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On Laplace–Carleson embedding theorems

Birgit Jacob* Jonathan R. Partington† Sandra Pott‡

Abstract

This paper gives embedding theorems for a very general class of weighted Bergman spaces: the results include a number of classical Carleson embedding theorems as special cases. Next, a study is made of Carleson embeddings in the right half-plane induced by taking the Laplace transform of functions defined on the positive half-line (these embeddings have applications in control theory): particular attention is given to the case of a sectorial measure or a measure supported on a strip, and complete necessary and sufficient conditions for a bounded embedding are given in many cases.

Keywords. Hardy space, weighted Bergman space, Laplace transform, Carleson measure

2000 Subject Classification. 30D55, 30E05, 47A57, 47D06, 93B05.

1 Introduction and Notation

Let w denote a weight function on the imaginary axis $i\mathbb{R}$, let μ be a positive regular Borel measure on the right half plane $\mathbb{C}_+$ and let $1 \le p, q \le \infty$. Embeddings of the form

$$L^p_w(i\mathbb{R}) \to L^q(\mathbb{C}_+, \mu), \tag{1}$$

where a locally integrable function f on the imaginary axis $i\mathbb{R}$ is mapped to its Poisson extension on the right half plane $\mathbb{C}_+$, are known as Carleson embeddings, and have been much studied in the literature. In linear control, another, related class of embeddings plays an important role, namely embeddings of the form

$$\mathcal{H}^p_{\beta,w}(0,\infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f = \int_0^\infty e^{-t\cdot} f(t)\,dt,$$

given by the Laplace transform $\mathcal{L}$. Here, $\mathcal{H}^p_{\beta,w}$ denotes the Sobolev space of index β and weight w: the case $\beta = 0$ corresponds to a weighted L^p space. We shall refer to such embeddings as Laplace–Carleson embeddings.

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In the easiest case, $\beta = 0$, $w \equiv 1$ and $p = q = 2$, the Laplace transform maps $\mathcal{H}^2_{0,1}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ isometrically up to a constant to $H^2(\mathbb{C}_+)$, which is a closed subspace of $L^2(i\mathbb{R})$, and we only have to deal with the unweighted classical Carleson embedding theorem for $p = 2$. This can be found in many places, for example, [5, 12].

Theorem 1.1 (Carleson embedding theorem) *Let μ be a positive regular Borel measure on the right half plane $\mathbb{C}_+$. Then the following are equivalent:*

1. *The natural embedding*

$$H^p(\mathbb{C}_+) \to L^p(\mathbb{C}_+, \mu)$$

is bounded for some (or equivalently, for all) $1 \le p < \infty$.

2. *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{C}_+} |k_\lambda(z)|^2 d\mu(z) \le C\|k_\lambda\|^2_{H^2} \text{ for all } \lambda \in \mathbb{C}_+,$$

where $k_\lambda(z) = \frac{1}{2\pi}\frac{1}{z+\bar{\lambda}}$ for $\lambda, z \in \mathbb{C}_+$.

3. $$\mu(Q_I) \le C|I| \text{ for all intervals } I \subset i\mathbb{R},$$

where Q_I denotes the Carleson square $Q_I = \{z = x + iy \in \mathbb{C}_+ : iy \in I, 0 < x < |I|\}$.

In this case, μ is called a Carleson measure.

A further relatively easy case is $p = q = 2$ and w a power weight, $w(t) = t^\alpha$ with $\alpha < 0$. This case corresponds to the classical embedding theorem for standard weighted Bergman spaces on the half plane by Duren, see e.g. [3]. For more general weights w, the Laplace–Carleson embedding corresponds to a new embedding theorem for weighted Bergman spaces A^2_ν on the half plane with a translation-invariant measure ν, which is the subject of Section 2.
In the case of general $1 \le p, q \le \infty$, $p > 2$, the Laplace–Carleson embeddings are very subtle even in case that $w = 1$, due to the oscillatory part of the Laplace transform integral kernel. A general characterization seems out of reach at the moment, but with additional conditions on the support of the measure μ, a full characterization can sometimes be given. This is the content of Section 3.
Our results have applications in terms of interpolation in certain spaces of holomorphic functions and also admissibility and controllability in diagonal semigroups, which will be presented elsewhere.

The reproducing kernel functions for $H^2(\mathbb{C}_+)$ are denoted by k_λ, $\lambda \in \mathbb{C}_+$, where $k_\lambda(s) = \frac{1}{2\pi}\frac{1}{s+\bar{\lambda}}$ for $s \in \mathbb{C}_+$, and satisfy $f(\lambda) = \langle g, k_\lambda \rangle$ for $f \in H^2(\mathbb{C}_+)$. Note that $\|k_\lambda\|^2 = \frac{1}{4\pi \operatorname{Re} \lambda}$.

2 Embedding theorems for weighted Bergman spaces

In this section, we will be interested in embeddings

$$A^p_\nu(\mathbb{C}_+) \hookrightarrow L^p(\mathbb{C}_+, \mu),$$

where $A^p_\nu(\mathbb{C}_+)$ is a weighted Bergman space defined below and ν is a translation-invariant positive regular Borel measure on $\overline{\mathbb{C}_+}$, that is, $\nu = \tilde{\nu} \otimes \lambda$, where λ denotes Lebesgue measure and $\tilde{\nu}$ is a positive regular Borel measure on $[0, \infty)$. This corresponds to the case of radial measures on the unit disc.
The investigation of such embeddings has a long history, starting with [2] for the case of the standard weighted Bergman space on $\mathbb{C}_+$ with $d\tilde{\nu}(t) = t^\beta dt$ for $\beta > -1$, respectively the standard weights $(1 - |z|)^\beta$ on the disc. Oleinik [13] observed already in 1974 that for measures $\rho(t)dt$, where the weight ρ decreases very fast towards $t = 0$, such as $\rho(t) = e^{-1/t^{1+\gamma}}$, $\gamma > 0$, it is not sufficient to compare the weights of Carleson squares $\mu(Q_I)$ and $\nu(Q_I)$ (or equivalently, to compare the measures of Euclidean balls D centered on the imaginary axis). Instead, in the example above one has to consider the measures of Euclidean balls away from the imaginary axis,

$$D_z = D\left(z, \frac{(\operatorname{Re} z)^{1+\gamma}}{(1 + \operatorname{Re} z)^\gamma}\right)$$

for $z \in \mathbb{C}_+$. Roughly speaking, the faster the weight $\rho(t)$ decreases for $t \to 0$, the smaller the radius of the ball D_z in relation to the distance of z to the imaginary axis, and the more detailed information on the measure μ is required. Recently, necessary and sufficient conditions have been found for the case even faster deceasing weights $\rho(t)$, such as double exponentials [14]. Our aim in this section is somewhat different: we want to find a class of measures $\tilde{\nu}$ as large as possible, for which a characterisation in terms of Carleson squares, and in terms of testing on powers of reproducing kernels k_z, $z \in \mathbb{C}_+$, is possible. Also, we do not want to put any continuity or smoothness conditions on the measure $\tilde{\nu}$. Clearly, we need a growth condition on $\tilde{\nu}$ in 0. This will be a (Δ_2)-condition in 0.

2.1 Carleson measure on Zen spaces

Let $\tilde{\nu}$ be a positive regular Borel measure on $[0, \infty)$ satisfying the following (Δ_2)-condition:

$$R := \sup_{t>0} \frac{\tilde{\nu}[0, 2t)}{\tilde{\nu}[0, t)} < \infty. \qquad (\Delta_2)$$

Let ν be the positive regular Borel measure on $\mathbb{C}_+ = [0, \infty) \times \mathbb{R}$ given by $d\nu = d\tilde{\nu} \otimes d\lambda$, where λ denotes Lebesgue measure. In this case, for $1 \le p < \infty$, we call

$$A^p_\nu = \left\{ f : \mathbb{C}_+ \to \mathbb{C} \text{ analytic} : \sup_{\varepsilon>0} \int_{\overline{\mathbb{C}_+}} |f(z + \varepsilon)|^p d\nu(z) < \infty \right\}$$

a Zen space on $\mathbb{C}_+$. If $\tilde{\nu}(\{0\}) > 0$, then by standard Hardy space theory, f has a well-defined boundary function $\tilde{f} \in L^p(i\mathbb{R})$, and we can give meaning to the expression $\int_{\overline{\mathbb{C}_+}} |f(z)|^p d\nu(z)$. Therefore, we write

$$\|f\|_{A^p_\nu} = \left(\int_{\overline{\mathbb{C}_+}} |f(z)|^p d\nu(z) \right)^{1/p}.$$

Clearly the space A^2_ν is a Hilbert space.
Well-known examples of Zen spaces are Hardy space $H^p(\mathbb{C}_+)$, where $\tilde{\nu}$ is the Dirac measure in 0, or the standard weighted Bergman spaces A^p_α, where $d\tilde{\nu}(t) = t^\alpha dt$, $\alpha > -1$. Some further examples constructed from Hardy spaces on shifted half planes were given by Zen Harper in [8, 9]. Note that by the (Δ_2)-condition, there exists $N \in \mathbb{N}$ such that $k_\lambda^N \in A^p_\nu$ for all $\lambda \in \mathbb{C}_+$ and all $1 \le p < \infty$. Here is our Embedding Theorem for Zen spaces.

