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RESEARCH ARTICLE

Meshfree Methods in Option Pricing

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A meshfree approximation scheme based on the radial basis function methods is presented for the numerical solution of the options pricing model. This work deals with the valuation of the European, Barrier, Asian, American options of a single asset and American options of multi assets. The option prices are modeled by the Black-Scholes equation. The θ-method is used to discretize the equation with respect to time. By the next step, the option price is approximated in space with radial basis functions (RBF), in particular, we consider multiquadric radial basis functions (MQ-RBF). In case of American options a penalty method is used, i.e. the free boundary is removed by adding a small and continuous penalty term to the Black-Scholes equation. Finally, we present a comparison of analytical and finite difference solutions and numerical results.

Keywords: meshfree methods; option pricing; radial basis functions; European option; Barrier option; Asian option; American option

AMS Subject Classification: 62P05; 49K20; 65M99

1. Introduction

An option is a derivative product representing a contract, which gives the buyer a right to buy (*call*) or sell (*put*) the underlying asset at prescribed price (*the strike price*) depending on the certain period of time or on a prescribed date (*exercise date*).

There are plenty of kinds of options in the market. In this work we focus on *Vanilla options*, i.e. an option without any nonstandard properties. A vanilla option can belong to different styles of options, namely the *European option* or the *American option*. The difference between these two styles of Vanilla options is the date of exercise: the European option can only be exercised at the end of its life on the maturity date, while the American option allows the holder the early exercise before the maturity date.

An analytical formula exists for the evaluation European call and put options. By assuming a risk-neutrality of the underlying asset price, Black and Scholes [1] showed that the European call option value satisfies a lognormal partial differential equation of diffusion type, which is known as the Black-Scholes equation. However, there is no analytical formula for the American option due to the free boundary.

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Until recently, there are only a few grid-based numerical methods for the valuation of the American options, but in this work we focus our attention on a *meshfree radial basis function* (RBF) approach as a spatial approximation for the numerical solution of the options value and its derivatives in the Black-Scholes equation.

Recently the meshfree RBF approximation for solving the Black-Scholes equation for both European and American options has been examined by a couple of authors. For instance, the meshfree RBF approach has been considered as a spatial approximation for the numerical solution of American option by Fasshauer *et al.* [2, 3]. Hon *et al.* [8, 9] examined the application of global RBFs to transform the Black-Scholes equation into a system of first-order ordinary differential equations with respect to time in order to approximate the numerical solution by known numerical schemes like, for example, the fourth-order Runge-Kutta method; optimizing the method parameters has been investigated by Pettersson *et al.* [13]. A RBF approximation for options value was also studied by Koc *et al.* [10], Goto *et al.* [7] and Marcozzi *et al.* [12].

Hon and Mao [9] developed a numerical scheme by applying the RBFs, particularly Hardy's multiquadric, as a spatial approximation for the numerical solution of the options value and its derivatives in the Black-Scholes equation. They showed that the method does not require the generation of a grid in contrast to the finite difference method. Moreover, the computational domain is composed of scattered collocation points. As we can see, the RBFs are infinitely continuously differentiable and this is the reason why the higher order partial derivatives of the options value can directly be computed by using the derivatives of the basis function.

Fasshauer, Khaliq and Voss [3] considered the Black-Scholes model for American basket options with a nonlinear penalty source term. A basket option is an option whose price is based on multiple underlying assets. A penalty method replaces a constrained optimization problem by a series of unconstrained problems whose solutions ideally converge to the solution of the original constrained problem. The unconstrained problems are formed by adding a term to the objective function that consists of a penalty parameter and a measure of violation of the constraints. The measure of violation is nonzero when the constraints are violated and is zero in the region where constraints are not violated. The problem can be solved on a fixed domain. The Gaussian radial function was used in this approach with a user-selectable shape parameter in the numerical tests.

In Section 2 the foundations of the option pricing are presented. Meshfree methods are presented in Section 3. In Section 4 and Section 5 the procedure of discretization is described and algorithms are given. Finally, all numerical results are presented in Section 6.

2. Option pricing

Here we describe briefly the basics of the European, the American, the Barrier, the Asian and the basket options.

2.1 *European Options*

An analytical formula exists for the evaluation of European call and put options [17]. By assuming a risk-neutrality of the underlying asset price, Black and Scholes showed in their pioneering work [17] that the European call option value satisfies a backward-in-time lognormal partial differential equation (PDE) of diffusion type, which is known as the Black-Scholes equation.

We consider an option, whose price $V(S,t)$ satisfies the following Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \; 0 \le t \le T, \; S \ge 0, \tag{1}$$

where r is the risk-free interest rate, σ is the volatility of the asset price S, $V(S,t)$ denotes the option value at time t and asset price S. The terminal condition at the time of expiry T is given as

$$V(S,T) = \begin{cases} \max\{E - S, 0\}, \text{ for a put option } P(S,t) = V(S,t), \\ \max\{S - E, 0\}, \text{ for a call option } C(S,t) = V(S,t), \end{cases} \tag{2}$$

where E is the strike price of the option.

The boundary conditions for a European call option read

$$C(0,t) = 0, \; C(S,t) \sim S \; as \; S \to \infty, \tag{3}$$

where $C(S,t)$ is the value of the European call option satisfying the corresponding equation (1).

The boundary conditions for a European put option are given as

$$P(0,t) = Ee^{-r(T-t)}, \; P(S,t) \sim 0 \; as \; S \to \infty, \tag{4}$$

where $P(S,t)$ is the value of the European put option satisfying the corresponding equation (1) with a constant interest rate r.

By the simple exponential substitution $S = e^y$ the PDE (1) and terminal condition (2) changes to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial y^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial y} - rU = 0, \tag{5}$$

$$U(y,T) = \begin{cases} \max\{E - e^y, 0\}, \text{ for a put option}, \\ \max\{e^y - E, 0\}, \text{ for a call option}. \end{cases} \tag{6}$$

2.2 *American Options*

The typical feature of an American option is that it allows for an early exercise before the maturity date leading to a free boundary problem. There is no explicit solution known for this case due to the free boundary.

A valuation of the American option is difficult because at each moment of time we have to determine not only the option value, but also the decision whether or not the option should be exercised for each value of S. This problem is known as a free boundary problem. In case of the American option it is unknown a priori where the boundary conditions should be applied since the optimal exercise price S_f is unknown.

The American option valuation problem can be specified by a few constraints. The first constraint says that the option value has to be greater than or equal to the payoff function since the arbitrage profit, which is given from an early exercise, should not be greater than zero. To avoid arbitrage opportunities the option should be exercised in the region where the option value is equal to the payoff function, or it has to satisfy the corresponding Black-Scholes equation where it transcends the payoff. Therefore another constraint requires that the Black-Scholes equation is replaced by an inequality. From the arbitrage it also follows that the option value has to be a continuous function of S.

