. ARNOLD, *On Absorbing boundary conditions for quantum transport equations*, Math. Meth. Num. Anal., 28 (1994), pp. 853–872.

[12] A. BARRY, J.BIELAK AND R. MACCAMY, *On absorbing boundary conditions for wave propagation*, J. Comput. Phys., 79 (1988), pp. 449–468.

[13] A. BAYLISS AND E. TURKEL, *Radiation boundary conditions for wave-like equations*, Comm. Pure Appl. Math., 33 (1980), pp. 707–725.

[14] A. BAYLISS AND E. TURKEL, *Far field boundary conditions for compressible flows*, J. Comput. Phys., 48 (1982), pp. 182–199.

[15] J.P. BÉRENGER, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys., 114 (1994), pp. 185-200.

[16] D. CARLSON AND H. SCHNEIDER, *Inertia theorems for matrices: the semidefinit case*, J. Math. Anal. Appl., 6 (1963), pp. 430–446.

[17] W.C. CHEW AND W.H. WEEDON, *A 3D perfectly matched medium from modified Maxwell's equations with stretched coordinates*, Microwave Opt. Technol. Lett., 7 (1994), pp. 599–604.

[18] F. COLLINO AND P. MONK, *Optimizing the perfectly matched layer*, Comput. Meth. Appl. Mech. Engrg., 164 (1998), pp. 157–171.

[19] T. COLONIUS, *Modeling artificial boundary conditions for compressible flow*, Ann. Rev. Fluid Mech., 36 (2004), pp. 315–345.

[20] J. DIAZ AND P. JOLY, *An analysis of higher-order boundary conditions for the wave equation*, SIAM J. Appl. Math., 65 (2005), pp. 1547–1575.

[21] G. DOETSCH, *Guide to the applications of the Laplace and z-transforms*, Van Nostrand Reinholf Company, 1971.

[22] M. EHRHARDT, *Absorbing Boundary Conditions for Hyperbolic Systems*, Numer. Math. Theor. Meth. 3 (2010), pp. 295–337.

[23] B. ENGQUIST AND L. HALPERN, *Far field boundary conditions for computation over long time*,

Appl. Num. Math., 4 (1988), pp. 21–45.

[24] B. ENGQUIST AND A. MAJDA, *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comput., 31 (1977), pp. 629–651.

[25] B. ENGQUIST AND A. MAJDA, *Radiation boundary conditions for acoustic and elastic wave calculations*, Comm. Pure Appl. Math., 32 (1979), pp. 313–357.

[26] M. GILES, *Nonreflecting boundary conditions for Euler equation calculations*, AIAA J., 28 (1990), pp. 2050–2058.

[27] D. GIVOLI, *Numerical methods for problems in infinite domains*, Elsevier, Amsterdam, 1992.

[28] D. GIVOLI, *High-order local non-reflecting boundary conditions: a review*, Wave Motion, 39 (2004), pp. 319–326.

[29] D. GIVOLI AND B. NETA, *High-order non-reflecting boundary conditions for dispersive waves*, Wave Motion, 37 (2003), pp. 257–271.

[30] D. GIVOLI, T. HAGSTROM AND I. PATLASHENKO, *Finite element formulation with high order absorbing boundary conditions for time-dependent waves*, Comput. Method. Appl. Mech., 195 (2006), pp. 3666–3690.

[31] J. GOODRICH AND T. HAGSTROM, *A comparison of two accurate boundary treatments for computational aeroacoustics*, Technical Report 97-1585, AIAA, 1999.

[32] M. GROTE AND J.B. KELLER, *On nonreflecting boundary conditions*, J. Comput. Phys., 122 (1995), pp. 231–243.

[33] M. GROTE AND C. KIRSCH, *Far-field evaluation with nonreflecting boundary conditions*, Proceedings of the Ninth Internat. Conf. on Hyperbolic Problems (HYP2002), Pasadena, USA, 2002.

[34] M. GROTE AND C. KIRSCH, *Nonreflecting Boundary Conditions for Time Dependent Multiple Scattering*, J. Comput. Phys., 221 (2007), pp. 41–62.

[35] B. GUSTAFSSON, H-O. KREISS AND J. OLIGER, *Time dependent problems and difference method*, ohn Wiley & Sons Inc., 1995.

[36] B. GUSTAFSSON, H-O. KREISS AND SUNDSTRÖM, *Stability theory of difference approximations for mixed initial boundary value problems*, Math. Comput., 26 (1972), pp. 649–686.

[37] M. GOLDBERG AND E. TADMOR, *Convenient stability criteria for difference approximations of hyperbolic initial boundary value problems II*, Math. Comput., 48 (1987), pp. 503–520.

[38] T. HAGSTROM, *On high-order radiation boundary conditions*, In B. Engquist and G. Kriegsmann, editors, IMA Volume on Computational Wave Propagation, Springer, New York, 1996, pp. 1–22.