Theorem 2.1 *Let $1 \le p < \infty$, let A^p_ν be a Zen space on $\mathbb{C}_+$, with measure $\nu = \tilde{\nu} \otimes \lambda$ as above, and let μ be a positive regular Borel measure on $\mathbb{C}_+$. Then the following are equivalent:*

1. *The embedding $A^p_\nu \hookrightarrow L^p(\mathbb{C}_+, \mu)$ is well-defined and bounded for one, or equivalently for all, $1 \le p < \infty$.*
2. *For one, or equivalently for all, $1 \le p < \infty$, and some sufficiently large $N \in \mathbb{N}$, there exists a constant $C_p > 0$ such that*
$$\int_{\mathbb{C}_+} |(k_\lambda(z))^N|^p d\mu(z) \le C_p \int_{\mathbb{C}_+} |(k_\lambda(z))^N|^p d\nu(z) \text{ for each } \lambda \in \mathbb{C}_+. \quad (2)$$
3. *There exists a constant $C > 0$ such that*
$$\mu(Q_I) \le C\nu(Q_I) \text{ for each Carleson square } Q_I. \quad (3)$$

Proof: The implication (1) $\Rightarrow$ (2) for fixed p is immediate.
For the implication (2) $\Rightarrow$ (3), we use a standard argument using the decay of reproducing kernels. Given an interval I in $i\mathbb{R}$, and let λ denote the centre of the Carleson square Q_I. Note that

$$|(k_\lambda(z))^N| \ge \frac{1}{(2\pi)^N (4 \operatorname{Re} \lambda)^N} \text{ for } z \in Q_I.$$

Hence
$$\int_{\mathbb{C}_+} |(k_\lambda(z))^N|^p d\mu(z) \ge \frac{1}{(8\pi)^{pN} (\operatorname{Re} \lambda)^{pN}} \mu(Q_I).$$

It only remains to estimate $\int_{\mathbb{C}_+} |(k_\lambda(z))^N|^p d\nu(z)$ in terms of $\nu(Q_I)$. Let R be the constant from the (Δ_2)-condition.
For $k \in \mathbb{N}$, let $2^k I$ denote the interval with the same centre as I and the 2^k fold length. By the (Δ_2)-condition, $\nu(Q_{2^k I}) \le R^k 2^k \nu(Q_I)$. Note that

$$|(k_\lambda(z))^N| \le \frac{1}{(2\pi)^N} \frac{1}{(2^{k-1} \operatorname{Re} \lambda)^N} \text{ for } z \in Q_{2^k I} \backslash Q_{2^{k-1} I}.$$

Hence

$$\|k_\lambda^N\|_{A^p_\nu}^p$$
$$\leq \sum_{k=0}^{\infty} \nu(Q_{2^k I}) \left(\frac{1}{(2\pi)^N} \frac{1}{(2^{k-1} \operatorname{Re} \lambda)^N} \right)^p \leq \nu(Q_I) \frac{1}{(2\pi)^{pN}} \frac{1}{(\operatorname{Re} \lambda)^{Np}} \sum_{k=0}^{\infty} \frac{2^k R^k}{2^{(k-1)Np}}.$$

Choosing N sufficiently large, depending on R, we find that the sum on the right converges to a constant $K_{N,p}$ depending on N and p. Hence

$$\frac{1}{(8\pi)^{pN} (\operatorname{Re} \lambda)^{pN}} \mu(Q_I) \leq \int_{\mathbb{C}_+} |(k_\lambda(z))^N|^p d\mu(z) \leq C_p \int_{\mathbb{C}_+} |(k_\lambda(z))^N|^p d\nu(z)$$
$$= C_p K_{N,p} \frac{1}{(2\pi)^{pN}} \frac{1}{(\operatorname{Re} \lambda)^{Np}} \nu(Q_I), \quad (4)$$

and we obtain $\mu(Q_I) \leq C_{N,p} \nu(Q_I)$, with a constant $C_{N,p}$ depending only on N and p (and hence on the (Δ_2)-condition constant R).
(3) $\Rightarrow$ (1):
Our strategy is to deduce the boundedness of the embedding from the classical Carleson Embedding Theorem via a suitable decomposition.
Suppose for the moment that $\tilde{\nu}(\{0\}) = 0$ and that there exists a strictly increasing sequence $(a_n)_{n \in \mathbb{Z}}$ in $\mathbb{R}_+$ such that

1. There exists $1 > c > 0$ with
$$\frac{a_{n+1} - a_n}{a_{n+1}} \geq c \text{ for all } n \in \mathbb{Z}; \quad (5)$$

2. $$(2R)^3 \tilde{\nu}([a_{n-1}, a_n)) \geq \tilde{\nu}([a_n, a_{n+1})) \geq 2R \tilde{\nu}([a_{n-1}, a_n)). \quad (6)$$

Write $\beta_n = \tilde{\nu}([a_n, a_{n+1}))$. Notice that by (6),

$$\sum_{n \in \mathbb{Z}} \beta_n \|f\|_{H^p_{a_{n+1}}}^p \leq \int_{\mathbb{C}_+} |f(z)|^p d\nu = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \int_{[a_n, a_{n+1})} |f(t + i\omega)|^p d\tilde{\nu}(t) d\omega$$
$$\leq \sum_{n \in \mathbb{Z}} \beta_n \|f\|_{H^p_{a_n}}^p \leq (2R)^3 \sum_{n \in \mathbb{Z}} \beta_n \|f\|_{H^p_{a_{n+1}}}^p.$$

Thus both $(\sum_{n \in \mathbb{Z}} \beta_n \|f\|_{H^p_{a_{n+1}}}^p)^{1/p}$ and $(\sum_{n \in \mathbb{Z}} \beta_n \|f\|_{H^p_{a_n}}^p)^{1/p}$ give us equivalent expressions for the A^p_ν-norm. In addition, notice that if μ, ν satisfy the Carleson-type Condition (3), then also μ, $\sum_{n \in \mathbb{Z}} \beta_n \delta_{a_n} \otimes \lambda$, satisfy the condition with the same constant C. Therefore, we assume without loss of generality that

$$\tilde{\nu} = \sum_{n \in \mathbb{Z}} \beta_n \delta_{a_n},$$

where δ_{a_n} denotes the Dirac measure at a_n. For simplicity of notation we will also assume that the constant C in the condition (3) equals 1.
Our next step is a decomposition of μ into $\sum_{n \in \mathbb{Z}} \mu_n$, where each μ_n satisfies the Carleson-type condition (3) with respect to $\nu_n = \beta_n \delta_{a_n} \otimes \lambda$.

Lemma 2.2 *Let $N \in \mathbb{Z}$ and suppose that μ is supported on the closed half-plane $\overline{\mathbb{C}_{a_N}}$. Then there exist positive regular Borel measures μ_n, $n \geq N$, such that*

1.
$$\mu = \sum_{n=N}^{\infty} \mu_n; \tag{7}$$

2.
$$\mu_n \text{ is supported on the closed half-plane } \overline{\mathbb{C}_{a_n}}; \tag{8}$$

3. *There exists a constant $C' > 0$ such that for all intervals $I \subset i\mathbb{R}$,*

$$\mu_n(Q_I) \leq C'\nu_n(Q_I) \quad (n > N), \quad \mu_N(Q_I) \leq C' \sum_{k=-\infty}^{N} \nu_k(Q_I). \tag{9}$$

Moreover, μ_n is a Carleson measure for the shifted half plane $\mathbb{C}_{a_{n-1}}$, with Carleson constant

$$\mathcal{C}_{a_{n-1}}(\mu_n) \leq \frac{C'}{c}\beta_n \quad (n > N), \quad \mathcal{C}_{a_{N-1}}(\mu_N) \leq \frac{C'}{c} \sum_{k=-\infty}^{N} \beta_k,$$

where c is the constant appearing in the definition of the sequence (a_n) above.

Proof of the lemma: First, we prove that μ_n, $n \geq N$, exist, satisfying Conditions (7), (8) and (9). By replacing ν_N with $(\sum_{n=-\infty}^{N} \beta_n)\delta_{a_N} \otimes \lambda$, we can assume without loss of generality that ν is supported on $\overline{\mathbb{C}_{a_N}}$.
We begin by constructing a family of Carleson rectangles, on which (9) can be checked. Fix a dyadic grid $\mathcal{D}_N$ of half-open intervals in $i\mathbb{R}$ with minimal intervals of length a_N, and denote these intervals on length a_N as intervals of generation 0. The remaining intervals in the family $\mathcal{D}_N$ will be parents, grandparents, etc of the minimal intervals.
We can assume without loss that

$$c \leq 1 - \frac{1}{\sqrt{2}}. \tag{10}$$

By (5),

$$a_{N+k+1} \geq \frac{a_{N+k}}{1-c}$$

for any $k \geq 0$. If

$$a_{N+k+1} \geq \frac{a_{N+k}}{(1-c)^2},$$

we choose an integer $l \geq 2$ such that

$$\frac{1}{1-c} \leq \gamma = \left(\frac{a_{N+k+1}}{a_{N+k}}\right)^{1/l} < \frac{1}{(1-c)^2}$$

and add $l-1$ intermediate points

$$\gamma a_{N+k}, \ldots, \gamma^{l-1} a_{N+k}$$

between a_{N+k} and a_{N+k+1}. In this way, we create a strictly increasing sequence $(b_j)_{j\geq 0}$ such that $b_0 = a_N$, all terms of the sequence $(a_n)_{n\geq N}$ appear as terms of the sequence $(b_j)_{j\geq 0}$, and

$$\frac{1}{1-c} b_j \leq b_{j+1} < \frac{1}{(1-c)^2} b_j \text{ for all } j. \tag{11}$$

The family $\mathcal{F}$ of Carleson squares we want to consider is formed by rectangles of the form $(0, b_{j+1}) \times I$, where $I \in \mathcal{D}_N$, $j \geq 0$, for which the eccentricity is bounded above and below by

$$\frac{1}{\sqrt{2}} < \frac{b_{j+1}}{|I|} \leq \sqrt{2}. \tag{12}$$

Such a rectangle is denoted by $Q_{I,j}$. Note that by (11) and (10), for each $I \in \mathcal{D}_N$ there exits $j \geq 0$ with (12), and for each $j \geq 0$ there exists exactly one size of intervals I in $\mathcal{D}_N$ such that (12) holds. Note that different Carleson squares in this family $\mathcal{F}$ can have the same base $I \in \mathcal{D}_N$. It is easy to see that any Carleson square Q over an interval in $i\mathbb{R}$ can be covered by a bounded number of elements in $\mathcal{F}$, with comparable base length. Therefore, up to a possible change of constant, it is sufficient to check the Carleson-type conditions (3) and (9) on the elements of $\mathcal{F}$.