The value $V(S,t)$ of the American option satisfies the following inequality

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \le 0. \tag{7}$$

The terminal condition at the time of expiry T is given as

$$V(S,T) = \begin{cases} \max\{E - S, 0\}, \text{ for a put option,} \\ \max\{S - E, 0\}, \text{ for a call option,} \end{cases} \tag{8}$$

where E is the strike price of the option.

By the same exponential substitution $S = e^y$ as in the European option case the inequality (7) and condition (8) can be changed to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial y^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial y} - rU \le 0, \tag{9}$$

$$U(y,T) = \begin{cases} \max\{E - e^y, 0\}, \text{ for a put option,} \\ \max\{e^y - E, 0\}, \text{ for a call option.} \end{cases} \tag{10}$$

2.3 *Barrier Options*

Barrier options can be "knock-out" or "knock-in" options. If the barrier price of the option equal the barrier K, the option is called knock-out in case it can be exercised unless the asset price S achieves the barrier K before expiry. The option is called knock-in in case it can be exercised if the asset price S passes the barrier K before expiry.

The knock-out options can be classified into "up-and-out" and "down-and-out" options. The up-and-out option becomes worthless if the barrier K is reached from below before expiry. The down-and-out option becomes worthless if the barrier K is reached from above before expiry. The knock-in options can be classified into "up-and-in" and "down-and-in" options. The up-and-in option is worthless unless the barrier K is reached from below before expire. The down-and-in option is worthless unless the barrier K is reached from above before expire.

Barrier options are attractive because they give more flexibility: the option premium can be reduced through the barrier option by not paying a premium to cover scenarios which are regarded as unlikely.

The value $C(S,t)$ of the down-and-out call barrier option with the barrier K satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \; S > K, \tag{11}$$

$$C(S,t) = 0, \; S \le K. \tag{12}$$

The terminal condition on the expiration date T is given as

$$C(S,T) \;=\; \max\{S-E,\;0\}. \tag{13}$$

If the asset price S reaches K, the option is worthless. Therefore,

$$C(K,t) \;=\; 0,\; S \;=\; K. \tag{14}$$

Consequently, the payoff X at the expiry date T satisfies

$$X \;=\; \begin{cases} \max\{S-E,0\}, & \text{if } S>K \text{ for all } t<T,\\ 0, & \text{if } S\le K \text{ at } t<T. \end{cases} \tag{15}$$

2.4 *Asian Options*

Asian options are averaging options whose terminal payoff depends on some form of averaging of the price of the underlying asset over a part or the whole of the option's life. For details we refer the reader to Kwok [11].

Asian options have the following advantages: Asian options reduce the risk of market manipulation; Asian options are typically cheaper than European or American options, because of the reduced volatility inherent in the option.

There are two main classes of Asian options, the "*fixed strike*" (average rate) and the "*floating strike*" (average strike) options. An average rate option is a cash settled option whose payoff is based on the difference between the average value of the asset during the period from the day of purchase and the expiration date and a strike price. An average strike option is a cash settled whose payoff is based on the difference between the average value of the asset during the period and the asset price at the expiration date. The terminal call payoff X is given as

$$X = \begin{cases} \max(A_T-E,0), & \text{for a fixed strike},\\ \max(S_T-A_T,0), & \text{for an average strike}. \end{cases} \tag{16}$$

Here S_T is the asset price at expiry, E is the strike price, A_T denotes some form of average of the price of the underlying asset over the averaging period $[0,T]$.

In the discrete case we consider an arithmetic average

$$A_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}, \tag{17}$$

where, S_{t_i} is the asset price at the discrete time points $t_i, i = 1, 2 \ldots, n$.

In the limit $n \to \infty$ the discrete sampled average become the continuous sampled average

$$A_T = \frac{1}{T}\int_0^T S_t dt. \tag{18}$$

Consider that the payoff of an option depends on an average strike of an asset

$$\frac{1}{t}\int_0^t S(\tau)d\tau. \tag{19}$$

Setting

$$I = \int_0^t S(\tau)d\tau, \tag{20}$$

we can obtain the following PDE for valuing Asian options [11]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0, \tag{21}$$

where r is the risk free interest rate, σ is the volatility of the stock price S, $V(S,t)$ is the option value at time t and stock price S.

The terminal payoff $V(S,I,T)$ is given by the following expression for put and call options

$$V(S,I,T) = \begin{cases} \max(S - \frac{1}{T}\int_0^t S(\tau)d\tau, 0), & \text{for a call option,} \\ \max(\frac{1}{T}\int_0^t S(\tau)d\tau) - S, 0), & \text{for a put option.} \end{cases} \tag{22}$$

2.5 *American Basket Options*

A basket option is an option whose price is based on several underlying assets. The basket option is a good opportunity for reducing several different risks at the same time and for this reason it is cheaper [2].

Consider an American basket option. The price of d assets at time t is denoted by

$$S(t) = (S_1(t), \ldots, S_d(t)). \tag{23}$$

For the American option early exercise is allowed, therefore this problem can be formulated as a free boundary problem that can be stated by a the Black-Scholes equation for multi-asset problems

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i S_j} + \sum_{i=1}^{d} rS_i \frac{\partial P}{\partial S_i} - rP = 0, \tag{24}$$

$$S_i > \underline{S_i}(t),\ i = 1,\ldots,d, 0 \le t < T, \tag{25}$$

where $P(S,t)$ is the value of the American put option, $\underline{S}(t) = (\underline{S_1}(t), \ldots, \underline{S_d}(t))$ is the free boundary, T is the time of expiry, σ_i denotes the volatility of the i-th underlying asset, r - the risk free interest rate (assumed to be fixed throughout the time period of interest), ρ_{ij} is the correlation between assets i and j.

The payoff function is given by

$$F(S) = max(E - \sum_{i=1}^{d} \alpha_i S_i, 0), \tag{26}$$

where E is the exercise price of the option and α_i are given constants.
The terminal condition reads

$$P(S,T) = F(S),\ S \in \Omega = \{(S_1, \ldots, S_d) :\ S_i > 0, i = 1, \ldots, d\}, \tag{27}$$

and along the free boundary

$$P(\underline{S}(T), t) = F(\underline{S}(t)),\ F(\underline{S}(T)) = 0. \tag{28}$$

The smooth pasting condition to determine the location of the free boundary is given by

$$\frac{\partial P}{\partial S_i}(\underline{S}, t) = -\alpha_i,\ i = 1, \ldots, d. \tag{29}$$

The boundary conditions read

$$\lim_{S_i \to \infty} P(S,t) = 0,\ P(S,t) = g_i(S,t),\ S_i \in \Omega_i,\ S \in \Omega,\ i = 1, \ldots, d, \tag{30}$$

where the Ω_i denote the boundaries of Ω along which the price S_i vanishes.
For the American option early exercise is allowed, therefore we have the following positivity constraint

$$P(S,t) - F(S) \geq 0, S \in \Omega. \tag{31}$$

3. Meshfree Methods

Computation with high-dimensional data is an important issue in many areas of science but a lot of traditional grid based numerical methods can not handle such problems. Meshfree methods are a better strategy when dealing with changes in the geometry of the domain of interest than classical discretization techniques such as finite differences, finite elements or finite volume methods. Moreover, the meshfree discretization is independent from a mesh, because these techniques are based only on a set of independent points.

