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Abstract

In this paper we construct infinitely many selfadjoint solutions of the control algebraic Riccati equation using invariant subspaces of the associated Hamiltonian. We do this under the assumption that the system operator is normal and has compact inverse, and that the Hamiltonian possesses a Riesz basis of invariant subspaces.

Keywords. Algebraic Riccati equation, Hamiltonian operator, infinite-dimensional system, Riesz basis, invariant subspace.

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1 Introduction

The algebraic Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0 \tag{1}$$

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appears in many control problems. For instance, it is directly linked to the solution of the linear quadratic optimal control problem. That is, the minimal nonnegative solution of (1) determines the optimal cost, and $-B^*X$ is the optimal feedback, see e.g. [5, 6, 7, 21, 22].

Since it is a quadratic operator equation, (1) may possess infinitely many solutions, even in the finite-dimensional case. A characterisation of all solutions of the algebraic Riccati equation can be presented using the Hamiltonian operator matrix

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

It is easy to see that X satisfies the algebraic Riccati equation if and only if its graph subspace $\Gamma(X) = \mathcal{R}\left(\begin{smallmatrix} I \\ X \end{smallmatrix}\right)$ is invariant under the Hamiltonian.

In the finite-dimensional setting this connection has led to a complete description of all solutions, see e.g. [3, 15, 17, 18]. In the infinite-dimensional setting, only some results in this direction are known: Kuiper and Zwart [14] studied the case where B and C are bounded and T is a Riesz-spectral operator. They obtained a characterisation of all bounded solutions in terms of the eigenvectors of T. Langer, Ran and van de Rotten [16] also studied the case of bounded B, C and used the symmetry of T with respect to an indefinite inner product to prove the existence of nonnegative and nonpositive solutions. In [23, 25] these two approaches were combined and extended to the case that BB^* and C^*C are unbounded closed operators on the state space.

In this paper, we use the connection between invariant subspaces of T and the Riccati equation (1) to construct infinitely many selfadjoint solutions under the following conditions:

(a) A is a normal operator with compact resolvent on a Hilbert space H and generates a C_0-semigroup;

(b) $B \in L(U, H_{-s})$, $C \in L(H_s, Y)$ with $0 \leq s \leq 1$, where $H_s \subset H \subset H_{-s}$ are the usual fractional domain spaces corresponding to A, and we consider duality with respect to the pivot space H.

From (b) we see that $BB^*, C^*C \in L(H_s, H_{-s})$. In particular, BB^* and C^*C map out of the state space H, and the Hamiltonian T is not of the class considered in [23, 25]. We consider T as an unbounded operator on $H \times H$ with domain of definition $\mathcal{D}(T) = \{v \in H_s \times H_s \,|\, Tv \in H \times H\}$ and make the following additional assumption:

(c) T as an operator on $H \times H$ has a compact resolvent and admits a finitely spectral Riesz basis of subspaces, i.e., a Riesz basis consisting of finite-dimensional spectral subspaces.

Such a Riesz basis exists for example if T admits a Riesz basis of generalised eigenvectors. On the other hand, the concept of finitely spectral Riesz bases of subspaces is more general as it allows for Hamiltonians whose (generalised) eigenvectors are complete but do not form a Riesz basis. We use assumption (c) to construct invariant subspaces of the Hamiltonian: For any $\sigma \subset \sigma(T)$ the closed subspace W_σ generated by all generalised eigenvectors corresponding to eigenvalues in σ is T-invariant.

In Theorem 4.6 we show that condition (c) holds if in addition to the assumptions (a) and (b) we have that $s < 1/2$, $C \in L(H,Y)$, A generates an analytic semigroup, and the eigenvalues of A satisfy suitable growth conditions. The idea for the proof is to decompose T as

$$T = S + R, \qquad S = \begin{pmatrix} A & -BB^* \\ 0 & -A^* \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ -C^*C & 0 \end{pmatrix}. \tag{2}$$

The generalised eigenvectors of S are given by explicit formulas and a theorem of Bari implies that they form a Riesz basis. A perturbation result then yields the Riesz basis for T.

Apart from the finitely spectral Riesz basis of subspaces, our main tool to construct solutions of the Riccati equation are the two indefinite inner products on $H \times H$ given by

$$\langle v|w\rangle = (J_1 v|w) \quad \text{with} \quad J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix},$$
$$[v|w] = (J_2 v|w) \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where $(\cdot|\cdot)$ is the standard scalar product on $H \times H$. The Hamiltonian is J_1-skew-symmetric, i.e. $\langle Tv|w\rangle = -\langle v|Tw\rangle$ for all $v, w \in \mathcal{D}(T)$, and J_2-dissipative, $\operatorname{Re}[Tv|v] \le 0$. These relations enable us to apply abstract results from [25] for (skew-) symmetric and dissipative operators on indefinite inner product spaces: The skew-symmetry of T implies that the spectrum $\sigma(T)$ is symmetric with respect to the imaginary axis. If now $\sigma(T) \cap i\mathbb{R} = \varnothing$ and $\sigma \subset \sigma(T)$ is skew-conjugate, i.e., σ contains exactly one eigenvalue from each skew-conjugate pair $(\lambda, -\overline{\lambda})$ in $\sigma(T)$, then $W_\sigma = W_\sigma^{\langle\perp\rangle}$. Here $W_\sigma^{\langle\perp\rangle}$ denotes the orthogonal complement of W_σ with respect to $\langle\cdot|\cdot\rangle$. The dissipativity of T implies that, with respect to $[\cdot|\cdot]$, the subspace $W_+ = W_{\sigma(T)\cap\mathbb{C}_-}$ corresponding to the spectrum in the left half-plane is nonnegative, while $W_- = W_{\sigma(T)\cap\mathbb{C}_+}$ is nonpositive.

Based on these results we prove Theorem 5.6 on the existence of solutions of the Riccati equation: Suppose that the assumptions (a), (b) and (c) hold, that the pair (A,B) is approximately controllable and that there are no non-observable eigenvalues of A on $i\mathbb{R}$. Then $\sigma(T) \cap i\mathbb{R} = \varnothing$ and for every skew-conjugate $\sigma \subset \sigma(T)$, the T-invariant subspace W_σ is the graph of a selfadjoint operator X on H,

$$W_\sigma = \Gamma(X) = \left\{ \begin{pmatrix} x \\ Xx \end{pmatrix} \,\Big|\, x \in \mathcal{D}(X) \right\};$$

in particular, X is a solution of (1). Moreover, the solution X_+ corresponding to W_+ is nonnegative, X_- is nonpositive.