The family $\mathcal{F}$ gives rise to a family $\mathcal{T}$ of right halves of the Carleson rectangles in $\mathcal{F}$, which we will call *tiles*, and which will form a decomposition of the closed half plane $\overline{\mathbb{C}_{a_N}}$ into disjoint sets. These are rectangles of the form $T_{I,j} = [b_j, b_{j+1}) \times I$, where $Q_{I,j} \in \mathcal{F}$, see Figure 1.

We say that an interval $I \in \mathcal{D}_N$ belongs to generation j, if it is the base for a Carleson square $Q_{I,j} \in \mathcal{F}$. Thus each $I \in \mathcal{D}_N$ belongs to at least one, and possibly more than one, generation.

The idea of the construction below is to define μ_N as the "largest possible part" of μ which can be dominated by ν_N, in terms of Condition (9). Recall that by the Carleson-type Condition (3) we are given, we have in particular

$$\mu(T_{I,0}) = \mu(Q_{I,0}) \leq \nu(Q_{I,0}) = \nu_N(Q_{I,0})$$

for all intervals I of generation 0 in $\mathcal{D}_N$. For such intervals, we define the remaining part of ν_N by

$$\nu_N^0|_{\{a_N\}\times I} = \frac{\nu_N(Q_{I,0}) - \mu(T_{I,0})}{\nu_N(Q_{I,0})} \nu_N|_{\{a_N\}\times I}.$$

This defines a measure ν_N^0 on $\{a_N\} \times i\mathbb{R}$.

In the second step, we define a measure ν_N^1 on $\{a_N\} \times i\mathbb{R}$ by letting

$$\nu_N^1|_{\{a_N\}\times I'} = \begin{cases} \frac{\nu_N^0(Q_{I',1}) - \mu(T_{I',1})}{\nu_N^0(Q_{I',1})} \nu_N|_{\{a_N\}\times I'} & \text{if} \quad \nu_N^0(Q_{I',1}) > \mu(T_{I',1}), \\ 0 & \text{if} \quad \nu_N^0(Q_{I',1}) \leq \mu(T_{I',1}), \end{cases}$$

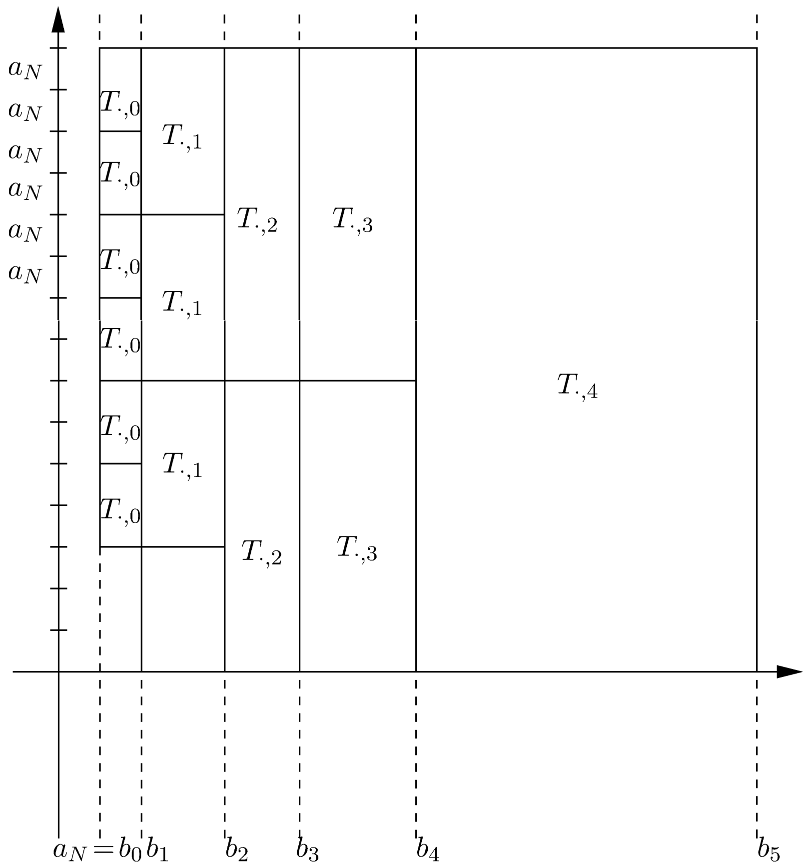

Figure 1: The tiles $T_{I,j}$

for intervals $I' \in \mathcal{D}_N$ of generation 1. In the next and all following steps, having already defined the measure ν_N^j on $\{a_N\} \times i\mathbb{R}$ for some $j \geq 0$, we let

$$\nu_N^{j+1}|_{\{a_N\}\times J} = \begin{cases} \frac{\nu_N^j(Q_{J,j+1}) - \mu(T_{J,j+1})}{\nu_N^j(Q_{J,j+1})} \nu_N|_{\{a_N\}\times J} & \text{if} \quad \nu_N^j(Q_{J,j+1}) > \mu(T_{J,j+1}), \\ 0 & \text{if} \quad \nu_N^j(Q_{J,j+1}) \leq \mu(T_{J,j+1}),\ \nu_N^j(Q_{J,j+1}) > 0, \\ 0 & \text{if} \quad \nu_N^j|_{\{a_N\}\times J} = 0, \end{cases} \tag{13}$$

for intervals of generation $j+1$ in $\mathcal{D}_N$, thereby defining the measure ν_N^{j+1} on $\{a_N\} \times i\mathbb{R}$. Depending on which case in (13) appears, we say that $(J, j+1)$ is of type 1, 2 or 3, respectively. We are finally ready to define the measure μ_N. We start by letting

$$\mu_N|_{T_{I,0}} = \mu|_{T_{I,0}}$$

for intervals $I \in \mathcal{D}_N$ of generation 0. For $T_{I,j+1} \in \mathcal{T}$, $j \geq 0$, let

$$\mu_N|_{T_{I,j+1}} = \begin{cases} \mu|_{T_{I,j+1}} & \text{if} \quad \nu_N^j(Q_{I,j+1}) > \mu(T_{I,j+1}), \\ \frac{\nu_N^j(Q_{I,j+1})}{\mu(T_{I,j+1})}\mu|_{T_{I,j+1}} & \text{if} \quad \nu_N^j(Q_{I,j+1}) \leq \mu(T_{I,j+1}), \nu_N^j(Q_{I,j+1}) > 0, \\ 0 & \text{if} \quad \nu_N^j|_{T_{I,j+1}} = 0. \end{cases}$$

Since the tiles in $\mathcal{T}$ form a decomposition of $\overline{\mathbb{C}_{a_N}}$, this defines a measure μ_N on $\overline{\mathbb{C}_{a_N}}$.
By construction, we have for each $Q_{I,j} \in \mathcal{F}$, (I,j) of type 2 or type 3:

$$\mu_N(Q_{I,j}) = \nu_N(\{a_N\} \times I).$$

If (I,j) is of type 1, then

$$\mu_N(Q_{I,j}) < \nu_N(\{a_N\} \times I).$$

Hence μ_N satisfies Condition 3 in Theorem 2.1 with respect to ν_N. Let us now look at the Carleson condition for $\mu - \mu_N$.
If (I,j) is of type 1, then $\mu_N(T_{I,j}) = \mu(T_{I,j})$ by construction, and $(\mu - \mu_N)(T_{I,j}) = 0$. Therefore, by decomposing the Carleson square $Q_{I,j}$ into

$$Q_{I,j} = T_{I,j} \cup \left(\bigcup_{I' \subset I, Q_{I',j-1} \in \mathcal{F}} Q_{I',j-1} \right)$$

and iterating if necessary, we see that we have to check Condition 3 only for Carleson squares $Q_{I,j}$ with (I,j) of type 2 or type 3. But in this case,

$$\mu_N(Q_{I,j}) = \nu_N(\{a_N\} \times I)$$

and thus by the original Condition 3 in Theorem 2.1 for μ and ν, $\mu - \mu_N$ satisfies Condition 3 with respect to $\nu - \nu_N$, again with constant 1 on Carleson squares in $\mathcal{F}$.
Finally, again by Condition (3) in Theorem 2.1 for μ and ν, we see that for all tiles $T_{I,j}$ contained in the strip $\{z \in \mathbb{C} : a_N \leq \operatorname{Re} z < a_{N+1}\}$, $\mu_N|_{T_{I,j}} = \mu|_{T_{I,j}}$. Therefore, $\mu - \mu_N$ is supported on the closed half-plane $\overline{\mathbb{C}_{a_{N+1}}}$. We can now make an induction step by applying the same procedure to the measures $\mu - \mu_N$, $\nu - \nu_N$ with respect to the half-plane $\overline{\mathbb{C}_{a_{N+1}}}$ to construct μ_{N+1}, etc. We thus obtain a decomposition

$$\mu = \sum_{n=N}^{\infty} \mu_n$$

satisfying the Conditions (7), (8) and (9) in the lemma.
It remains to be shown that for each $n > N$, μ_n is a Carleson measure with respect to the shifted half-plane $\mathbb{C}_{a_{n-1}}$, with the appropriate estimate of the Carleson constant. Let Q be a Carleson square in $\mathbb{C}_{a_{n-1}}$ over the interval I. Recall that by (9), there exists $C' > 0$ with $\mu_n(\tilde{Q}) \leq C'\nu_n(\tilde{Q})$ for each Carleson square $\tilde{Q}$ in $\mathbb{C}_+$ and that $(a_n - a_{n-1}) \geq c a_n$.