The scattered data fitting problem is one of the fundamental problems in approximation theory and data modelling in general. We refer the reader to Fasshauer [4], [5] for a more detailed view, how the meshfree approximation method can be applied to PDEs.

Here we write very basic concepts adopted from [4], [5].

Problem 3.1 [4] *Given data* $(x_j, y_j), j = 1, \ldots, N$ *with* $x_j \in \mathbb{R}^s, y_j \in \mathbb{R}$ *find a continuous function* $\mathcal{P}f$ *such that* $\mathcal{P}f(x_j) = y_j,\ j = 1, \ldots, N$.

Here the x_j are the measurement location (or data sites), and the y_j are the corresponding measurements (or data values). These values are obtained by sampling a data function f at the data sites, $y_j = fx_j$, $j = 1, \dots, N$, x_j lies in a s-dimensional space $\mathbb{R}^s$.

We assume in the sequel that the function $\mathcal{P}f$ is a linear combination of certain *basis functions* B_k

$$\mathcal{P}f(x) = \sum_{k=1}^{N} c_k B_k(x), \quad x \in \mathbb{R}^s. \tag{32}$$

Hence, we have to solve the following linear system $Ac = y$, where the entries of the interpolation matrix $A \in \mathbb{R}^{N\times N}$ are given by $a_{jk} = B_k(x_j)$, $j,k = 1, \dots, N$, $c = [c_1, \dots, c_N]^\top$, $y = [y_1, \dots, y_N]^\top$.

Problem 3.1 is well-posed, i.e. a solution to the problem will exist and be unique, if and only if the matrix A is non-singular.

Definition 3.2 *A function* $\Phi : \mathbb{R}^s \to \mathbb{R}$ *is called* radial *provided there exists a univariate function* $\varphi : [0,\infty) \to \mathbb{R}$ *such that* $\Phi(x) = \varphi(r)$, *where* $r = \|x\|$, *and* $\|\cdot\|$ *is some norm on* $\mathbb{R}^s$ *— usually the Euclidean norm.*

Fasshauer [5] showed, that if the univariate function φ is completely monotone and not a constant function, then it leads to a strictly positive definite *radial function* on any $\mathbb{R}^s$, and can be used as a *basic function* to generate bases for the problem.

4. Discretization

For evaluating the prices of European, Barrier, Asian, American and multi-asset American options with radial basis functions we consider discretization methods which were proposed in Goto *et al.* [7] and Fasshauer *et al.* [2].

4.1 *The Case of European Options*

It is well-known that the Black-Scholes equation (1) holds for the option price $V(S,t)$ with asset price S at time t, where the volatility σ is assumed to be constant between the date of purchase and the expiration date.

If the differential operator is abbreviated as

$$F_1 = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r, \tag{33}$$

the PDE (1) becomes

$$\frac{\partial V(S,t)}{\partial t} + F_1 V(S,t) = 0,\ t \in [0,T]. \tag{34}$$

The backward-in-time parabolic PDE (34) is supplied with the terminal conditions (2). An application of the θ-method for the time discretization of (34) leads to

$$\frac{V(S,t+\Delta t)-V(t)}{\Delta t}+(1-\theta)F_1V(S,t+\Delta t)+\theta F_1V(S,t)=0, \qquad (35)$$

where $0 \leq \theta \leq 1$ denotes the implicitness parameter.
After rearranging terms in (35) we obtain

$$[1+(1-\theta)\Delta tF_1]V(S,t+\Delta t)=[1-\theta\Delta tF_1]V(S,t), \qquad (36)$$

i.e.

$$H_1V(S,t+\Delta t)=G_1V(S,t), \qquad (37)$$

where

$$H_1=1+(1-\theta)F_1,\ G_1=1-\theta\Delta tF_1.$$

The multi-quadric radial basis function (MQ - RBF), which will be used for the approximation of the option price $V(S,t)$, is given as [7]

$$\phi(S,S_j)=\sqrt{c^2+\|S-S_j\|^2}, \qquad (38)$$

where S_j is the asset price at the collocation point j for approximating the option price V and $\|S-S_j\|$ denotes the radial distance of each of the N scattered data points S_j. The parameter c is positive and it is called shape parameter. The value of c has dual effects on stability and accuracy of the approximation: as c is increased, so does the accuracy, but only at the cost of ill-conditioning of the matrix of the RBF. This effect is known as the trade-off principle.

Therefore, the approximation for the option price $V(S,t)$ by the RBF is given as

$$V(S,t)\simeq\sum_{j=1}^{N}\lambda_j^t\phi(S,S_j), \qquad (39)$$

where N is the total number of the collocation points at the date t, λ_j^t denote the unknown parameters depending on time t, $\lambda_j^t=\lambda_j(t)$, $\lambda_j^{t+\Delta t}=\lambda_j(t+\Delta t)$, where Δt is the time-step size.

After substituting the ansatz (39) into equation (37) we get

$$\sum_{j=1}^{N}\lambda_j^{t+\Delta t}H_1\phi(S,S_j)=\sum_{j=1}^{N}\lambda_j^tG_1\phi(S,S_j). \qquad (40)$$

Starting from the terminal condition (2), the coefficients λ are determined from the numerical result by using any backward time integration scheme at each time step $T-\Delta t$.

4.2 *The Case of Barrier Options*

Consider the down-and-out option with the expiration price E and the barrier K. It means that the option becomes worthless if the barrier K is reached from above before expiry. The price of the option satisfies (11). The terminal condition is given as (13). It was shown, that the payoff function X for the barrier option is given as (15). Obviously, the discretized equation derived from equation (11) is the same as for the European option, equation (40), because the PDE is identical.

4.3 *The Case of Asian Options*

For Asian options the payoff depends on an average strike of an asset S, given as

$$\frac{1}{t}\int_0^t S(\tau)\,d\tau. \tag{41}$$

Let us set

$$I = \int_0^t S(\tau)\,d\tau, \tag{42}$$

therefore, the following PDE for the Asian option price V holds

$$\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{43}$$

By using the substitution [7]

$$V(S,R,t) = S \cdot H(R,t), \tag{44}$$

where H denotes the option price and R is defined as

$$R = \frac{1}{S}\int_0^t S(\tau)\,d\tau = \frac{I}{S}, \tag{45}$$

equation (43) leads to the backward-in-time convection-diffusion equation

$$\frac{\partial H}{\partial t} + F_2 H = 0. \tag{46}$$

Here, the operator F_2 is defined as

$$F_2 = \frac{1}{2}\sigma^2 R^2 \frac{\partial^2}{\partial R^2} + (1 - rR)\frac{\partial}{\partial R}. \tag{47}$$