In general, the solutions X will be unbounded. One consequence is that the Riccati equation (1) is only formally satisfied; instead we have

$$A^*Xx + X(Ax - BB^*Xx) + C^*Cx = 0 \quad \text{for all} \quad x \in D,$$

where D is a dense subset of $\mathcal{D}(X)$. On the other hand, Theorem 7.4 yields the existence of bounded solutions: If T has a Riesz basis of generalised eigenvectors whose stable part is quadratically close to an orthonormal system, then X is bounded whenever $\sigma \cap \mathbb{C}_+$ is finite. In particular X_+ is bounded then. We derive a sufficient condition for the existence of such Riesz bases in Theorem 7.8.

The article is structured as follows: In Section 2 we recall the notions of Riesz bases, Riesz bases of subspaces and finitely spectral Riesz bases of subspaces for arbitrary operators on Hilbert spaces. We state the theorem of Bari on the existence of Riesz bases, the invariance of the spaces W_σ, and a perturbation result for finitely spectral Riesz bases of subspaces.

Section 3 contains the general assumptions (a), (b), (c) and the definition of the Hamiltonian and the spaces H_s. In Section 4 we consider the special case that C is bounded. We study the generalised eigenvectors of T and S, see (2), and show that under additional conditions they form Riesz bases.

In Section 5 we then introduce the indefinite inner products $\langle\cdot|\cdot\rangle$ and $[\cdot|\cdot]$, show the J_1-skew-symmetry and J_2-dissipativity of T, derive the properties of the spaces W_σ with respect to the inner products, and finally use this to construct the solutions of the Riccati equation.

The controllability and observability conditions in Theorem 5.6 are actually formulated as conditions on the eigenvectors of A with respect to $\ker B^*$ and $\ker C$. In Section 6 we define suitable notions of controllability and observability for non-admissible inputs and outputs, and we prove that in our setting they can be reformulated in terms of the eigenvectors of A. Section 7 is devoted to the existence of bounded solutions, and Section 8 finally contains an application of our results to the one-dimensional heat equation with boundary control.

Let us give some remarks on the notation: We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers including zero. $\mathbb{C}_+$ is the open right half-plane and $\mathbb{C}_-$ the open left half-plane of the complex plane. On a Hilbert space, we write $(x|y)$ for the scalar product of two vectors. By contrast, (x, y) is the pair consisting of the two elements x and y, so $(x, y) \in H \times H$ for $x, y \in H$. For two normed spaces V and W, the set of bounded linear operators $T : V \to W$ is denoted by $L(V, W)$, and $L(V) = L(V, V)$.

2 Riesz bases of eigenvectors

Let us first recall the notions of Riesz bases and of Riesz bases of subspaces, see e.g. [9, 25]. Let V be a separable Hilbert space. Recall that a sequence $(v_k)_{k\in\mathbb{N}}$ in V is called complete if $\operatorname{span}\{v_k \mid k \in \mathbb{N}\} \subset V$ is dense. For a sequence of subspaces $(V_k)_{k\in\mathbb{N}}$ of V we denote by $\sum_{k\in\mathbb{N}} V_k = \operatorname{span}\bigcup_{k\in\mathbb{N}} V_k$ the subspace generated by the sequence $(V_k)_{k\in\mathbb{N}}$, i.e., the set of all finite sums of elements from the V_k. We say that $(V_k)_{k\in\mathbb{N}}$ is complete if $\sum_{k\in\mathbb{N}} V_k \subset V$ is dense.

Definition 2.1 (i) A sequence $(v_k)_{k\in\mathbb{N}}$ in V is called a *Riesz basis* of V if there exists an isomorphism $\Phi \in L(V)$ such that $(\Phi v_k)_{k\in\mathbb{N}}$ is an orthonormal basis of V.

(ii) A sequence $(V_k)_{k\in\mathbb{N}}$ of closed subspaces of V is called a *Riesz basis of subspaces* of V if there exists an isomorphism $\Phi \in L(V)$ such that $(\Phi(V_k))_{k\in\mathbb{N}}$ is a complete sequence of pairwise orthogonal subspaces.

The sequence $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis if and only if $(v_k)_{k\in\mathbb{N}}$ is complete and there are constants $m, M > 0$ such that

$$m\sum_{k=0}^{n}|\alpha_k|^2 \le \left\|\sum_{k=0}^{n}\alpha_k v_k\right\|^2 \le M\sum_{k=0}^{n}|\alpha_k|^2, \quad \alpha_k \in \mathbb{C},\, n \in \mathbb{N}. \tag{3}$$

Similarly, the sequence of closed subspaces $(V_k)_{k\in\mathbb{N}}$ is a Riesz basis of subspaces of V if and only if $(V_k)_{k\in\mathbb{N}}$ is complete and there exist constants $m, M > 0$ such that

$$m\sum_{k=0}^{n}\|x_k\|^2 \le \left\|\sum_{k=0}^{n}x_k\right\|^2 \le M\sum_{k=0}^{n}\|x_k\|^2, \quad x_k \in V_k,\, n \in \mathbb{N}. \tag{4}$$

If $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis of V, then every $x \in V$ has a unique representation $x = \sum_{k=0}^{\infty}\alpha_k v_k$, $\alpha_k \in \mathbb{C}$, and the convergence of the series is unconditional. Similarly, for a Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$ every $x \in V$ has a unique unconditional expansion $x = \sum_{k=0}^{\infty} x_k$, $x_k \in V_k$.

It is clear that, if $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis of V and $V_j = \operatorname{span}\{v_{k_j}, \dots, v_{k_{j+1}-1}\}$, $0 = k_0 < k_1 < \dots$, then $(V_j)_{j\in\mathbb{N}}$ is a Riesz basis of finite-dimensional subspaces. In the opposite direction, we have the following result:

Lemma 2.2 *Let $(V_j)_{j\in\mathbb{N}}$ be a Riesz basis of finite-dimensional subspaces of V. For each j let $(v_{j1}, \dots, v_{jr_j})$ be a basis of V_j and $\Phi_j : V_j \to V_j$ an isomorphism* *such that $(\Phi_j v_{jk})_{k=1,\dots,r_j}$ is an orthonormal basis of V_j. Then $(v_{jk})_{j\in\mathbb{N},k=1,\dots,r_j}$ is a Riesz basis of V if and only if*

$$\sup_{j\in\mathbb{N}} \|\Phi_j\| < \infty, \qquad \sup_{j\in\mathbb{N}} \|\Phi_j^{-1}\| < \infty. \tag{5}$$