If the sidelength $|I|$ of Q is less than $a_n - a_{n-1}$, then Q has empty intersection with the support of μ_n, and $\mu_n(Q) = 0$. If $|I| \geq a_n - a_{n-1}$, then Q can be covered by a Carleson square $\tilde{Q}$ in $\mathbb{C}_+$ with sidelength at most $\frac{1}{c}|I|$. Thus

$$\mu_n(Q) \leq \mu_n(\tilde{Q}) \leq C'\nu_n(\tilde{Q}) \leq C'\frac{1}{c}\beta_n|I|,$$

and we obtain the desired result. The result for μ_N is shown in the same way.
■

Now let $f \in A^p_\nu$. Note that by the (Δ_2)-condition, each of the norms $\|f\|_{H^p_{a_n}}$, denoting the norms on the Hardy spaces $H^p(\mathbb{C}_{a_n})$ of the shifted half planes $\mathbb{C}_{a_n}$, is finite. Restricting μ to some closed half-plane $\overline{\mathbb{C}_{a_N}}$, and using the decomposition in Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{\overline{\mathbb{C}_{a_N}}} |f(z)|^p d\mu(z) \\
= \;& \sum_{n=N}^{\infty} \int_{\mathbb{C}_+} |f(z)|^p d\mu_n(z) \\
\leq \;& C_p \frac{C'}{c} \sum_{n=N}^{\infty} \beta_n \|f\|^p_{H^p_{a_{n-1}}} \\
\leq \;& (2R)^3 C_p \frac{C'}{c} \sum_{n=N}^{\infty} \beta_{n-1} \|f\|^p_{H^p_{a_{n-1}}} \\
= \;& (2R)^3 C_p \frac{C'}{c} \sum_{n=N}^{\infty} \tilde{\nu}([a_{n-1}, a_n)) \|f\|^p_{H^p_{a_{n-1}}} \\
\leq \;& (2R)^3 C_p \frac{C'}{c} \|f(z)\|^p_{A^p_\nu}.
\end{aligned}$$

Here, we use that for any $f \in A^p_\nu$, the map

$$r \mapsto \int_{-\infty}^{\infty} |f(r+it)|^p dt$$

is non-increasing. Using that

$$\int_{\mathbb{C}_+} |f(z)|^p d\mu(z) = \sup_{N\in\mathbb{Z}} \int_{\overline{\mathbb{C}_{a_N}}} |f(z)|^p d\mu(z),$$

we obtain the desired estimate.
Now we can finish the proof of the theorem by showing that a sequence of positive numbers $(a_n)_{n\in\mathbb{Z}}$ with the required properties (5), (6) exists, and that we can also treat the case $\nu(\{0\}) > 0$. Let R be the (Δ_2) constant of the measure $\tilde{\nu}$, and let F be the function given by,

$$F : [0,\infty) \to \mathbb{R}, \quad F(r) = \tilde{\nu}([0,r)).$$

F is left continuous, and also right continuous up to countably many jumps. By the (Δ_2)-condition,

$$F(2t) \leq RF(t),$$

so in particular, a jump of F at t may be no more than $(R-1)F(t)$.
If $\tilde{\nu}(\{0\})=0$, then let for $n\in\mathbb{Z}$

$$a_n=\sup\{r\geq 0: F(r)\leq (2R)^{2n}\},$$

provided that the supremum is finite. If the supremum is infinite, we stop the sequence at the corresponding n.
By the right continuity of F, $a_n>0$ for all $n\in\mathbb{Z}$, and by the condition on jumps of F,

$$(2R)^{2n}\geq F(a_n)\geq \frac{1}{R}(2R)^{2n}$$

and therefore

$$\frac{\tilde{\nu}([0,a_{n+1}))}{\tilde{\nu}([0,a_n))}\geq \frac{(2R)^{2n+2}}{R(2R)^{2n}}\geq 4R.$$

Hence $a_{n+1}\geq 2a_n$ and

$$\frac{a_{n+1}-a_n}{a_{n+1}}\geq \frac{1}{2}.$$

Furthermore,

$$(2R)^{2n}(4R-1)\leq \tilde{\nu}([a_n,a_{n+1}))=F(a_{n+1})-F(a_n)\leq (2R)^{2n+2},$$

hence

$$2R\leq \frac{\tilde{\nu}([a_n,a_{n+1}))}{\tilde{\nu}([a_{n-1},a_n))}\leq (2R)^3.$$

If $\tilde{\nu}(\{0\})\neq 0$, then let $a_0=0$, $\beta_0=\tilde{\nu}(\{0\})$, and let

$$a_n=\sup\{r\geq 0: F(r)\leq (2R)^{2(n+1)}\tilde{\nu}(\{0\})\}\text{ for } n\in\mathbb{N}.$$

We see in the same way as before that properties (5), (6) hold, if we replace $\tilde{\nu}([a_0,a_1))$ by $\tilde{\nu}((a_0,a_1))$. We write $\beta_1=\tilde{\nu}((a_0,a_1))$ and $\beta_n=\tilde{\nu}([a_{n-1},a_n))$ for $n\geq 2$. Then

$$\beta_0\|f\|^p_{H^p_{a_0}}+\sum_{n=1}^{\infty}\beta_n\|f\|^p_{H^p_{a_{n+1}}}\leq \int_{\mathbb{C}_+}|f(z)|^p d\nu(z)\leq \beta_0\|f\|^p_{H^p_{a_0}}+\sum_{n=1}^{\infty}\beta_n\|f\|^p_{H^p_{a_n}},$$

and the same construction as before applies. If the sequence (a_n) is finite to the right, an analogous argument can be made. ■

The following proposition is elementary and appears for special cases in [8, 9]. Partial results are also given in [1, 4].

Proposition 2.3 *Let A^2_ν be a Zen space, and let $w:(0,\infty)\to\mathbb{R}_+$ be given by*

$$w(t)=2\pi\int_0^\infty e^{-2rt}d\tilde{\nu}(r)\qquad (t>0).$$

Then the Laplace transform defines an isometric map $\mathcal{L}: L^2_w(0,\infty)\to A^2_\nu$.

Note that the existence of the integral is guaranteed by the (Δ_2)-condition.

Proof: Let $f \in L^2_w(0,\infty)$. Then

$$\begin{aligned}
& \sup_{\varepsilon>0} \int_{\mathbb{C}_+} |\mathcal{L}f(z+\varepsilon)|^2 d\nu(z) \\
= & \sup_{\varepsilon>0} \int_0^\infty \|(\mathcal{L}f)(\varepsilon + r + \cdot)\|^2_{L^2(i\mathbb{R})} d\tilde{\nu}(r) \\
= & \sup_{\varepsilon>0} \int_0^\infty \|\mathcal{F}(e^{-(r+\varepsilon)\cdot} f)\|^2_{L^2(\mathbb{R})} d\tilde{\nu}(r) \\
= & \sup_{\varepsilon>0} \int_0^\infty 2\pi \|e^{-(r+\varepsilon)\cdot} f\|^2_{L^2(0,\infty)} d\tilde{\nu}(r) \\
= & \sup_{\varepsilon>0} \int_0^\infty |f(t)|^2 2\pi \int_0^\infty e^{-2(r+\varepsilon)t} d\tilde{\nu}(r) dt \\
= & \int_0^\infty |f(t)|^2 w(t) dt
\end{aligned}$$

by isometry of the Fourier transform and the dominated convergence theorem. ■

Here is a Laplace–Carleson Embedding Theorem, which is an immediate consequence.

Theorem 2.4 *Let A^2_ν be a Zen space, $\nu = \tilde{\nu} \otimes d\lambda$, and let $w : (0,\infty) \to \mathbb{R}_+$ be given by*

$$w(t) = 2\pi \int_0^\infty e^{-2rt} d\tilde{\nu}(r) \qquad (t > 0). \tag{14}$$

Then the following are equivalent:

1. *The Laplace transform $\mathcal{L}$ given by $\mathcal{L}f(z) = \int_0^\infty e^{-tz} f(t)dt$ defines a bounded linear map*
$$\mathcal{L} : L^2_w(0,\infty) \to L^2(\mathbb{C}_+, \mu).$$
2. *For a sufficiently large $N \in \mathbb{N}$, there exists a constant $C > 0$ such that*
$$\int_{\mathbb{C}_+} \left|(\mathcal{L}t^{N-1}e^{-\lambda t})(z)\right|^2 d\mu(z) \le C \int_{\mathbb{C}_+} |t^{N-1}e^{-\lambda t}|^2 w(t)dt \text{ for each } \lambda \in \mathbb{C}_+.$$
3. *There exists a constant $C > 0$ such that*
$$\mu(Q_I) \le C\nu(Q_I) \text{ for each Carleson square } Q_I.$$

Proof: Noticing that $\mathcal{L}(t^{N-1}e^{-\lambda t})$ is a scalar multiple of $(k_\lambda)^N$, this follows immediately from Theorem 2.1. ■

2.2 Hankel operators on Zen spaces

The boundedness of Hankel operators on Zen spaces can be deduced from a Carleson measure condition on the symbol, in analogy to the classical proof of the Fefferman–Stein duality theorem via Carleson measures (see e.g. [5], page 239). We will require some adaptations of classical calculations for the case of the disk to the case of the half-plane, and start by introducing the required notation.
Let $\tilde{k}_\lambda(z)$ denote the normalized kernel and let

$$p_\lambda(t) = |\tilde{k}_\lambda(t)|^2 = \frac{1}{\pi}\frac{\operatorname{Re}\lambda}{(t - \operatorname{Im}\lambda)^2 + \operatorname{Re}\lambda^2}$$

denote the Poisson kernel for the right half plane $\mathbb{C}_+$.