The payoff function on the expiration date $t = T$ for a call option is given as

$$V(S,R,T) = \Big(S - \frac{1}{T}\int_0^t S(\tau)\,d\tau\Big)^+. \tag{48}$$

Using the equation (44) and (45) in expression for the payoff (48) leads to

$$S \cdot H(R,T) = S \cdot \Big(1 - \frac{R}{T}\Big)^+.$$

The terminal condition for equation (46) is given as

$$H(R,T) = \Big(1 - \frac{R}{T}\Big)^+. \tag{49}$$

The approximation for the option price $H(R,t)$ by the RBF is given as

$$H(R,t) \simeq \sum_{j=1}^{N} \lambda_j^t \phi(R,R_j), \tag{50}$$

where N is the total number of the collocation points at the date t, λ_j denote the unknown parameters, and MQ-RBF $\phi(R,R_j)$ is given as

$$\phi(R,R_j) = \sqrt{c^2 + \|R - R_j\|^2}. \tag{51}$$

As in the case of the European option the discretized equation reads

$$\sum_{j=1}^{N} \lambda_j^{t+\Delta t} H_2 \phi(R,R_j) = \sum_{j=1}^{N} \lambda_j^t G_2 \phi(R,R_j), \tag{52}$$

where

$$H_2 = 1 + (1-\theta)\Delta t F_2, \quad G_2 = 1 - \theta \Delta t F_2.$$

4.4 *The Case of American Options and Multi-asset American Options*

We apply the meshfree approximation schemes for the solution of the American option problem and for the solution of multi-asset American option problems. According to Fasshauer, Khaliq and Voss [2], we will use a penalty method to remove the free and moving boundary and a linearly implicit θ - method for the time discretization.

We consider the equation (24) with conditions described in Section 2.5 which are satisfied for multi-asset problems.

For eliminating the moving boundary, we will use a penalty term, which was proposed by Fasshauer, Khaliq and Voss [2]. The penalty term was chosen such that the solution stays above the payoff function as the solution approaches expiry and small enough so that the PDE still resembles the Black-Scholes equation very closely. Therefore, the penalty term has the following form

$$\frac{\epsilon C}{P_\epsilon + \epsilon - q}, \tag{53}$$

where $0 < \epsilon \ll 1$ is a small regularization parameter, $C \geq rE$ is a positive constant, and the barrier function is given as

$$q(S) = E - \sum_{i=1}^{d} \alpha_i S_i. \tag{54}$$

After adding the penalty term (53) to the equation (24) the PDE becomes

$$\frac{\partial P_\epsilon}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P_\epsilon}{\partial S_i S_j} + \sum_{i=1}^{d} r S_i \frac{\partial P_\epsilon}{\partial S_i} - rP_\epsilon + \frac{\epsilon C}{P_\epsilon + \epsilon - q} = 0, \tag{55}$$

where $S \in \Omega, \quad 0 \leq t \leq T$.

By using the RBF approach the following expression for the value of the option is obtained

$$P(S,t) = \sum_{j=1}^{N} a_j(t)\phi(\|S - x_j\|).$$

Here the MQ-RBF is used

$$\phi(\|S - x_j\|) = \sqrt{c^2 + \|S - x_j\|^2}. \tag{56}$$

5. Algorithms

In this section as an example of the computational procedure we consider an algorithm for evaluating the Asian option's value. Algorithms for other cases observed in this work have the same structure with specified equations for certain kind of options. For details we refer the reader to Goto *et al* [7] and Fasshauer *et al* [2].

(1) H_{max} is chosen big enough and N collocation points are taken uniformly for the asset price H in the interval $[0, H_{max})$.
(2) The time-step size $\Delta t = T/M$ is chosen and the time interval $[0,T]$ is discretized with the time-step Δt, where $t = 0$ is the date of purchase, $t = T$ is the exercise date and M denotes the number of time steps.
(3) The option price $H(R,T)$ at the expiration date $t = T$ is calculated from the terminal condition (49).
(4) The parameter λ_j^T on the expiration date T is calculated from the equation (50) on $H(R,T)$.
(5) $t \leftarrow T - \Delta t$.

(6) To obtain the unknown coefficients λ_j^t equation (52) is solved.
(7) $t \leftarrow t - \Delta t$.
(8) If $t > 0$, the process returns to the step 6.
(9) To obtain the fair price of the option $H(R, 0)$, λ_j^0 is substituted into equation (50).

6. Results

Here we will present results of our numerical experiments using the procedures described before. In our experiments we use different values of the support radius c and a couple of RBF for the radial basis function approximation of the option price V_t: the Multiquadric (MQ) RBF (38) [7], the Quadratic Matern (QMat) RBF, Wendland's (Wend) RBF and Inverse multiquadric (IMQ) RBF. The detailed description of these RBFs can be found in Fasshauer [5].

6.1 *European Options*

We consider a European call option with strike price E and expiration date T. The parameters used in numerical example are presented in Table 1.

Table 1. Parameters for numerical analysis

Maximum asset price value	$S_{max} = 30$
Number of asset data points	$N = 121$
Number of time steps	$M = 100$
Time-step size	$\Delta t = 0.005$
Expiration date	$T = 0.5$ $(year)$
Exercise price	$E = 10.0$
Risk free interest rate	$r = 0.05$
Volatility	$\sigma = 0.2$
Crank-Nicolson method	$\theta = 0.5$
Support radius	$c = 0.01$

The numerical results for the RBF approximation with different RBF obtained with the value of the support parameter $c = 0.01$ are illustrated by Figure 1 in comparison with the analytical solution for the European call option. .

The numerical results obtained after increasing the parameter c to $c = 1.0$ are illustrated in Figure 2.

The computational error ϵ_{RMSE} was measured as the root mean square error (RMSE) and calculated in the form

$$\epsilon_{RMSE} = \frac{1}{\sqrt{N}} \sqrt{\sum_{j=1}^{N} |\ V(S_j, t)_{RBF} - V(S_j, t)_{Analytical}\ |^2}, \tag{57}$$

where $V(S_j, t)_{RBF}$ and $V(S_j, t)_{Analytical}$ denote the numerical solution by using the RBF approximation and the theoretical solution, respectively.