Proof. If (5) holds, then the estimates

$$\begin{aligned}
\Big\|\sum_{k=1}^{r_j} \alpha_k v_{jk}\Big\|^2 &\le \|\Phi_j^{-1}\|^2 \Big\|\sum_{k=1}^{r_j} \alpha_k \Phi_j v_{jk}\Big\|^2 = \|\Phi_j^{-1}\|^2 \sum_{k=1}^{r_j} |\alpha_k|^2,\\
\sum_{k=1}^{r_j} |\alpha_k|^2 &= \Big\|\sum_{k=1}^{r_j} \alpha_k \Phi_j v_{jk}\Big\|^2 \le \|\Phi_j\|^2 \Big\|\sum_{k=1}^{r_j} \alpha_k v_{jk}\Big\|^2
\end{aligned}$$

together with (4) imply (3). Obviously, $(v_{jk})_{jk}$ is complete and hence it is a Riesz basis. On the other hand, if $(v_{jk})_{jk}$ is a Riesz basis, we have

$$\begin{aligned}
\Big\|\Phi_j \sum_{k=1}^{r_j} \alpha_k v_{jk}\Big\|^2 &= \sum_{k=1}^{r_j} |\alpha_k|^2 \le \frac{1}{m} \Big\|\sum_{k=1}^{r_j} \alpha_k v_{jk}\Big\|^2,\\
\Big\|\Phi_j^{-1} \sum_{k=1}^{r_j} \alpha_k \Phi_j v_{jk}\Big\|^2 &= \Big\|\sum_{k=1}^{r_j} \alpha_k v_{jk}\Big\|^2 \le M \sum_{k=1}^{r_j} |\alpha_k|^2 = M \Big\|\sum_{k=1}^{r_j} \alpha_k \Phi_j v_{jk}\Big\|^2,
\end{aligned}$$

which yields (5). □

Theorem 2.3 (Bari [9, Theorem VI.2.3]) *Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of V and let $v_k \in V$ be such that*

(i) $(v_k)_{k\in\mathbb{N}}$ is quadratically close to $(e_k)_{k\in\mathbb{N}}$, i.e. $\sum_{k=0}^\infty \|v_k - e_k\|^2 < \infty$,

(ii) $(v_k)_{k\in\mathbb{N}}$ is ω-linearly independent, i.e., for all $\alpha_k \in \mathbb{C}$, $\sum_k |\alpha_k|^2 < \infty$ we have the implication

$$\sum_{k=0}^\infty \alpha_k v_k = 0 \quad \Rightarrow \quad \alpha_k = 0 \ \ \forall k.$$

Then $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis of V.

For use in Section 7 we also need the following variant of Bari's Theorem; for yet another variant see [10, Lemma 1]. Note that we do not require $(e_k)_{k\in\mathbb{N}}$ to be a basis.

Lemma 2.4 *Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal system of V and let $(v_k)_{k\in\mathbb{N}}$ be quadratically close to $(e_k)_{k\in\mathbb{N}}$. Then there exists $k_0 \in \mathbb{N}$ such that $(v_k)_{k\ge k_0}$ is a Riesz basis of $\overline{\operatorname{span}\{v_k \,|\, k \ge k_0\}}$.*

Proof. We extend $(e_k)_k$ to an orthonormal basis $(e_k)_{k\in\mathbb{N}} \cup (f_k)_{k\in J}$ of V with $J \subset \mathbb{N}$ appropriate. Then we choose $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \|v_k - e_k\|^2 < 1. \tag{6}$$

Now we can define a linear operator T on V by setting $Tf_k = 0$, $Te_k = 0$ for $k < k_0$, and $Te_k = v_k - e_k$ for $k \geq k_0$. From (6) it is easy to see that T is bounded with $\|T\| < 1$. Therefore $I + T$ is an isomorphism and $(I + T)e_k = v_k$ for $k \geq k_0$, which proves the claim. □

Corollary 2.5 *Let $(v_k)_{k\in\mathbb{N}}$ be a complete, ω-linearly independent sequence in V. If there exists $k_1 \in \mathbb{N}$ and an orthonormal system $(e_k)_{k\geq k_1}$ such that*

$$\sum_{k=k_1}^{\infty} \|v_k - e_k\|^2 < \infty,$$

then $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis of V.

Proof. By the previous lemma there exists $k_0 \geq k_1$ such that $(v_k)_{k\geq k_0}$ is a Riesz basis of $U = \overline{\operatorname{span}\{v_k \,|\, k \geq k_0\}}$. Let $W = \operatorname{span}\{v_0, \dots, v_{k_0-1}\}$. Then $U \cap W = \{0\}$ by the ω-linearly independence of $(v_k)_k$. The completeness implies that the algebraically direct sum $U \oplus W \subset V$ is dense. Since U is closed and W is finite dimensional, $U \oplus W$ is also closed and hence $U \oplus W = V$. This in turn implies that $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis of V. □

Definition 2.6 Let T be a linear operator on V with compact resolvent. A Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$ is called *finitely spectral* for T if

(i) all V_k are finite-dimensional, T-invariant, satisfy $V_k \subset \mathcal{D}(T)$, and

(ii) the sets $\sigma(T|_{V_k})$ are pairwise disjoint.

In other words, a Riesz basis $(V_k)_{k\in\mathbb{N}}$ is finitely spectral for T if and only if the V_k are spectral subspaces corresponding to finite disjoint sets of eigenvalues of T.

Let us denote by $\mathcal{L}(\lambda)$ the *generalised eigenspace* or root subspace of T corresponding to an eigenvalue $\lambda \in \sigma_p(T)$, i.e.

$$\mathcal{L}(\lambda) = \bigcup_{k\in\mathbb{N}} \ker(T - \lambda)^k.$$

We say that a sequence $(x_1, \dots, x_n)$ in $\mathcal{L}(\lambda)$ is a *Jordan chain* if $Tx_1 = \lambda x_1$ and $(T - \lambda)x_{k+1} = x_k$.

An example for the existence of a finitely spectral Riesz bases of subspaces is the case that T admits a Riesz bases of generalised eigenvectors:

Lemma 2.7 *Let T have compact resolvent, and let λ_k be the pairwise distinct eigenvalues of T.*

(i) T admits a Riesz basis of generalised eigenvectors $(v_j)_{j\in\mathbb{N}}$ if and only if $(\mathcal{L}(\lambda_k))_{k\in\mathbb{N}}$ is a finitely spectral Riesz basis of subspaces for T. In this case we have
$$\mathcal{L}(\lambda_k) = \operatorname{span}\{v_j \mid v_j \in \mathcal{L}(\lambda_k)\}.$$

(ii) If $(V_k)_{k\in\mathbb{N}}$ is a finitely spectral Riesz basis of subspaces and almost all V_k are eigenspaces, then T admits a Riesz basis of eigenvectors and finitely many Jordan chains.