Theorem 2.5 *Let A^2_ν be a Zen space, $d\nu = d\tilde{\nu} \otimes d\lambda$ and let $N \in \mathbb{N}$ with $\int_0^\infty \frac{1}{(1+r)^N} d\tilde{\nu}(r) < \infty$. Let $b : \mathbb{C}_+ \to \mathbb{C}$ be analytic, $b \in H^2(\mathbb{C}_+)$. If the measure*

$$|b'(z)|^2 \operatorname{Re} z \int_0^{\operatorname{Re} z} (\operatorname{Re} z - r) d\tilde{\nu}(r) dA(z)$$

is a ν-Carleson measure on $\mathbb{C}_+$, then the Zen Hankel operator

$$A^2_\nu \to \overline{A^2_\nu}, \quad f \mapsto Q_\nu \bar{b} f$$

defines a bounded linear operator. Here, Q_ν denotes the orthogonal projection $Q_\nu : L^2(\mathbb{C}_+, d\nu) \to \overline{A^2_\nu}$.

Proof: For the proof, we follow the lines of the proof of Fefferman's Duality Theorem in [5]. Suppose that $|b'(z)|^2[\int_0^{\operatorname{Re} z} \operatorname{Re}(z-r) d\tilde{\nu}(r)]dA(z)$ is a ν-Carleson measure and let $f, g \in A^2_\nu$. Then

$$\begin{aligned} \langle Q_\nu \bar{b} f, \bar{g}\rangle_{A^2_\nu} &= \int_0^\infty \int_{i\mathbb{R}} \bar{b}(r+t) f(r+t) g(r+t) dt d\tilde{\nu}(r) \\ &= \int_0^\infty \int_{\mathbb{C}_+} \bar{b}'(z+r)(f(r+\cdot)g(r+\cdot))'(z) \operatorname{Re} z dA(z) d\tilde{\nu}(r), \end{aligned}$$

where we use the Littlewood–Paley identity

$$\int_0^\infty \int_{i\mathbb{R}} |f(r+t)|^2 dt d\tilde{\nu}(r) = \int_0^\infty \int_{\mathbb{C}_+} |f'(z+r)|^2 \operatorname{Re} z dA(z) d\tilde{\nu}(r)$$

and its polarization. Now

$$\begin{aligned}
&\left|\int_0^\infty \int_{\mathbb{C}_+} \bar{b}'(z+r) f(r+z) g'(r+z) \operatorname{Re} z dA(z) d\tilde{\nu}(r)\right| \\
&\leq \Big(\int_0^\infty \int_{\mathbb{C}_+} |\bar{b}'(z+r)|^2 |f(r+z)|^2 \operatorname{Re} z dA(z) d\tilde{\nu}(r)\Big)^{1/2} \\
&\qquad\qquad \Big(\int_0^\infty \int_{\mathbb{C}_+} |g'(z+r)|^2 \operatorname{Re} z dA(z) d\tilde{\nu}(r)\Big)^{1/2} \\
&= \Big(\int_{\mathbb{C}_+} |\bar{b}'(z)|^2 |f(z)|^2 \Big[\int_0^{\operatorname{Re} z} \operatorname{Re}(z-r) d\tilde{\nu}(r)\Big] dA(z)\Big)^{1/2} \\
&\qquad\qquad \Big(\int_0^\infty \int_{\mathbb{C}_+} |g'(z+r)|^2 \operatorname{Re} z dA(z) d\tilde{\nu}(r)\Big)^{1/2} \\
&\lesssim \|f\|_{A^2_\nu} \|g\|_{A^2_\nu},
\end{aligned}$$

and the second term $\int_0^\infty \int_{\mathbb{C}_+} \bar{b}'(z+r) f'(r+z) g(r+z) \operatorname{Re} z dA(z) d\tilde{\nu}(r)$ is estimated accordingly. Hence the Hankel operator is bounded. ∎

3 $L^p - L^q$ embeddings

As mentioned above, there is no known full characterization of boundedness of Laplace–Carleson embeddings

$$L^p(0,\infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f = \int_0^\infty e^{-t\cdot} f(t) dt.$$

However, characterizations are possible in case $p \leq 2$, $p' < q$ and in some cases with additional information on the support on the measure. We list some results for natural spectral inclusion conditions which appear naturally in operator semigroups. In the cases we consider here, the oscillatory part of the Laplace transform can be discounted, and a full characterization of boundedness can be achieved.
First, let us make the following simple observation.

Proposition 3.1 *Let μ be a positive regular Borel measure on $\mathbb{C}_+$, let $1 \leq p, q < \infty$ and suppose that the Laplace–Carleson embedding*

$$\mathcal{L} : L^p(0,\infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded. Then there exists a constant $C_{p,q} > 0$ such that for all intervals $I \subset i\mathbb{R}$,

$$\mu(Q_I) \leq C_{p,q} |I|^{q/p'} \text{ if } p > 1, \qquad \mu(Q_I) \leq C_{p,q} \text{ if } p = 1. \tag{15}$$

Proof: Let Q_I be a Carleson square with centre λ_I. Note that

$$\|(\mathcal{L}e^{-\cdot\bar{\lambda}_I})\|^q_{L^q(\mathbb{C}_+,\mu)} \geq \int_{Q_I} |(\mathcal{L}e^{-\cdot\bar{\lambda}_I})|^q d\mu \geq \frac{1}{(4 \operatorname{Re} \lambda_I)^q} \mu(Q_I) = \frac{1}{2^q |I|^q} \mu(Q_I),$$

and

$$\|e^{-\cdot\lambda_I}\|_p^p = \frac{1}{p\,\mathrm{Re}\,\lambda_I} = \frac{2}{p|I|}$$

for $1 \le p, q < \infty$. Hence there exists a constant $C_{p,q} > 0$ such that

$$\mu(Q_I) \le C_{p,q}|I|^{q/p'} \text{ if } p > 1 \text{ and } \mu(Q_I) \le C_{p,q} \text{ if } p = 1.$$

This concludes the proof. ∎

The proposition immediately yields the following theorem:

Theorem 3.2 *Let μ be a positive regular Borel measure supported in the right half-plane $\mathbb{C}_+$, and let $1 < p' \le q < \infty$, $p \le 2$. Then the following are equivalent:*

1. *The Laplace–Carleson embedding*

$$\mathcal{L} : L^p(0,\infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded.

2. *There exists a constant $C > 0$ such that*

$$\mu(Q_I) \le C|I|^{q/p'} \text{ for all intervals } I \subset i\mathbb{R} \tag{16}$$

3. *There exists a constant $C > 0$ such that $\|\mathcal{L}e^{-\cdot z}\|_{L^q_\mu} \le C\|e^{-\cdot z}\|_{L^p}$ for all $z \in \mathbb{C}_+$.*

Proof: Obviously (1) ⇒ (3), and (3) ⇒ (2) by the Proposition.
This leaves (2) ⇒ (1), which is also easy: By the Hausdorff–Young inequality, the map $\mathcal{L} : L^p(0,\infty) \to L^{p'}(i\mathbb{R})$ is bounded. By Duren's theorem, the Poisson extension $L^{p'}(i\mathbb{R}) \to L^q_\mu(\mathbb{C}_+)$ is bounded, given the Carleson condition (15). The composition of both gives the boundedness of the Laplace transform $\mathcal{L} : L^p(0,\infty) \to L^q(\mathbb{C}_+, \mu)$. This concludes the proof. ∎

The case $p > 2$ is much more complicated. We give two special cases here, that of the measure μ being supported in a strip and that of μ being supported in a sector.

3.1 Sectorial measures

If the measure μ is supported on a sector $S(\theta) = \{z \in \mathbb{C}_+ : |\arg z| < \theta\}$ for some $0 < \theta < \frac{\pi}{2}$, then the oscillatory part of the Laplace transform can be discounted, and a full characterization of boundedness can be achieved (see also [7], Theorem 3.2 for an alternative characterization by means of a different measure).

Theorem 3.3 *Let μ be a positive regular Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$, and let $q \ge p > 1$. Then the following are equivalent:*

1. The Laplace–Carleson embedding

$$\mathcal{L} : L^p(0,\infty) \to L^q(\mathbb{C}_+,\mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded.

2. There exists a constant $C > 0$ such that $\mu(Q_I) \leq C|I|^{q/p'}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0.

3. There exists a constant $C > 0$ such that $\|\mathcal{L}e^{-\cdot z}\|_{L^q_\mu} \leq C\|e^{-\cdot z}\|_{L^p}$ for all $z \in \mathbb{R}_+$.

4. There exists a constant $C > 0$ such that $\|\mathcal{L}e^{-\cdot 2^n}\|_{L^q_\mu} \leq C\|e^{-\cdot 2^n}\|_{L^p}$ for all $n \in \mathbb{N}$.