The maximum computational error ϵ_{max} is calculated as

$$\epsilon_{max} = \max\left(|\ V(S_j, t)_{RBF} - V(S_j, t)_{Analytical}\ |\right). \tag{58}$$

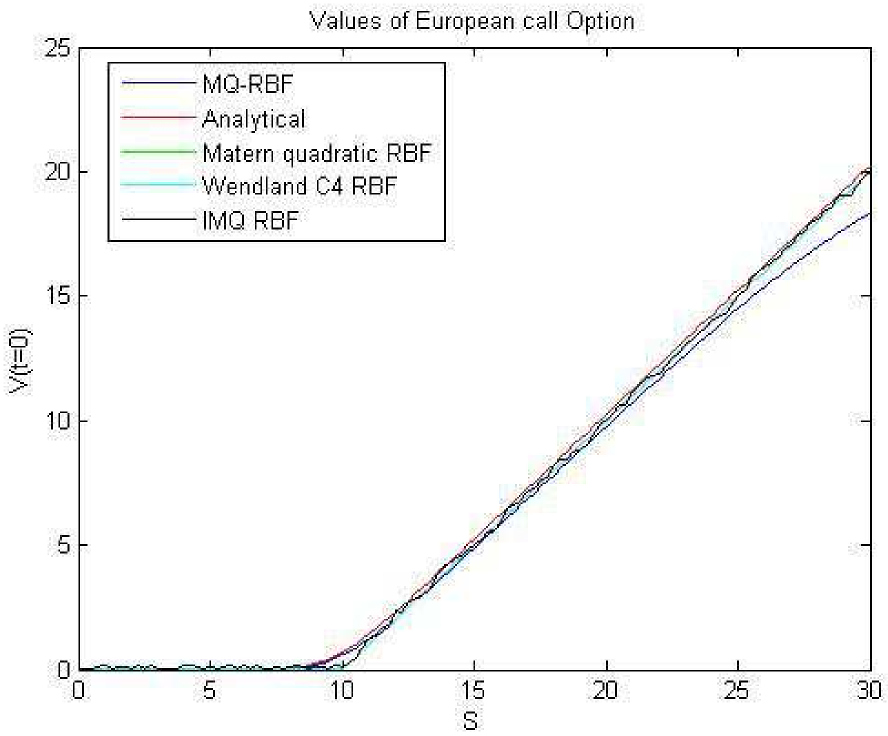

Figure 1. *Values of the European call option, obtained by meshfree methods in comparison with the analytical solution for the European call option with parameters listed in Table 1. Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the support parameter* $c = 0.01$.

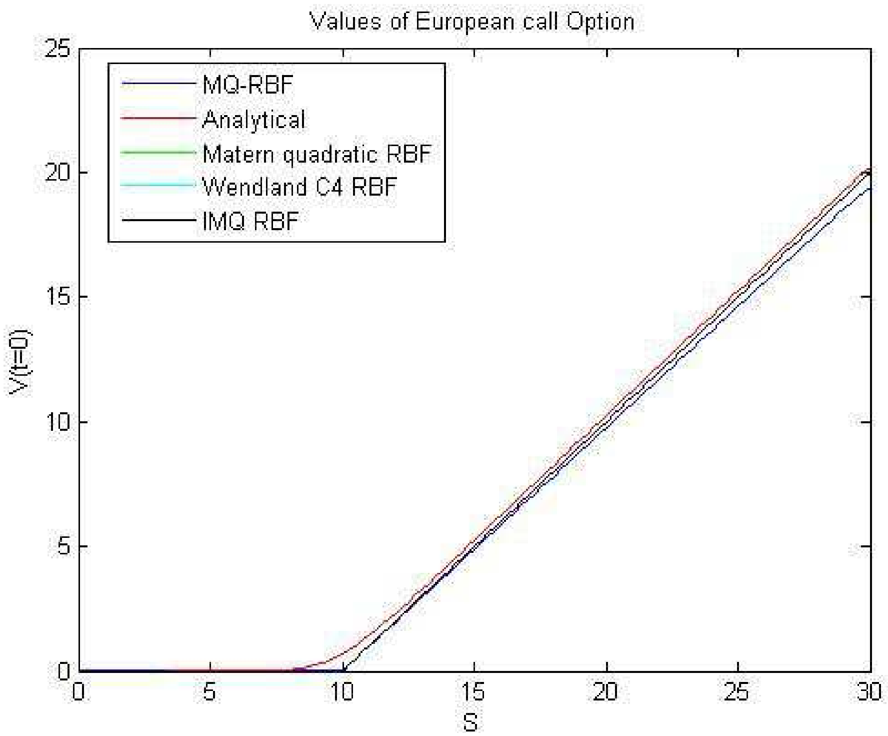

Figure 2. *Values of the European call option, obtained by meshfree methods in comparison with the analytical solution for the European call option with parameters listed in Table 1. Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the support parameter* $c = 1.0$

The RMSE and maximum computational error for different kinds of RBFs and the various values of the parameter c are represented in Table 2.

Table 2. The RMSE and the maximum computational error

c	ϵ_{RMSE}^{MQ}	ϵ_{RMSE}^{QMat}	ϵ_{RMSE}^{Wend}	ϵ_{RMSE}^{IMQ}	ϵ_{max}^{MQ}	ϵ_{max}^{QMat}	ϵ_{max}^{Wend}	ϵ_{max}^{IMQ}
0.01	0.6205	0.2413	0.2413	0.2276	1.9245	0.6889	0.6889	0.5947
1.0	0.4416	0.2414	0.2414	0.2414	0.8284	0.6889	0.6889	0.6889

We can conclude based upon the Table 2, that for the European option the IMQ-RBF gives the best result in comparison with the analytical solution, while the MQ-RBF gives results with the largest error. The change of the parameter c affects deeply MQ-RBF, but the Quadratic Matern RBF and the Wendland's RBF stay insensitive to this change.

6.2 *Barrier Options*

We consider a down-and-out call option with strike price E and barrier K. Parameters used in this numerical example are listed in Table 3.

Table 3. Parameters for numerical analysis

Maximum asset price value	$S_{max} = 30$
Number of asset data points	$N = 121$
Number of time steps	$M = 100$
Time-step size	$\Delta t = 0.005$
Expiration date	$T = 0.5$ $(year)$
Exercise price	$E = 10.0$
Barrier	$K = 9.0$
Risk free interest rate	$r = 0.05$
Volatility	$\sigma = 0.2$
Crank-Nicolson method	$\theta = 0.5$
Support parameter	$c = 0.01$

The numerical results for the RBF approximation with different RBF are illustrated by Figure 3 in comparison with the analytical solution for the down-and-out call Option.

The numerical results obtained after increasing of the parameter c to $c = 1.0$ are illustrated in Figure 4.

The RMSE and maximum computational error for different kinds of RBFs and the various values of the parameter c are represented in Table 4.

Table 4. The RMSE and the maximum computational error

c	ϵ_{RMSE}^{MQ}	ϵ_{RMSE}^{QMat}	ϵ_{RMSE}^{Wend}	ϵ_{RMSE}^{IMQ}	ϵ_{max}^{MQ}	ϵ_{max}^{QMat}	ϵ_{max}^{Wend}	ϵ_{max}^{IMQ}
0.01	0.6207	0.2303	0.2303	0.2199	1.9245	0.6166	0.6166	0.5430
1.0	0.4358	0.2303	0.2303	0.2303	0.8284	0.6166	0.6166	0.6166

In case of the Barrier option we observe the same effect as in the European option case: the IMQ-RBF gives the best result in comparison with the analytical solution and the MQ-RBF has the largest error. The change of the parameter c influences strongly on the results, obtained with the MQ-RBF, but the Quadratic Matern RBF and the Wendland's RBF stay insensitive to this change.