Proof. (i): Since T has a compact resolvent, all $\mathcal{L}(\lambda_k)$ are finite-dimensional, each eigenvalue λ_k is isolated, and there exist the Riesz projections P_k onto $\mathcal{L}(\lambda_k)$. If $(v_j)_{j\in\mathbb{N}}$ is a Riesz basis of generalised eigenvectors of T, we set
$$V_k = \operatorname{span}\{v_j \mid j \in N_k\} \quad \text{where} \quad N_k = \{j \in \mathbb{N} \mid v_j \in \mathcal{L}(\lambda_k)\}.$$
Then the N_k are finite, pairwise disjoint, and $\mathbb{N} = \bigcup_k N_k$. Consequently $(V_k)_{k\in\mathbb{N}}$ is a Riesz basis of subspaces and $V_k \subset \mathcal{L}(\lambda_k)$. For $x \in \mathcal{L}(\lambda_k)$, the expansion $x = \sum_{j\in\mathbb{N}} \alpha_j v_j$ yields
$$x = P_k x = \sum_{j\in N_k} \alpha_j v_j.$$
Hence $V_k = \mathcal{L}(\lambda_k)$, and $(V_k)_{k\in\mathbb{N}}$ is finitely spectral for T. If on the other hand $(\mathcal{L}(\lambda_k))_{k\in\mathbb{N}}$ is a Riesz basis of subspaces, then the choice of an orthonormal basis in each $\mathcal{L}(\lambda_k)$ yields the desired Riesz basis $(v_j)_{j\in\mathbb{N}}$ by Lemma 2.2.

(ii): We choose an orthonormal basis in each V_k that is an eigenspace. In the finitely many remaining V_k, we have the Jordan canonical form of the restrictions $T|_{V_k}$ and may choose bases consisting of Jordan chains. In view of Lemma 2.2, the collection of these bases is a Riesz basis. □

Remark 2.8 (i) Note that the Riesz basis $(v_j)_j$ of Lemma 2.7(i) does not necessarily consist of Jordan chains. We also remark that the conditions in 2.7(i) are equivalent to T being a discrete spectral operator in the sense of Dunford and Schwartz, see [8, 23].

(ii) The notion of a finitely spectral Riesz basis of subspaces is more general than the one of a Riesz basis of generalised eigenvectors, see e.g. [25, Example 3.7]. Moreover, in [25] finitely spectral Riesz bases of subspaces are investigated without the assumption that T has a compact resolvent. Instead, the weaker property that $\sum_k V_k$ is a core for T is used.

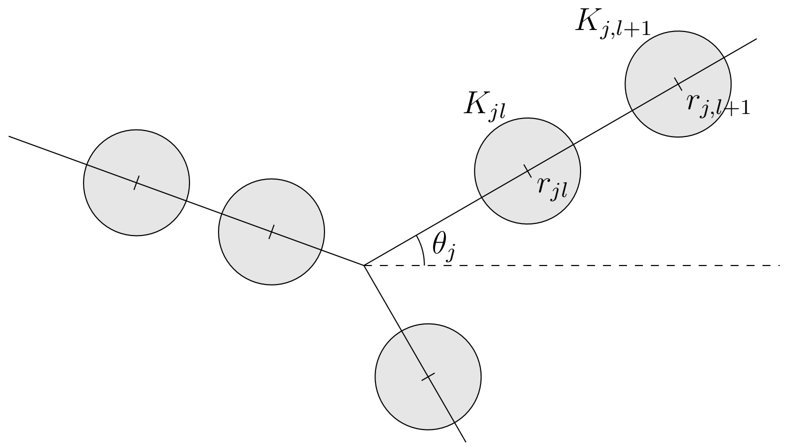

Figure 1: The location of the spectrum of S in Theorem 2.10

A finitely spectral Riesz basis of subspaces yields a representation of T with respect to the subspaces V_k. Moreover, it implies the existence of T-invariant subspaces associated with arbitrary subsets of the point spectrum:

Proposition 2.9 *Let T have compact resolvent and a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$. Then*

$$\begin{aligned}
\mathcal{D}(T) &= \left\{ x = \sum_{k=0}^{\infty} x_k \,\middle|\, x_k \in V_k,\ \sum_{k=0}^{\infty} \|Tx_k\|^2 < \infty \right\}, \\
Tx &= \sum_{k=0}^{\infty} Tx_k \quad \text{for} \quad x = \sum_{k=0}^{\infty} x_k \in \mathcal{D}(T),\ x_k \in V_k, \\
V_k &= \sum_{\lambda\in\sigma(T|_{V_k})} \mathcal{L}(\lambda).
\end{aligned}$$

For every $\sigma \subset \sigma_p(T)$, the subspace

$$W_\sigma = \overline{\sum_{\lambda\in\sigma} \mathcal{L}(\lambda)} \tag{7}$$

is T-invariant and $(T-z)^{-1}$-invariant for every $z \in \varrho(T)$.

Proof. See Proposition 3.5 and Corollaries 3.6 and 3.11 in [25]. □

Theorem 2.10 *Let S be an operator on V with compact resolvent and a Riesz basis of Jordan chains. Suppose that almost all eigenvalues of S lie inside discs*

$$K_{jl} = \left\{\lambda \in \mathbb{C} \,\middle|\, |\lambda - e^{i\theta_j} r_{jl}| \leq \alpha\right\}, \qquad j = 1, \dots, n,\ l \in \mathbb{N},$$

see Figure 1, with $0 \leq \theta_j < 2\pi$, $\alpha \geq 0$, and $r_{jl} \geq 0$ such that

$$\lim_{l\to\infty} r_{j,l+1} - r_{jl} = \infty, \qquad j = 1, \dots, n.$$

Then for any $R \in L(V)$ the operator $T = S + R$ has compact resolvent and admits a finitely spectral Riesz basis of subspaces.

If moreover almost all eigenvalues of S are simple and almost all K_{jl} contain exactly one eigenvalue, then T admits a Riesz basis of eigenvectors and finitely many Jordan chains.