Proof: (2) $\Rightarrow$ (1) For $n \in \mathbb{N}$, let

$$T_n = \{x + iy \in \mathbb{C}_+ : 2^{n-1} < x \leq 2^n, -2^{n-1} < y \leq 2^{n-1}\}.$$

That is, T_n is the right half of the Carleson square Q_{I_n} over the interval $I_n = \{y \in \mathbb{R}, |y| \leq 2^{n-1}\}$. The T_n are obviously pairwise disjoint.
Without loss of generality we assume $0 < \theta < \arctan(\frac{1}{2})$, in which case $S(\theta) \subseteq \bigcup_{n=-\infty}^{\infty} T_n$.
Now let $z \in T_n$ for some $n \in \mathbb{Z}$. Then we obtain, for $f \in L^p(0,\infty)$,

$$|\mathcal{L}f(z)| \leq \int_0^\infty |e^{-zt}||f(t)|dt \leq \int_0^\infty |e^{-2^{n-1}t}||f(t)|dt \leq C_\Theta 2^{-n+1} Mf(2^{-n+1}),$$

where $C_\Theta > 0$ is a constant dependent only on the integration kernel $\Theta(t) = \chi_{[0,\infty)}(t+1)e^{-t-1}$ and Mf is the Hardy–Littlewood maximal function. We refer to e.g. [15], page 57, equation (16) for a pointwise estimate between the maximal function induced by the kernel Θ and M. We can easily dominate Θ by a positive, radial, decreasing L^1 function here. Consequently,

$$\begin{aligned}
\int_{S(\theta)} |\mathcal{L}f(z)|^q d\mu(z) &\leq \sum_{n=-\infty}^{\infty} 2^{q(-n+1)} (Mf(2^{-n+1}))^q \mu(T_n) \\
&\leq C_\Theta^q \sum_{n=-\infty}^{\infty} 2^{q(-n+1)} 2^{nq/p'} (Mf(2^{-n+1}))^q \\
&= C_\Theta^q \sum_{n=-\infty}^{\infty} 2^{q/p'} (2^{(-n+1)} (Mf(2^{-n+1})^p))^{q/p} \\
&\leq C_\Theta^q 2^{q/p'} \left(\sum_{n=-\infty}^{\infty} 2^{(-n+1)} (Mf(2^{-n+1}))^p \right)^{q/p} \\
&\lesssim \|f\|_{L^p}^q.
\end{aligned}$$

Note that in the case $1 < p \leq 2$, (but not for $p > 2$) this result can also easily be deduced from the Hausdorff–Young inequality and Duren's Theorem [3].

(4) ⇒ (2) Let $I \subset i\mathbb{R}$ be an interval which is symmetric about 0. We can assume without loss that $|I| = 2^n$. It is easy to see that

$$|(\mathcal{L}e^{-2^{n-1}\cdot})(z)| = \left|\frac{1}{2^{n-1}+z}\right| \geq \frac{1}{2^{n+1}} \text{ for } z \in Q_I.$$

Thus

$$\begin{aligned}\mu(Q_I) &\leq (2^{n+1})^q \int_{\mathbb{C}_+} |(\mathcal{L}e^{-2^{n-1})(z)\cdot}|^q d\mu(z) \\ &\leq C(2^{n+1})^q \|e^{-2^{n-1}\cdot}\|_p^q \approx 2^{nq}2^{-nq/p} = 2^{nq/p'}.\end{aligned}$$

(1) ⇒ (3) and (3) ⇒ (4) are obvious. ■

Remark 3.4 *Let μ, θ, p and q be as in Theorem 3.3. In [7], Theorem 3.2, essentially the equivalence of the following statements is shown for discrete measures:*

1. *The Laplace–Carleson embedding*

$$\mathcal{L} : L^p(\mathbb{R}_+) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded.

2. *There exists a constant $C > 0$ such that $\tilde{\mu}(Q_I) \leq C|I|^{q/p}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0, where $d\tilde{\mu}(z) = |z|^q d\mu(\frac{1}{z})$.*

The proof also uses the method of the maximal function.

Now let us consider the case $p > q$ for sectorial measures μ. We will, among others, obtain a condition in terms of the balayage S_μ of μ (compare this with the characterization of bounded $H^p(\mathbb{C}_+) \to L^q(\mathbb{C}_+, \mu)$ embeddings for $p > q$ in [10]). Recall that the balayage S_μ of a positive Borel measure μ on $\mathbb{C}_+$ is given by $S_\mu(t) = \int_{\mathbb{C}_+} p_z(t) d\mu(z)$.
To look at a dyadic version, let

$$T_n = \{x + iy \in \mathbb{C}_+ : 2^{n-1} < x \leq 2^n, -2^{n-1} < y \leq 2^{n-1}\}$$

as in the previous proof, and for $k \in \mathbb{Z}$, let $T_{n,k} = T_n + ik2^n$, so the $T_{n,k}$ are the translates of T_n parallel to the imaginary axis. Similarly, let $I_{n,k} = I_n + k2^n$ be the translates of $I_n := \{y \in \mathbb{R} : |y| \leq 2^{n-1}\}$. The $\{T_{n,k} : n, k \in \mathbb{Z}\}$ then form a dyadic tiling of the right half plane. We write $S_n = \cup_{k\in\mathbb{Z}} T_{n,k} = \{z \in \mathbb{C} : 2^{n-1} < \operatorname{Re} z \leq 2^n\}$. Let

$$S_\mu^d(t) = \sum_{n,k\in\mathbb{Z}} \chi_{I_{n,k}}(t) \frac{\mu(T_{n,k})}{2^n}.$$

S_μ^d is called the dyadic balayage of μ. Note that $S_\mu^d \leq 2\pi S_\mu$ pointwise, since

$$\begin{aligned}S_\mu(t) &= \int_{\mathbb{C}_+} p_z(t) d\mu(z) \geq \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \chi_{I_{n,k}}(t) \int_{T_{n,k}} p_z(t) d\mu(z) \\ &\geq \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \chi_{I_{n,k}}(t) \inf_{z\in T_{n,k}} \{p_z(t)\} \mu(T_{n,k}) \geq \frac{1}{2\pi} \sum_{n,k\in\mathbb{Z}} \chi_{I_{n,k}}(t) \frac{\mu(T_{n,k})}{2^n}. \quad (17)\end{aligned}$$

In the special case that μ is sectorial with opening angle $\theta < \pi/2$, the measure μ is supported on $\cup_{n\in\mathbb{Z}} T_n$, and we get a particularly simple form of the dyadic balayage S^d_μ, namely

$$S^d_\mu(t) = \sum_{n=-\infty}^{\infty} \chi_{I_n}(t)\frac{\mu(T_n)}{2^n} = \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty} \chi_{I_n\setminus I_{n-1}}(t)\frac{\mu(T_{n+k})}{2^{n+k}} = \sum_{k=0}^{\infty} S^d_{\mu,k}(t),$$

where

$$S^d_{\mu,k}(t) = \sum_{n=-\infty}^{\infty} \chi_{I_n\setminus I_{n-1}}(t)\frac{\mu(T_{n+k})}{2^{n+k}} = S^d_{\mu,0}(2^k t).$$

Let us now look at an estimate from above for S_μ in terms of the $S^d_{\mu,k}$. Let $t \in I_n\setminus I_{n-1}$. Then

$$\max_{z\in T_k} p_z(t) \le \begin{cases} \frac{2^k}{\pi 2^{2n}} & \text{if} \quad n > k, \\ \frac{1}{\pi 2^k} & \text{if} \quad n \le k. \end{cases}$$

This easily implies that for $t \in I_n\setminus I_{n-1}$,

$$\begin{aligned} S_\mu(t) &\le \frac{1}{\pi}\left(\sum_{j=-\infty}^{-1} \frac{2^{n+j}}{\pi 2^{2n}}\mu(T_{n+j}) + \sum_{j=0}^{\infty} \frac{1}{2^{n+j}}\mu(T_{n+j})\right) \\ &= \frac{1}{\pi}\left(\sum_{j=-\infty}^{-1} 2^{2j}\frac{\mu(T_{n+j})}{2^{n+j}} + \sum_{j=0}^{\infty} \frac{\mu(T_{n+j})}{2^{n+j}}\right) \\ &= \frac{1}{\pi}\left(\sum_{j=-\infty}^{-1} 2^{2j} S^d_{\mu,0}(2^j t) + \sum_{j=0}^{\infty} S^d_{\mu,0}(2^j t)\right). \end{aligned}$$

Hence, for $t \in \mathbb{R}$

$$\begin{aligned} S_\mu(t) &\le \frac{1}{\pi}\left(\sum_{j=-\infty}^{-1} 2^{2j} S^d_{\mu,0}(2^j t) + \sum_{j=0}^{\infty} S^d_{\mu,0}(2^j t)\right) \qquad (18) \\ &= \frac{1}{\pi}\left(\sum_{j=-\infty}^{-1} 2^{2j} S^d_{\mu,0}(2^j t) + S^d_\mu(t)\right). \end{aligned}$$

Theorem 3.5 *Let μ be a positive regular Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$ and let $1 \le q < p < \infty$. Then the following are equivalent:*

1. *The embedding*

$$\mathcal{L} : L^p(\mathbb{R}_+) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded.

2. *The sequence $(2^{-nq/p'}\mu(S_n))$ is in $\ell^{p/(p-q)}(\mathbb{Z})$.*

3. The sequence $(2^{n/p}\|\mathcal{L}k_{2^n}\|_{L^q_\mu})$ *is in* $\ell^{qp/(p-q)}(\mathbb{Z})$.

If $p' < q$*, then the above is also equivalent to*

4. $t^{q(2-p)/p} S_\mu \in L^{p/(p-q)}(\mathbb{R})$.

Remark. If $p' \geq q$, then the sweep S_μ may be infinite everywhere, so we cannot expect a characterisation in terms of S_μ.