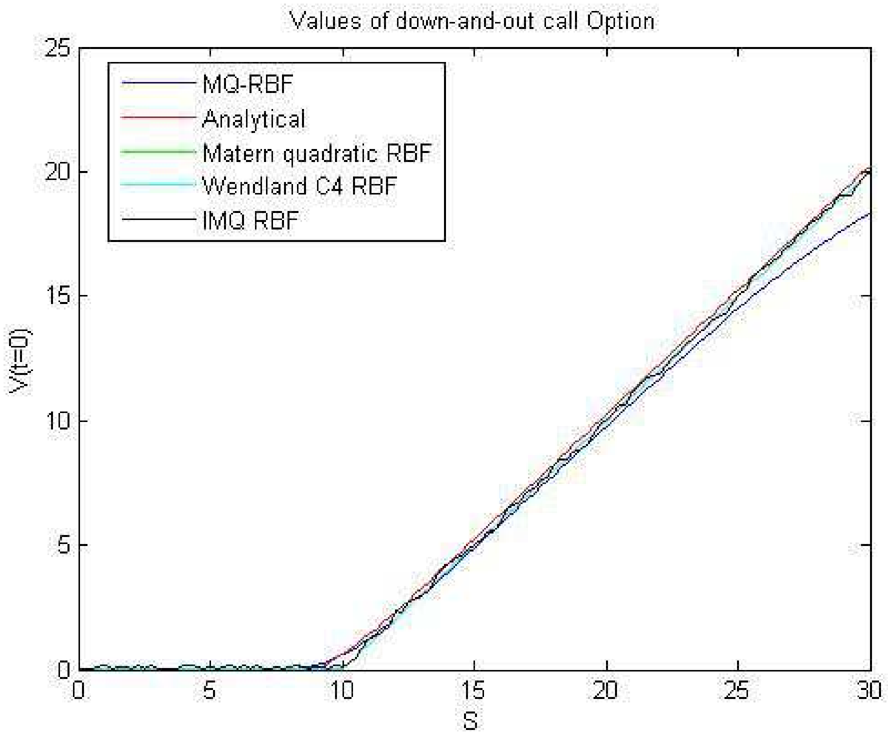

Figure 3. *Values of the down-and-out call option, obtained by meshfree methods in comparison with the analytical solution for the down-and-out call option with parameters listed in Table 3. Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the support parameter* $c = 0.01$.

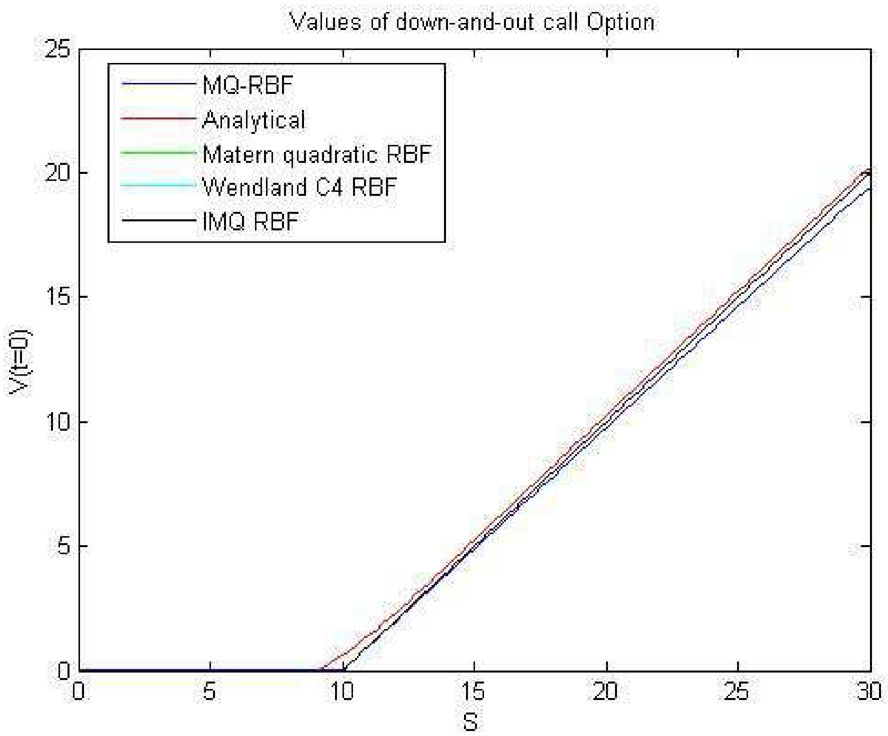

Figure 4. *Values of the down-and-out call option, obtained by meshfree methods in comparison with the analytical solution for the down-and-out call option with parameters listed in Table 3. Here the MQ RBF, the Quadratic Matern RBF, the Wendland's RBF and the IMQ RBF were used with the RBF parameter* $c = 1.0$

6.3 *Asian Options*

We consider an Asian average strike call option. Parameters used in numerical example are listed in Table 5.

Table 5. Parameters for numerical analysis

Maximum R	$R_{max} = 1.0$
Number of asset data points	$N = 101$
Number of time steps	$M = 1000$
Time-step size	$\Delta t = 0.0005$
Expiration date	$T = 0.5$ $(year)$
Risk free interest rate	$r = 0.1$
Volatility	$\sigma = 0.4$
Crank-Nicolson method	$\theta = 0.5$
Support parameter	$c = 0.06$

The numerical results for the RBF approximation with different RBF are illustrated by Figure 5 in comparison with the solution obtained by finite difference method (FD), particularly, Crank-Nicolson method. Here H_{MQ} denotes the result obtained with the MQ-RBF, H_{QMat} is the result obtained with the Quadratic Matern RBF, H_{Wend} is the result obtained with the Wendland's RBF and H_{IMQ} denotes the result obtained with the IMQ RBF.