Proof. By Proposition 6.6 in [24] there exists an isomorphism $\Phi \in L(V)$ such that $\Phi S \Phi^{-1} = S_0 + R_0$ where S_0 is normal with compact resolvent, R_0 is bounded, all eigenvalues of S_0 lie on the line segments

$$L_{jl} = \left\{e^{i\theta_j} x \,\middle|\, r_{jl} - \alpha \leq x \leq r_{jl} + \alpha\right\}, \qquad j = 1, \dots, n,\ l \in \mathbb{N}.$$

Moreover for almost all pairs (j, l) the sums of the algebraic multiplicities of the eigenvalues of S_0 in L_{jl} and of S in K_{jl}, respectively, are the same. Theorem 6.2 in [24] now implies that

$$\Phi T \Phi^{-1} = S_0 + R_0 + \Phi R \Phi^{-1}$$

has compact resolvent and a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$. If we also have that almost all eigenvalues of S are simple and almost all K_{jl} contain exactly one eigenvalue, then we even obtain that almost all V_k are one-dimensional. Lemma 2.7 thus yields a Riesz basis of eigenvectors and finitely many Jordan chains. Since Φ is an isomorphism, the same results hold for T. $\square$

3 The Hamiltonian

From now on we consider the following setting: Let A be the generator of a C_0-semigroup on a Hilbert space H such that A is normal and has a compact resolvent. So there is an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H consisting of eigenvectors,

$$Ae_k = \lambda_k e_k, \qquad A^* e_k = \overline{\lambda}_k e_k,$$

and the eigenvalues satisfy $\sup_k \operatorname{Re} \lambda_k < \infty$ and $\lim_{k\to\infty} |\lambda_k| = \infty$.

Along with A we consider the usual fractional domain spaces H_s: For $0 \leq s \leq 1$ let $H_s = \mathcal{D}((|A|+I)^s)$ be equipped with the graph norm, and let H_{-s} be the completion of H with respect to the norm $\|(|A|+I)^{-s}\cdot\|$. Hence, as a consequence of the spectral theorem for A,

$$H_s = \Big\{ \sum_k \alpha_k e_k \,\Big|\, \sum_k |\lambda_k|^{2s}|\alpha_k|^2 < \infty \Big\}, \qquad -1 \leq s \leq 1,$$

and an equivalent norm on H_s is given by

$$\|x\|_s^2 = \sum_k (|\lambda_k|+1)^{2s}|\alpha_k|^2, \qquad x = \sum_k \alpha_k e_k \in H_s.$$

In particular $H_s \subset H_t$ for $1 \geq s \geq t \geq -1$. The operators A, A^* have bounded extensions

$$A, A^* : H_s \to H_{s-1}, \qquad 0 \leq s \leq 1,$$

which we denote again by A, A^*. Similarly, the scalar product on H extends to a sesquilinear form

$$(\cdot|\cdot)_{s,-s} : H_s \times H_{-s} \to \mathbb{C}.$$

Via this extension we can identify H_{-s} with the dual space of H_s, i.e., H_{-s} is the dual of H_s with respect to the pivot space H.

Let us consider input and output operators $B \in L(U, H_{-s})$, $C \in L(H_s, Y)$ with $0 \leq s \leq 1$ and Hilbert spaces U, Y. Using duality with respect to H, we have $B^* \in L(H_s, U)$, $C^* \in L(Y, H_{-s})$ and hence $BB^*, C^*C \in L(H_s, H_{-s})$. The Hamiltonian operator matrix is now

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

For $v = (x, y) \in H_s \times H_s$ the product $Tv \in H_{-1} \times H_{-1}$ is well defined,

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax - BB^*y \\ -C^*Cx - A^*y \end{pmatrix}.$$

We want to consider T as an unbounded operator on $H \times H$, that is, we consider T with domain of definition

$$\mathcal{D}(T) = \{ v \in H_s \times H_s \,|\, Tv \in H \times H \}.$$

Then $\mathcal{D}(T) \subset H_{1-s} \times H_{1-s}$: e.g. $Ax - BB^*y \in H$ and $BB^*y \in H_{-s}$ imply $Ax \in H_{-s}$ and hence $x \in H_{1-s}$.

4 Eigenvectors of the Hamiltonian

In this section we derive conditions on B and C which imply that the Hamiltonian has a Riesz basis of generalised eigenvectors and hence also a finitely spectral Riesz basis of subspaces.

In addition to the general setting, we assume here that $B \in L(U, H_{-s})$ with $s < 1/2$ and $C \in L(H, Y)$. We decompose the Hamiltonian as

$$T = S + R, \qquad S = \begin{pmatrix} A & -BB^* \\ 0 & -A^* \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ -C^*C & 0 \end{pmatrix}. \tag{8}$$

Hence $\mathcal{D}(S) = \mathcal{D}(T)$, $R \in L(H \times H)$.

For any $\lambda \in \mathbb{C}$ let us denote by $(A - \lambda)^+$ the Moore-Penrose pseudoinverse of $A - \lambda$:

$$(A - \lambda)^+ x = \sum_{\substack{k \in \mathbb{N} \\ \lambda_k \neq \lambda}} \frac{1}{\lambda_k - \lambda} (x|e_k) e_k.$$

For $\lambda \in \varrho(A)$ this is simply the resolvent $(A - \lambda)^+ = (A - \lambda)^{-1}$. Just as the resolvent, the pseudoinverse admits extensions to the fractional power spaces H_s, e.g.

$$(A - \lambda)^+ : H_{-s} \to H_{1-s}.$$

We have now explicit expressions for the (generalised) eigenvectors of S: Let

$$v_k = \begin{pmatrix} e_k \\ 0 \end{pmatrix}, \qquad w_k = \begin{pmatrix} x_k \\ e_k \end{pmatrix}, \qquad x_k = (A + \overline{\lambda}_k)^+ BB^* e_k. \tag{9}$$

Lemma 4.1 *We have $v_k, w_k \in \mathcal{D}(S)$ and*

$$Sv_k = \lambda_k v_k, \qquad (S + \overline{\lambda}_k)^2 w_k = 0.$$

Moreover

$$(S + \overline{\lambda}_k) w_k = \begin{pmatrix} y_k \\ 0 \end{pmatrix}$$

where y_k is the orthogonal projection of $-BB^*e_k$ onto $\ker(A + \overline{\lambda}_k)$. *In particular $Sw_k = -\overline{\lambda}_k w_k$ if $-\overline{\lambda}_k \notin \sigma(A)$.*

Proof. First note that at least $v_k, w_k \in H_{1-s} \times H_1$, and hence $Sv_k, Sw_k \in H_{-1} \times H_{-1}$ are well defined. The first equation is immediate from $Ae_k = \lambda_k e_k$. By definition of the pseudoinverse $(A + \overline{\lambda}_k)^+$, $I - (A + \overline{\lambda}_k)(A + \overline{\lambda}_k)^+$ is the orthogonal projection onto $\ker(A + \overline{\lambda}_k)$ and hence

$$(S + \overline{\lambda}_k) w_k = \begin{pmatrix} (A + \overline{\lambda}_k) x_k - BB^* e_k \\ (-A^* + \overline{\lambda}_k) e_k \end{pmatrix} = \begin{pmatrix} y_k \\ 0 \end{pmatrix}.$$

Consequently $(S + \overline{\lambda}_k)^2 w_k = 0$. In particular we have $Sv_k, Sw_k \in H \times H$ and thus $v_k, w_k \in \mathcal{D}(S)$. $\square$

One consequence of the previous lemma is that for w_k to be a proper generalised eigenvector of S, it is necessary that $-\overline{\lambda}_k \in \sigma(A)$, i.e. that A has the skew-conjugate pair of eigenvalues $(\lambda_k, -\overline{\lambda}_k)$.