Proof: We will start by showing (2) $\Rightarrow$ (1). Recall that $S(\theta)$ is contained in $\bigcup_{n,k\in\mathbb{Z},|k|\leq N}^{\infty} T_{n,k}$ for some $N \in \mathbb{N}$. Suppose that (2) holds. Now as in the proof of Theorem 3.3, we obtain for $z \in S_n$

$$|\mathcal{L}f(z)| \leq \int_0^\infty |e^{-zt}||f(t)|dt \leq \int_0^\infty |e^{-2^{n-1}t}||f(t)|dt \leq C_\Theta 2^{-n+1} Mf(2^{-n+1}).$$

Note that $\chi_{I_{-n}} Mf(2^{-n+1}) \leq \chi_{I_{-n}} Mf$. Consequently

$$\begin{aligned}
&\int_{\mathbb{C}_+} |\mathcal{L}f(z)|^q d\mu(z) \\
\lesssim & \sum_{n=-\infty}^{\infty} 2^{q(-n+1)} Mf(2^{-n+1})^q \mu(S_n) \\
& \sum_{n=-\infty}^{\infty} 2^{q(-n+1)} Mf(2^{-n+1})^q \mu(S_n) \\
\leq & \left(\sum_{n=-\infty}^{\infty} 2^{(-n+1)} Mf(2^{-n+1})^p \right)^{q/p} \left(\sum_{n=-\infty}^{\infty} 2^{(q/p')(-n+1)(p/q)'} \mu(S_n)^{(p/q)'} \right)^{1/(p/q)'} \\
\lesssim & \|f\|^q_{L^p} \|(2^{-nq/p'} \mu(S_n))\|_{(p/q)'}
\end{aligned}$$

by the boundedness of the Hardy–Littlewood maximal function on $L^p(0,\infty)$.
(1) $\Rightarrow$ (2) For $\lambda > 0$ we write $\tilde{k}_\lambda$ for the $L^p(0,\infty)$ function given by

$$\tilde{k}_\lambda(t) = \lambda^{1/p} e^{-\lambda t} \qquad (t \geq 0),$$

noting that $\|\tilde{k}_\lambda\|_{L^p} \asymp 1$.
From a result of Gurariĭ and Macaev in [6], it can be deduced that

$$\left\| \sum_{n\in\mathbb{Z}} \alpha_n \tilde{k}_{2^n} \right\|_p \approx \left(\sum_{n\in\mathbb{Z}} |\alpha_n|^p \right)^{1/p} \tag{19}$$

for any l^p sequence (α_n).
More precisely, we claim that there are constants $A, B > 0$ such that for all scalars (α_k) we have

$$A \sum_n |\alpha_n|^p \leq \left\| \sum_n \alpha_n \tilde{k}_{2^n} \right\|_p^p \leq B \sum_n |\alpha_n|^p$$

By the change of variable $x = e^{-t}$ we have

$$\left\|\sum_n \alpha_n \tilde{k}_{2^n}\right\|_p^p = \int_0^1 \left|\sum_n \alpha_n 2^{n/p} x^{2^n - 1/p}\right|^p dx.$$

Recall that a sequence (m_j) is a *lacunary sequence* if $\inf m_{j+1}/m_j = r > 1$. Now, the result of Gurariĭ and Macaev in [6] asserts the following: If $(n_j + 1/p)$ is lacunary, then the sequence of functions $(t \mapsto (n_j + 1/p)^{1/p} t^{n_j})$ in $L^p(0,1)$ is equivalent to the standard basis of ℓ^p.
Writing $n_j = 2^j - 1/p$ for $j \in \mathbb{Z}$, we have the conditions of the Gurariĭ–Macaev theorem, and the claim follows.

Now, denoting by $(\Omega, d\Omega)$ the probability space of sequences (ϵ_n) taking values in $-1, 1$ with equal probability, equipped with the standard product σ-algebra and probability measure, we obtain

$$\begin{aligned}
\left(\sum_n |\alpha_n|^p\right)^{q/p} &\gtrsim \int_\Omega \|\mathcal{L}(\sum_n \varepsilon_n \alpha_n \tilde{k}_n)\|_{L^q_\mu}^q d\Omega(\varepsilon) \\
&= \int_{\mathbb{C}_+}\int_\Omega \left|\sum_n \varepsilon_n \alpha_n 2^{n/p} \frac{1}{2^n + z}\right|^q d\mu(z) d\Omega(\varepsilon) \\
&\approx \int_{\mathbb{C}_+} \left(\sum_n |\alpha_n|^2 2^{2n/p} \frac{1}{|2^n + z|^2}\right)^{q/2} d\mu(z) \\
&= \sum_k \int_{S_k} \left(\sum_n |\alpha_n|^2 2^{2n/p} \frac{1}{|2^n + z|^q}\right)^{q/2} d\mu(z) \\
&\gtrsim \sum_n \int_{S_n} |\alpha_n|^q 2^{nq/p} \frac{1}{2^{qn}} d\mu(z) \\
&= \sum_n |\alpha_n|^q 2^{-nq/p'} \mu(S_n).
\end{aligned}$$

Here, we have used the fact that μ is supported in a sector $S(\theta)$ in the last inequality. Thus $(2^{-nq/p'}\mu(S_n))$ is a $l^{(p/q)'}$ sequence, and we have proved the desired implication.
A simple argument, again using sectoriality, shows that

$$\|\mathcal{L}k_{2^n}\|_q^q \gtrsim \frac{1}{2^{nq}} \mu(S_n).$$

Hence (3) $\Rightarrow$ (2).
For (1) $\Rightarrow$ (3), note that for any n,

$$\operatorname{Re} \mathcal{L}k_{2^n} \gtrsim |\mathcal{L}k_{2^n}(z)| \quad \text{for } z \in S(\theta).$$

Hence for any sequence $(\alpha_n) \in l^{p/q}(\mathbb{Z})$, $\alpha_n \geq 0$ for all n,

$$\begin{aligned}
\sum_n \alpha_n \|2^{n/p}\mathcal{L}k_{2^n}\|^q_{L^q(\mu)} &\leq \int_{\mathbb{C}_+} \left(\sum_n \alpha_n^{1/q} 2^{n/p} |\mathcal{L}k_{2^n}(z)| \right)^q d\mu(z) \\
&\lesssim \int_{\mathbb{C}_+} \left(\mathrm{Re} \sum_n \alpha_n^{1/q} 2^{n/p} \mathcal{L}k_{2^n}(z) \right)^q d\mu(z) \\
&\leq \int_{\mathbb{C}_+} \left| \mathcal{L}(\sum_n \alpha_n^{1/q} 2^{n/p} k_{2^n})(z) \right|^q d\mu(z) \\
&\lesssim \left\| \sum_n \alpha_n^{1/q} 2^{n/p} k_{2^n} \right\|_p^q \\
&\lesssim \|\alpha_n^{1/q}\|_p^q = \|\alpha_n\|_{p/q}
\end{aligned}$$

by (19). This proves the equivalence of the first three statements.
(2) $\Rightarrow$ (4) Again, we can assume without loss that $0 < \theta < \pi/2$, in which case $S(\theta) \subseteq \cup_{n=-\infty}^{\infty} T_n$. If $t^{q(2-p)/p} S_\mu \in L^{p/p-q}(\mathbb{R})$, then by (17) $t^{q(2-p)/p} S^d_{\mu,0} \in L^{p/(p-q)}(\mathbb{R})$, and

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} 2^{-nqp/(p'(p-q))} \mu(S_n)^{p/(p-q)} \\
= & \sum_{n=-\infty}^{\infty} 2^n 2^{-n(p-2)q/(p-q)} \frac{\mu(S_n)^{p/(p-q)}}{2^{np/(p-q)}} \\
\approx & \sum_{n=-\infty}^{\infty} \int_{I_{n+1}\setminus I_n} \left| t^{-(p-2)q/p} S^d_{\mu,0} \right|^{p/(p-q)} dt \\
= & \|t^{-(p-2)q/p} S^d_{\mu,0}\|^{p/(p-q)}_{p/(p-q)} < \infty.
\end{aligned}$$

Thus (2) holds.
Conversely, if $(2^{-nq/p'}\mu(S_n)) \in l^{p/(p-q)}$, then $t^{-(p-2)q/p} S^d_{\mu,0} \in L^{p/p-q}(\mathbb{R})$ by the above calculation. By (18),

$$\begin{aligned}
& \|t^{-(p-2)q/p} S_\mu\|_{p/(p-q)} \\
\leq & \frac{1}{\pi} \left(\|t^{-(p-2)q/p} S^d_\mu(t)\|_{p/(p-q)} + \sum_{k=-\infty}^{-1} 2^{2k} \|t^{-(p-2)q/p} S^d_{\mu,0}(2^k t)\|_{p/(p-q)} \right).
\end{aligned}$$

One sees easily that

$$\begin{aligned}
2^{2k}\|t^{-(p-2)q/p} S^d_{\mu,0}(2^k t)\|_{p/(p-q)} &= 2^{2k} 2^{-k(p-q)/p} 2^{k(p-2)q/p} \|t^{-(p-2)q/p} S^d_{\mu,0}\|_{p/(p-q)} \\
&= 2^{k(q+1-q/p)} \|t^{-(p-2)q/p} S^d_{\mu,0}\|_{p/(p-q)},
\end{aligned}$$

thus the second term in the sum converges and is controlled by the expression $\|t^{-(p-2)q/(p-q)} S^d_{\mu,0}\|_{p/(p-q)}$. For the first term, write

$$S^d_\mu(t) = \sum_n \chi_{I_n}(t) \frac{\mu(T_n)}{2^n} = \sum_{k=0}^{\infty} \sum_n \chi_{I_n \setminus I_{n-1}}(t) \frac{\mu(T_{n+k})}{2^{n+k}}.$$

as before. For each $k \geq 0$, it follows that

$$\begin{aligned}
& \int \left(t^{q(2-p)/p} \sum_n \chi_{I_n \setminus I_{n-1}}(t) \frac{\mu(T_{n+k})}{2^{n+k}} \right)^{p/(p-q)} dt \\
\lesssim & \sum_n 2^n 2^{nq(2-p)/(p-q)} 2^{-(n+k)p/(p-q)} \mu(T_{n+k})^{p/(p-q)} \\
= & \sum_n 2^{n(q/p-q+1)p/(p-q)} 2^{-(n+k)p/(p-q)} \mu(T_{n+k})^{p/(p-q)} \\
= & \sum_n 2^{n(-q/p'+1)p/(p-q)} 2^{-(n+k)p/(p-q)} \mu(T_{n+k})^{p/(p-q)} \\
= & 2^{-k(1-q/p')p/(p-q)} \sum_n 2^{(n+k)(-q/p')p/(p-q)} \mu(T_{n+k})^{p/(p-q)} \\
\lesssim & 2^{-k(1-q/p')p/(p-q)}.
\end{aligned}$$

Hence $t^{-(p-2)q/p} S_\mu \in L^{p/(p-q)}(\mathbb{R})$. This concludes the proof. ■