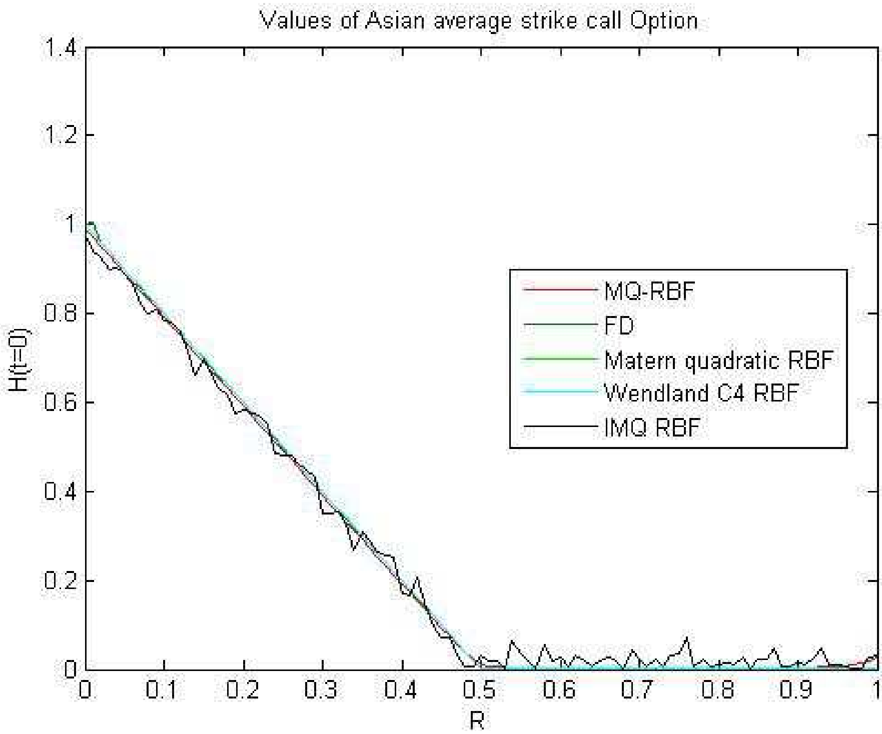

Figure 5. *Values of the Asian average strike call option, obtained by meshfree methods with parameters listed in Table 5, in comparison with the finite difference method with implicitness parameter* $\theta = 0.5$ *(Crank-Nicolson scheme). Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the RBF parameter* $c = 0.06$.

The numerical results obtained after increasing of the parameter c to $c = 0.09$ are illustrated by Figure 6.

The RMSE and maximum computational error for different kinds of RBFs and the various values of the parameter c are represented in Table 6.

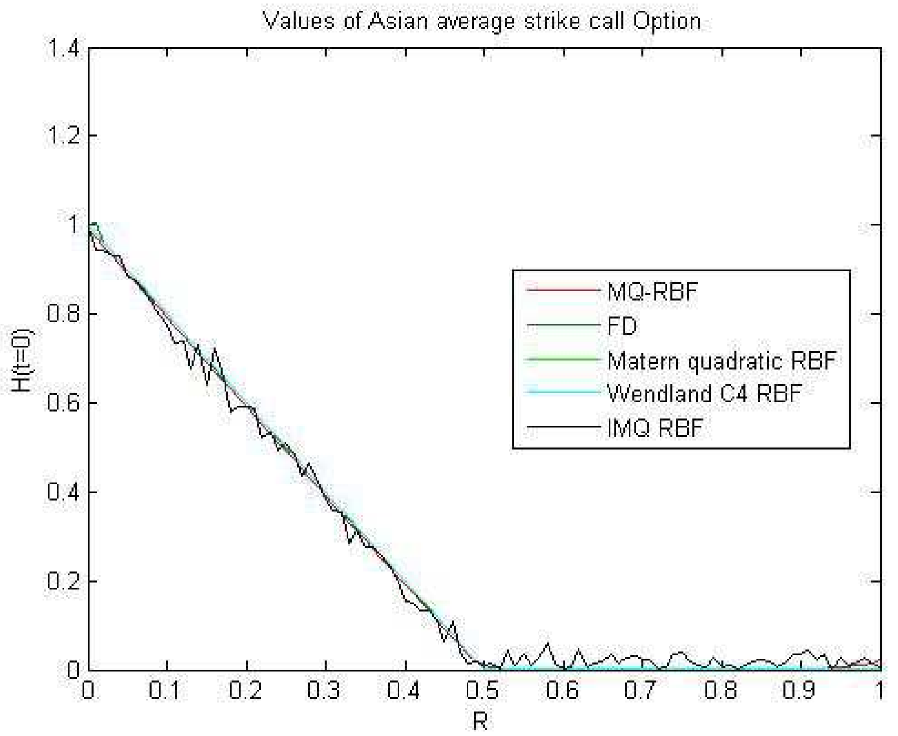

Figure 6. *Values of the Asian average strike call option, obtained by meshfree methods with parameters listed in Table 5, in comparison with the finite difference method with implicitness parameter $\theta = 0.5$ (Crank-Nicolson scheme). Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the RBF parameter $c = 0.09$.*

Table 6. The RMSE and the maximum computational error

c	ϵ_{RMSE}^{MQ}	ϵ_{RMSE}^{QMat}	ϵ_{RMSE}^{Wend}	ϵ_{RMSE}^{IMQ}	ϵ_{max}^{MQ}	ϵ_{max}^{QMat}	ϵ_{max}^{Wend}	ϵ_{max}^{IMQ}
0.06	0.0075	0.0025	0.0025	0.0262	0.0334	0.0239	0.0239	0.0735
0.09	0.0071	0.0025	0.0025	0.0253	0.0333	0.0239	0.0239	0.0627

Here we can see that for an Asian option the IMQ-RBF gives wavy results so it is impossible to apply this RBF for the Asian option. The IMQ-RBF is also the most sensitive for the change of the parameter c.

6.4 *American Options*

We consider an American put option and the parameters used in numerical example are listed in Table 7.

The numerical results for the RBF approximation with different RBF are shown are illustrated in Figure 7 in comparison with the solution obtained by finite difference method (FD), particularly, Crank-Nicolson method.

The numerical results obtained after increasing of the parameter c to $c = 1.0$ are illustrated by Figure 8.

The RMSE and maximum computational error for different kinds of RBFs relative to FD method and the various values of the parameter c are represented in Table 8.

For the American option the MQ-RBF gives the best result with the smallest maximum error. The IMQ-RBF, the Quadratic Matern RBF and the Wendland's RBF gives almost the same value of RMSE and maximum error. They stay almost insensitive to the change of the parameter c, while the MQ-RBF gives smaller value

Table 7. Parameters for numerical analysis

Maximum asset price	$S_{max} = 30$
Number of asset data points	$N = 101$
Number of time steps	$M = 100$
Time-step size	$\Delta t = 0.005$
Expiration date	$T = 0.5$ $(year)$
Exercise price	$E = 10.0$
Risk free interest rate	$r = 0.05$
Volatility	$\sigma = 0.2$
Crank-Nicolson method	$\theta = 0.5$
Constant in the penalty term	$C = 0.9$
Regularization parameter in the penalty term	$\epsilon = 0.001$