Lemma 4.2 *S has a compact resolvent and $\sigma(S) = \sigma(A) \cup \sigma(-A^*)$.*

Proof. The previous lemma implies $\lambda_k, -\overline{\lambda}_k \in \sigma_p(S)$ for all k, i.e. $\sigma(A) \cup \sigma(-A^*) \subset \sigma_p(S)$. On the other hand, let $z \in \varrho(A) \cap \varrho(-A^*)$ and consider

$$G = \begin{pmatrix} (A-z)^{-1} & -(A-z)^{-1}BB^*(A^*+z)^{-1} \\ 0 & -(A^*+z)^{-1} \end{pmatrix}.$$

We aim to show that G is a compact operator on $H \times H$ and that it is the inverse of $S - z$. The operator

$$H_{s-1} \xrightarrow{(A-z)^{-1}} H_s \xrightarrow{BB^*} H_{-s} \xrightarrow{(A^*+z)^{-1}} H_{1-s}$$

is bounded. Since $(A-z)^{-1}$ and $(A^* + z)^{-1}$ are compact as operators on H and since the imbeddings $H \hookrightarrow H_{s-1}$ and $H_{1-s} \hookrightarrow H$ are also compact, G is a compact operator on $H \times H$. Since $\mathcal{R}(G) \subset H_{1-s} \times H_1$, the product SG is well defined and we calculate

$$(S-z)G = \begin{pmatrix} A-z & -BB^* \\ 0 & -(A^*+z) \end{pmatrix} G = I_{H \times H}$$

and $G(S-z) = I_{\mathcal{D}(S)}$. Consequently, $\mathcal{R}(G) = \mathcal{D}(S)$, $z \in \varrho(S)$ and $(S-z)^{-1} = G$. We conclude that S has compact resolvent and $\sigma_p(S) = \sigma(S) = \sigma(A) \cup \sigma(-A^*)$. $\square$

Lemma 4.3 *For $z \in \varrho(A)$, the resolvent $(A-z)^{-1}$ considered as an operator in $L(H_{-s}, H)$ has the norm*

$$\|(A-z)^{-1}\|_{L(H_{-s},H)} = \sup_{k \in \mathbb{N}} \frac{(|\lambda_k| + 1)^s}{|\lambda_k - z|}. \tag{10}$$

Proof. For $x = \sum_k \alpha_k e_k \in H_{-s}$ we have

$$\|x\|_{-s}^2 = \sum_k (|\lambda_k| + 1)^{-2s} |\alpha_k|^2.$$

Hence

$$\|(A-z)^{-1}x\|^2 = \sum_k \Big|\frac{\alpha_k}{\lambda_k - z}\Big|^2 \le \sup_k \frac{(|\lambda_k|+1)^{2s}}{|\lambda_k - z|^2}\|x\|_{-s}^2,$$

which implies the estimate "$\le$" in (10). Equality now follows from a consideration of the cases $x = e_k$. $\square$

Let us now consider the situation that almost all eigenvalues λ_k of A lie in a sector in the open left half-plane, i.e., that A generates an analytic semigroup.

Lemma 4.4 *Suppose that almost all eigenvalues λ_k of A lie in a sector in the open left half-plane. Then there exist $k_0 \in \mathbb{N}$, $c_0 \in \mathbb{R}$ such that for $k \ge k_0$ we have*

$$|\lambda_k| \ge 1, \quad -\overline{\lambda}_k \notin \sigma(A) \quad \textit{and} \quad \sup_{j\in\mathbb{N}} \frac{(|\lambda_j|+1)^s}{|\lambda_j + \overline{\lambda}_k|} \le \frac{c_0}{|\lambda_k|^{1-s}}.$$

Proof. By assumption there exist $k_1 \in \mathbb{N}$, $c_1 \in \mathbb{R}$ such that

$$k \ge k_1 \quad \Rightarrow \quad \operatorname{Re}\lambda_k < 0,\ |\operatorname{Im}\lambda_k| \le c_1|\operatorname{Re}\lambda_k|.$$

Hence

$$|\lambda_k| \le \sqrt{1+c_1^2}\,|\operatorname{Re}\lambda_k| \quad \text{for} \quad k \ge k_1.$$

Moreover $-\overline{\lambda}_k \in \sigma(A)$, i.e. $-\overline{\lambda}_k = \lambda_j$ for some j is possible for at most finitely many k. Let now $r = \max\{|\lambda_0|, \dots, |\lambda_{k_1-1}|, 1\}$ and choose $k_0 \ge k_1$ such that $k \ge k_0$ implies $|\lambda_k| \ge 2r$ and $-\overline{\lambda}_k \notin \sigma(A)$. For $k \ge k_0$ and $j < k_1$ we then have

$$\frac{(|\lambda_j|+1)^s}{|\lambda_j + \overline{\lambda}_k|} \le \frac{(r+1)^s}{|\lambda_k| - |\lambda_j|} \le \frac{(r+1)^s}{|\lambda_k| - r} \le \frac{2(r+1)^s}{|\lambda_k|} \le \frac{2(r+1)^s}{(2r)^s|\lambda_k|^{1-s}}.$$

For the case $j \ge k_1$ we use the following estimate for $\tau \ge 1$:

$$\sup_{t\ge 0} \frac{(t+1)^s}{t+\tau} \le \frac{2^s}{\tau^{1-s}}.$$

Indeed for $0 \le t \le \tau$ we have

$$\frac{(t+1)^s}{t+\tau} \le \frac{(\tau+1)^s}{\tau} \le \frac{2^s\tau^s}{\tau},$$

and for $t \ge \tau$,

$$\frac{(t+1)^s}{t+\tau} \le \frac{2^s t^s}{t} = \frac{2^s}{t^{1-s}} \le \frac{2^s}{\tau^{1-s}}.$$

Now for $k \geq k_0$, $j \geq k_1$ we obtain

$$\frac{(|\lambda_j|+1)^s}{|\lambda_j + \overline{\lambda}_k|} \leq \frac{(|\lambda_j|+1)^s}{|\operatorname{Re}\lambda_j + \operatorname{Re}\lambda_k|} = \frac{(|\lambda_j|+1)^s}{|\operatorname{Re}\lambda_j| + |\operatorname{Re}\lambda_k|}$$
$$\leq \sqrt{1+c_1^2}\,\frac{(|\lambda_j|+1)^s}{|\lambda_j|+|\lambda_k|} \leq \sqrt{1+c_1^2}\,\frac{2^s}{|\lambda_k|^{1-s}}.$$

□

Theorem 4.5 *Suppose that almost all eigenvalues λ_k of A lie in a sector in the open left half-plane and let $B \in L(U, H_{-s})$ with $s < 1/2$. If*

$$\sum_{\substack{k=0\\ \lambda_k \neq 0}}^{\infty} \frac{1}{|\lambda_k|^{2(1-2s)}} < \infty, \tag{11}$$

then $(v_k, w_k)_{k\in\mathbb{N}}$ given by (9) is a Riesz basis. Here all v_k and almost all w_k are eigenvectors of S. Moreover, S admits a Riesz basis of eigenvectors and at most finitely many Jordan chains of length at most 2.