3.2 A counterexample

Let μ denote the measure on the interval $[1, \infty)$ defined by $d\mu(x) = dx/\sqrt{x}$. Clearly, μ is sectorial and contained in a shifted half-plane. Moreover μ satisfies the estimate that for a Carleson square Q of size h one has $\mu(Q) \leq 2h^{1/2}$.
Nonetheless, the Laplace–Carleson embedding $\mathcal{L} : L^2(0, \infty) \to L^1(\mu)$ is unbounded (or equivalently, in this case, the Carleson embedding $H^2(\mathbb{C}_+) \to L^1(\mu)$ is unbounded). This can be seen by noting that μ does not satisfy the condition of [11, Theorem C], since the function $t \mapsto \int_{\Gamma(t)} x^{-1} d\mu(x)$ behaves as $x^{-1/2}$ and thus does not lie in L^2. (Here $\Gamma(t)$ may be taken to be the interval $[t, \infty)$.)
It is constructive to give an explicit counterexample, following the reasoning of the proof of [11, Theorem C]. (Note that counterexamples in the case of the disc are simpler, and can be found in [16].)
Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ t^{-1/2}(1 + \log |t|)^{-1} & \text{if } |t| \geq 1. \end{cases}$$

Thus $\phi \in L^2(\mathbb{R})$ and there is a function $F \in H^2$ with $\operatorname{Re} F(it) = \phi(t)$. Now, if $F \in L^1(\mu)$ we would have

$$\int \operatorname{Re} F \, d\mu \leq \int |F| \, d\mu < \infty,$$

from which, as in [11] we could conclude (by writing $\operatorname{Re} F(x, 0)$ in terms of the Poisson kernel) that

$$A := \int_{-\infty}^{\infty} \phi(t) \int_1^{\infty} \frac{x}{x^2 + t^2} \frac{dx}{\sqrt{x}} \, dt < \infty,$$

and hence

$$\int_{-\infty}^{\infty} \phi(t)|t|^{-1/2}\, dt = \int_{-\infty}^{\infty} \phi(t) \int_{|t|}^{\infty} \frac{x}{2x^2} \frac{dx}{\sqrt{x}}\, dt \leq A < \infty,$$

which is a contradiction.

3.3 Measures supported in a strip

Theorem 3.6 *Let μ be a positive regular Borel measure supported in a strip $\mathbb{C}_{\alpha_1,\alpha_2} = \{z \in \mathbb{C} : \alpha_2 \geq \operatorname{Re} z \geq \alpha_1\}$ for some $\alpha_2 \geq \alpha_1 > 0$, and let $1 < p' \leq q < \infty$, $q \geq 2$. Then the following are equivalent:*

1. *The embedding*

$$\mathcal{L} : L^p(0,\infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded, with a bound only depending on the Carleson–Duren constant C and the ratio $\frac{\alpha_2}{\alpha_1}$.

2. *There exists a constant $C > 0$ such that*

$$\mu(Q_I) \leq C|I|^{q/p'} \text{ for all intervals } I \subset i\mathbb{R}. \qquad (20)$$

3. *There exists a constant $C > 0$ such that $\|\mathcal{L}e^{-\cdot z}\|_{L^q_\mu} \leq C\|e^{-\cdot z}\|_{L^p}$ for all $z \in \mathbb{C}_{\alpha_1,\alpha_2}$.*

Proof: Again, obviously (1) ⇒ (3). To show (3) ⇒ (2), we have to remember that the argument in Proposition 3.1 only works for Carleson squares Q_I with centre $\lambda_I \in \mathbb{C}_{\alpha_1,\alpha_2}$. Any Carleson square with centre $\lambda_I \in \mathbb{C}_{\alpha_1/3,\alpha_2}$ can be covered by a Carleson square of at most triple sidelength with centre in $\mathbb{C}_{\alpha_1,\alpha_2}$, any Carleson square with centre in $\lambda_I \in \mathbb{C}_{0,\alpha_1/3}$ has nonempty intersection with the support of μ. If Q_I is a Carleson square with centre $\lambda_I \in \mathbb{C}_{\alpha_2}$, then its intersection with the support of μ can be covered by at most $\left\lceil \frac{2\operatorname{Re}\lambda_I}{\alpha_2} \right\rceil + 1$ Carleson squares with centre in $\mathbb{C}_{\alpha_1,\alpha_2}$. Hence

$$\mu(Q_I) \lesssim \frac{|I|}{\alpha_2} \alpha_2^{q/p'} \leq |I|^{p/q'}.$$

This leaves (2) ⇒ (1).
Consider the line parallel to the imaginary axis $i\mathbb{R} + \alpha_1/2$. Note that

$$\mathcal{L} : L^p(\mathbb{R}_+) \to L^2(i\mathbb{R} + \frac{\alpha_1}{2})$$

is bounded, since

$$\begin{aligned}\|\mathcal{L}f\|_{L^2(i\mathbb{R}+\frac{\alpha_1}{2})} = \|\mathcal{L}(e^{-\alpha_1/2t}f)\|_{L^2(i\mathbb{R})} &= \|e^{-\alpha/2t}f\|_2 \\ &\leq \|e^{-\alpha_1 t}\|_{p/(p-2)}^{1/2}\|f\|_p \lesssim \alpha_1^{\frac{2-p}{2p}}\|f\|_p.\end{aligned}$$

Since the measure μ is supported in $\mathbb{C}_{\alpha_1,\alpha_2}$, by the Carleson condition (15) we have for each Carleson square in $Q_I = \{z \in \mathbb{C} : i\,\mathrm{Im}\, z \in I, \alpha_1/2 < \mathrm{Re}\, z < \alpha_1/2 + |I|\}$ in $\mathbb{C}_{+,\alpha_1/2}$:

$$\mu(Q_I) \leq C|I|^{q/p'} \leq C|I|^{q(1/p'-1/2)}|I|^{q/2} \leq C\alpha_2^{q(1/p'-1/2)}|I|^{q/2}.$$

Thus the Poisson embedding

$$L^2(i\mathbb{R} + \frac{\alpha_1}{2}) \to L^q(\mathbb{C}_+, \mu)$$

is bounded by Duren's Theorem, with constant $C\alpha_2^{1/p'-1/2}$. Again, composing both maps yields the Laplace transform

$$\mathcal{L} : L^p(\mathbb{R}_+) \to L^q(\mathbb{C}_+, \mu_\alpha)$$

with norm bound $C\alpha_1^{\frac{2-p}{2p}}\alpha_2^{1/2-1/p} = C\left(\frac{\alpha_2}{\alpha_1}\right)^{1/2-1/p}$. ■

3.4 Sobolev spaces

In this subsection, we will be interested in embeddings

$$\mathcal{H}^2_\beta(0,\infty) \hookrightarrow L^p(\mathbb{C}_+, \mu),$$

where for $\beta > 0$ the space $\mathcal{H}^2_\beta(0,\infty)$ is given by

$$\begin{aligned} \mathcal{H}^p_\beta(0,\infty) &= \left\{ f \in L^p(\mathbb{R}_+) : \int_0^\infty |(\frac{d}{dx})^\beta f(t)|^p dt < \infty \right\}, \\ \|f\|^p_{\mathcal{H}^p_\beta} &= \|f\|^p_p + \|(\frac{d}{dx})^\beta f\|^p_p. \end{aligned}$$

Here $(\frac{d}{dx})^\beta f$ is defined as a fractional derivative via the Fourier transform. It is now easy to find versions of Theorems 3.3 and 3.5 for Sobolev spaces.

Corollary 3.7 *Let μ be a positive Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$, and let $q \geq p > 1$. Then the following are equivalent:*

1. *The embedding*

$$\mathcal{L} : \mathcal{H}^p_\beta(0,\infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

   is well-defined and bounded.

2. *There exists a constant $C > 0$ such that $\mu_{q,\beta}(Q_I) \leq C|I|^{q/p'}$ for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0. Here, $d\mu_{q,\beta}(z) = (1 + \frac{1}{|z|^{q\beta}})d\mu(z)$.*

3. *There exists a constant $C > 0$ such that $\|\mathcal{L}e^{-\cdot z}\|_{L^q_\mu} \leq C\|e^{-\cdot z}\|_{\mathcal{H}^p_\beta}$ for all $z \in \mathbb{R}_+$.*

Proof: Follows immediately from Theorem 3.3 and basic properties of the Laplace transform. ■

Corollary 3.8 *Let μ be a positive regular Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, $0 < \theta < \frac{\pi}{2}$ and let $1 \leq q < p$, $\beta \geq 0$. Suppose that $S_{\tilde{\mu}_{\beta,q}} \in L^{p/(p-q)}$. Then the embedding*

$$\mathcal{L} : \mathcal{H}^p_\beta(0, \infty) \to L^q(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f,$$

is well-defined and bounded.

Proof: Follows immediately from Theorem 3.5. ■

Laplace–Carleson embeddings of Sobolev spaces $\mathcal{H}^2_\beta$ are easily understood by means of Theorem 1.1:

Theorem 3.9 *Let μ be a positive Borel measure on the right half plane $\mathbb{C}_+$ and let $\beta > 0$. Then the following are equivalent:*

1. *The Laplace–Carleson embedding*

$$\mathcal{H}^2_\beta(0, \infty) \to L^2(\mathbb{C}_+, \mu)$$

   is bounded.

2. *The measure $|1 + z|^{-2\beta} d\mu(z)$ is a Carleson measure on $\mathbb{C}_+$.*

Proof: The proof is a simple reduction to the Carleson embedding theorem. Note that the map

$$\mathcal{H}^2_\beta(\mathbb{R}_+) \to H^2(\mathbb{C}_+), \quad f \mapsto (1 + z)^\beta \mathcal{L}f,$$

is an isomorphism. The remainder follows from the holomorphy of $(1+z)^\beta$ and $\mathcal{L}f$ on $\mathbb{C}_+$, and a density argument. ■

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