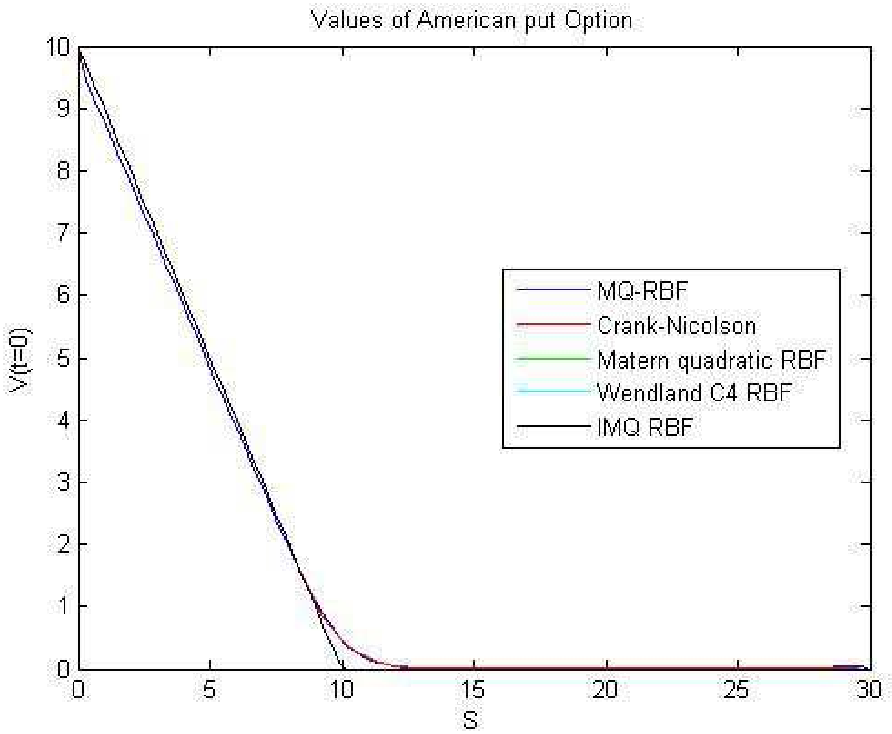

Figure 7. *Values of the American put option, obtained by meshfree methods with parameters listed in Table 7, in comparison with the finite difference method with implicitness parameter* $\theta = 0.5$ *(Crank-Nicolson scheme). Here the MQ RBF, the Quadratic Matern RBF, the Wendland's RBF and the IMQ RBF were used with the support parameter* $c = 0.02$.

Table 8. The RMSE and the maximum computational error

c	ϵ^{MQ}_{RMSE}	ϵ^{QMat}_{RMSE}	ϵ^{Wend}_{RMSE}	ϵ^{IMQ}_{RMSE}	ϵ^{MQ}_{max}	ϵ^{QMat}_{max}	ϵ^{Wend}_{max}	ϵ^{IMQ}_{max}
0.02	0.0899	0.0796	0.0796	0.0796	0.2469	0.4093	0.4093	0.4093
1.0	0.1180	0.0796	0.0796	0.0796	0.4186	0.4093	0.4093	0.4093

of the maximum error and RMSE with smaller parameter c.

Conclusions and Future Work

In this work, we analyzed European, Barrier, Asian and American options via the meshless RBF approach to obtain the approximate value of the option price. It is necessary to use accurate, fast methods with very low memory requirements,

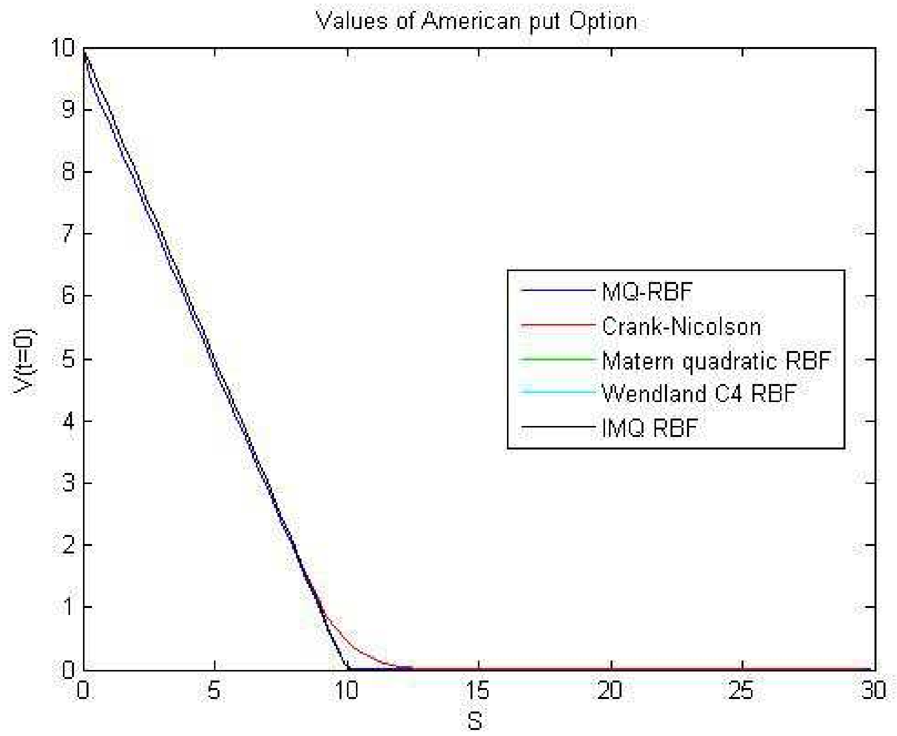

Figure 8. *Values of the American put option, obtained by meshfree methods with parameters listed in Table 7, in comparison with the finite difference method with implicitness parameter* $\theta = 0.5$ *(Crank-Nicolson scheme). Here the MQ RBF, the Quadratic Matern RBF, the Wendland's RBF and the IMQ RBF were used with the RBF parameter* $c = 1.0$

because the financial markets are becoming more and more complex.

The RBF approximation with infinitely smooth RBFs can be spectrally accurate, meaning that the required number of node points for a certain desired accuracy is relatively small. Moreover, the method has a meshfree nature that makes it relatively easy to use also in higher dimensions. These methods in contrast to the traditional simulation algorithms use the geometry of the simulated object directly for calculations. The interpolation problem with RBFs becomes insensitive to the dimension of the space in wich the data sites lie, instead a multivariate function, whose complexity will increase with increasing space dimension, we can use the same univariate function for all choices of dimension. We analyzed the results obtained by Goto *et al.* [7] and by Fasshauer *et al.* [2]. In [7] it was shown that the results of the RBF approximation agreed well with the theoretical solution. Fasshauer [2] got results for American options and multi-asset American options problems comparable to the finite difference method with fewer degrees of freedom.

In our study, we examined and implemented the RBF approximation to obtain the fair price of the European, Barrier, Asian and American options. As a new development, besides MQ-RBF, we investigated different kinds of RBFs such as the Quadric Matern RBF and Wendland's RBF.

The meshfree approach can be more accurate and stable method compared with the Finite Difference method and can be applied to solve PDEs. In our work, the best results were obtained in case of the Asian option for the solution of the reduced equation, where the fair price of the option is purely comparable with the values, obtained by the finite difference method.

Finally, we note that the results obtained by the meshfree method strongly depends on the choice of the RBF, implemented for the approximation of the value of the option. In our investigation, the smallest root-mean-square error was obtained with the MQ-RBF in comparison with Quadric Matern and Wendland's RBF, however the largest maximum error was obtained in the case of the MQ-RBF.

The choice of the RBF's shape parameter c also affects to the accuracy of the method, however we observed, that the influence of the change of the RBF's parameter c is the most intense for the MQ-RBF while for the other examined RBFs the RMSE and the maximum error remain almost unaltered.

In our further research, we will examine the influence of the different parameters of the method to the accuracy, particularly, to find the optimal value of the RBF's shape parameter c. It would also be very interesting for us to find RBFs that yield better results than the RBFs examined here.

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