Proof. To show that $(v_k, w_k)_k$ from (9) is a Riesz basis, we want to apply Theorem 2.3 of Bari using the orthonormal basis

$$\left(v_k, \begin{pmatrix} 0 \\ e_k \end{pmatrix}\right)_{k\in\mathbb{N}}$$

of $H \times H$. Since $(v_k, w_k)_{k\in\mathbb{N}}$ is obviously ω-linearly independent, it suffices to show that

$$\sum_{k=0}^{\infty} \left\| w_k - \begin{pmatrix} 0 \\ e_k \end{pmatrix} \right\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2 < \infty.$$

Let k_0, c_0 as in Lemma 4.4. For $k \geq k_0$ we have

$$\|x_k\| \leq \|(A+\overline{\lambda}_k)^{-1}\|_{L(H_{-s},H)}\, \|BB^*\|_{L(H_s,H_{-s})}\, \|e_k\|_s,$$

$\|e_k\|_s = (|\lambda_k|+1)^s$ and, by the previous lemmas,

$$\|(A+\overline{\lambda}_k)^{-1}\|_{L(H_{-s},H)} = \sup_{j\in\mathbb{N}} \frac{(|\lambda_j|+1)^s}{|\lambda_j+\overline{\lambda}_k|} \leq \frac{c_0}{|\lambda_k|^{1-s}}.$$

Therefore, using (11), we obtain

$$\sum_{k\geq k_0} \|x_k\|^2 \leq c_0^2\, \|BB^*\|^2 \sum_{k\geq k_0} \frac{(|\lambda_k|+1)^{2s}}{|\lambda_k|^{2(1-s)}} \leq 2^{2s} c_0^2\, \|BB^*\|^2 \sum_{k\geq k_0} \frac{|\lambda_k|^{2s}}{|\lambda_k|^{2(1-s)}} < \infty.$$

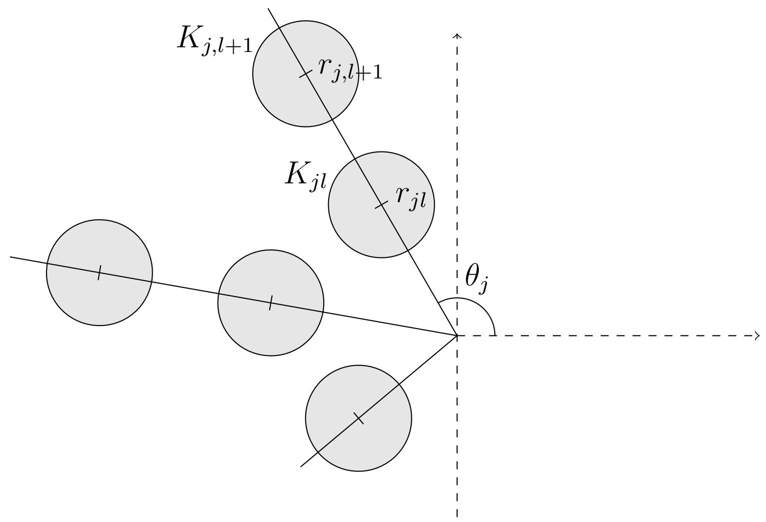

Figure 2: The location of almost all eigenvalues of A in Theorem 4.6

Due to the sectoriality assumption on the λ_k, the spectrum $\sigma(A)$ contains at most finitely many skew-conjugate pairs of eigenvalues, and Lemma 4.1 thus implies that almost all w_k are eigenvectors. The final assertion is now a consequence of Lemma 2.7: each generalised eigenspace $\mathcal{L}(\lambda)$ of S is spanned by some v_k, w_k; hence almost all $\mathcal{L}(\lambda)$ are eigenspaces, and the remaining ones contain Jordan chains of length at most two. $\square$

Theorem 4.6 *Let A generate an analytic semigroup, let $B \in L(U, H_{-s})$ with $s < 1/2$ and $C \in L(H, Y)$. Suppose that*

$$\sum_{\substack{k=0 \\ \lambda_k \neq 0}}^{\infty} \frac{1}{|\lambda_k|^{2(1-2s)}} < \infty$$

and that almost all λ_k lie inside discs

$$K_{jl} = \big\{\lambda \in \mathbb{C} \,\big|\, |\lambda - e^{i\theta_j} r_{jl}| \leq \alpha\big\}, \qquad j = 1, \dots, n,\ l \in \mathbb{N},$$

see Figure 2, where $\pi/2 < \theta_j < 3\pi/2$, $\alpha \geq 0$, and $r_{jl} \geq 0$ such that

$$\lim_{l \to \infty} r_{j,l+1} - r_{jl} = \infty, \qquad j = 1, \dots, n.$$

Then T has a compact resolvent and a finitely spectral Riesz basis of subspaces.

If moreover almost all eigenvalues of A are simple and almost all K_{jl} contain exactly one eigenvalue of A, then T even has a Riesz basis of eigenvectors and finitely many Jordan chains.

Proof. This a direct application of Theorem 2.10 to $T = S + R$ since S has compact resolvent and a Riesz basis of Jordan chains and $R \in L(H \times H)$. Note that since $\sigma(S) = \{\lambda_k, -\overline{\lambda}_k \,|\, k \in \mathbb{N}\}$, almost all eigenvalues of S lie in the discs K_{jl} and $-K_{jl}^* = \{-\bar{z} \,|\, z \in K_{jl}\}$. □

Remark 4.7 The assumptions of the previous theorem imply that B is an admissible control operator, compare Proposition 6.1.

5 Solutions of the Riccati equation

We return to the general setting of Section 3, i.e. $B \in L(U, H_{-s})$, $C \in L(H_s, Y)$, $0 \le s \le 1$. Let T have compact resolvent and a finitely spectral Riesz basis of subspaces.