[39] T. HAGSTROM, *Radiation boundary conditions for the numerical simulation of waves*, Acta Numerica, 8 (1999), pp. 47–106.

[40] T. HAGSTROM, *New results on absorbing layers and radiation boundary conditions*, Topics in Computational Wave Propagation, M. Ainsworth et al. eds., Springer-Verlag, 2003, pp. 1–42.

[41] T. HAGSTROM, *A new construction of perfectly matched layers for hyperbolic systems with applications to the linearized Euler equations*, in: G. Cohen et al. (eds.), Proceedings of the Sixth International Conference on Mathematical and Numerical Aspects of Wave Propagation Phenomena, Springer, New York, 2003, pp. 125–129.

[42] T. HAGSTROM, A. MAR-OR AND D. GIVOLI, *High-order local absorbing conditions for the wave equation: Extensions and improvements*, J. Comput. Phys., 227 (2008), pp. 3322–3357.

[43] T. HAGSTROM AND T. WARBURTON, *Complete Radiation Boundary Conditions: Minimizing the Long Time Error Growth of Local Methods*, SIAM J. Numer. Anal., 47 (2009), pp. 3678–3704.

[44] T. HAGSTROM, D. GIVOLI AND T. WARBURTON, *Radiation boundary conditions for time–dependent waves based on complete plane wave expansions*, J. Comput. Appl. Math., 234

(2010), pp. 1988–1995.

[45] R.L. HIGDON, *Initial boundary value problems for linear hyperbolic systems*, SIAM Review 28 (1986), pp. 177–217.

[46] R.A. HORN AND C.R. JOHNSON, *Matrix analysis*, Cambridge University Press, 1990.

[47] F.Q. HU, *Absorbing boundary conditions*, Inter. J. Comput. Fluid Dyn., 18 (2004), pp. 513–522.

[48] F.Q. HU, X.D. LI AND D.K. LIN, *PML absorbing boundary condition for nonlinear aeroacoustics problems*, AIAA paper (2006), pp. 2006–2521.

[49] M. ISRAELI AND S.A. ORSZAG, *Approximation of radiation boundary conditions*, J. Comput. Phys., 41 (1981), pp. 115–135.

[50] R. KOSLOFF AND D. KOSLOFF, *Absorbing boundary conditions for wave propagation problems*, J. Comput. Phys., 63 (1986), pp. 363–376.

[51] H.-O. KREISS, *Stability theory for difference approximations of mixed initial boundary value problems*, Math. Comput., 22 (1968), pp. 703–714.

[52] H.-O. KREISS, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math., 23 (1970), pp. 277–298.

[53] H.-O. KREISS AND J. LORENZ, *Initial boundary value problems and the Navier-Stokes equations*, Academic Press Inc., 1989.

[54] D. KRÖNER, *Absorbing Boundary Conditions for the Linearized Euler Equations in 2-D*, Math. Comput., 57 (1991), pp. 153–167.

[55] R.J. LEVEQUE, *Finite volume methods for hyperbolic problems*, Cambridge University Press, 2002.

[56] S. OSHER, *Stability of difference approximations of dissipative type for mixed initial-boundary value problems, I*, Math. Comput., 23 (1969), pp. 335–340.

[57] R.D. RICHTMYER AND K.W. MORTON, *Difference methods for initial-value problems*, Wiley-interscience, 1967.

[58] C.W. ROWLEY AND T. COLONIUS, *Discretely nonreflectiing boundary conditions for linear hyperbolic systems*, J. Comput. Phys., 157 (2000), pp. 500–538.

[59] J.C. STRIKWERDA, *Finite difference schemes and partial differential equations*, Chapman and Hall, 1989.

[60] J. SZEFTEL, *A nonlinear approach to absorbing boundary conditions for the semilinear wave equation*, Math. Comp., 75 (2006), pp. 565–594.

[61] J. SZEFTEL, *Absorbing boundary conditions for nonlinear scalar partial differential equations*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 3760–3775.

[62] J.W. THOMAS, *Numerical partial differential equations, finite difference methods*, Springer, 1995.

[63] L.N. TREFETHEN AND L. HALPERN, *Well-posedness of one-way wave equations and absorbing boundary conditions*, Math. Comp., 176 (1986), pp. 421–435.

[64] S.V. TSYNKOV, *Numerical solution of problems on unbounded domains. A review*, Appl. Numer. Math., 27 (1998), pp. 465–532.

[65] E. TURKEL AND A. YEFET, *Absorbing PML boundary layers for wave-like equations*, Appl. Numer. Math., 27 (1998), pp. 533–557.

[66] M.A. ZAHID AND M.N. GUDDATI, *Padded continued fraction absorbing boundary conditions for dispersive waves*, Comput. Meth. Appl. Mech. Engrg., 195 (2006), pp. 3797–3819.