As the main tool to prove the existence of solutions of the Riccati equation we will use two indefinite inner products on $H \times H$, which are connected to the Hamiltonian. Let

$$\langle v|w\rangle = (J_1 v|w) \quad \text{with} \quad J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix},$$
$$[v|w] = (J_2 v|w) \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where $(\cdot|\cdot)$ denotes the standard scalar product on $H \times H$. Since both J_1 and J_2 are selfadjoint involutions, each of the two inner products $\langle\cdot|\cdot\rangle$ and $[\cdot|\cdot]$ give $H \times H$ the structure of a Krein space. We refer to [1, 4, 13] for more results about Krein spaces.

Lemma 5.1 *The Hamiltonian T is J_1-skew-symmetric, i.e.*

$$\langle Tv|w\rangle = -\langle v|Tw\rangle \quad \textit{for all} \quad v, w \in \mathcal{D}(T),$$

and J_2-dissipative, i.e. $\mathrm{Re}[Tv|v] \le 0$ *for all $v \in \mathcal{D}(T)$. In fact*

$$\mathrm{Re}\Big[T\begin{pmatrix} x \\ y \end{pmatrix} \,\Big|\, \begin{pmatrix} x \\ y \end{pmatrix}\Big] = -\|B^*y\|_U^2 - \|Cx\|_Y^2 \le 0, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(T). \tag{12}$$

Proof. Let $(x, y), (\tilde{x}, \tilde{y}) \in \mathcal{D}(T)$. Then

$$x, y \in H_s, \qquad Ax,\ A^*y,\ BB^*y,\ C^*Cx \in H_{-s},$$
$$Ax - BB^*y,\ -C^*Cx - A^*y \in H,$$

and the same holds for $\tilde{x}, \tilde{y}$. We can thus rearrange the indefinite inner product using the extended scalar products $(\cdot|\cdot)_{-s,s}$ and $(\cdot|\cdot)_{s,-s}$ as follows:

$$\begin{aligned}
\Big\langle T\begin{pmatrix}x\\y\end{pmatrix}\Big|\begin{pmatrix}\tilde{x}\\\tilde{y}\end{pmatrix}\Big\rangle &= \Big\langle\begin{pmatrix}Ax - BB^*y\\-C^*Cx - A^*y\end{pmatrix}\Big|\begin{pmatrix}\tilde{x}\\\tilde{y}\end{pmatrix}\Big\rangle\\
&= i(Ax - BB^*y|\tilde{y}) - i(-C^*Cx - A^*y|\tilde{x})\\
&= i(Ax|\tilde{y})_{-s,s} - i(BB^*y|\tilde{y})_{-s,s} + i(C^*Cx|\tilde{x})_{-s,s} + i(A^*y|\tilde{x})_{-s,s}\\
&= i(x|A^*\tilde{y})_{s,-s} - i(y|BB^*\tilde{y})_{s,-s} + i(x|C^*C\tilde{x})_{s,-s} + i(y|A\tilde{x})_{s,-s}\\
&= i(x|C^*C\tilde{x} + A^*\tilde{y}) - i(y| - A\tilde{x} + BB^*\tilde{y})\\
&= \Big\langle\begin{pmatrix}x\\y\end{pmatrix}\Big|\begin{pmatrix}-A\tilde{x} + BB^*\tilde{y}\\C^*C\tilde{x} + A^*\tilde{y}\end{pmatrix}\Big\rangle = -\Big\langle\begin{pmatrix}x\\y\end{pmatrix}\Big|T\begin{pmatrix}\tilde{x}\\\tilde{y}\end{pmatrix}\Big\rangle.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\Big[T\begin{pmatrix}x\\y\end{pmatrix}\Big|\begin{pmatrix}x\\y\end{pmatrix}\Big] &= (Ax - BB^*y|y) + (-C^*Cx - A^*y|x)\\
&= (Ax|y)_{-s,s} - (BB^*y|y)_{-s,s} - (C^*Cx|x)_{-s,s} - (A^*y|x)_{-s,s}\\
&= (Ax|y)_{-s,s} - (B^*y|B^*y)_U - (Cx|Cx)_Y - (y|Ax)_{s,-s}
\end{aligned}$$

and hence (12). □

As in the Hilbert space case, the adjoint of T with respect to the indefinite inner product $\langle\cdot|\cdot\rangle$ is defined as the maximal operator $T^{\langle*\rangle}$ on $H \times H$ such that

$$\langle Tv|w\rangle = \langle v|T^{\langle*\rangle}w\rangle \quad \text{for all} \quad v \in \mathcal{D}(T),\, w \in \mathcal{D}(T^{\langle*\rangle}).$$

Lemma 5.2 *The Hamiltonian is J_1-skew-selfadjoint, $T = -T^{\langle*\rangle}$, and its spectrum $\sigma(T)$ is symmetric with respect to the imaginary axis.*

Moreover, we have $\sigma(T) \cap i\mathbb{R} = \varnothing$ if and only if

$$\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B^* = \{0\} \qquad \forall t \in \mathbb{R}. \tag{13}$$

Proof. Since T has compact resolvent, there exist $z, -\bar{z} \in \varrho(T)$. As in the Hilbert space situation, this together with the J_1-skew-symmetry of T implies the J_1-skew-selfadjointness. The general property $\lambda \in \sigma(T) \Leftrightarrow \overline{\lambda} \in \sigma(T^{\langle*\rangle})$ then yields the claimed symmetry of $\sigma(T)$.

For the second assertion let first $it \in \sigma(T) \cap i\mathbb{R}$ and $Tv = itv$ with $v = (x, y) \in \mathcal{D}(T)$. Then

$$(A - it)x - BB^*y = 0, \qquad -C^*Cx - (A^* + it)y = 0, \tag{14}$$

and (12) implies

$$0 = \mathrm{Re}(it[v|v]) = \mathrm{Re}[Tv|v] = -\|B^*y\|^2 - \|Cx\|^2.$$

Hence $B^*y = 0$, $Cx = 0$ and thus also $(A - it)x = (A^* + it)y = 0$. If on the other hand $x \in \ker(A-it) \cap \ker C$ and $y \in \ker(A^*+it) \cap \ker B^*$, then (14) holds and implies $v = (x, y) \in \mathcal{D}(T)$, $Tv = itv$. $\square$

The symmetry of $\sigma(T)$ with respect to $i\mathbb{R}$ implies that for $\sigma(T) \cap i\mathbb{R} = \varnothing$, $\sigma(T)$ consists of skew-conjugate pairs of eigenvalues only. In this case we say that a subset $\sigma \subset \sigma(T)$ is *skew-conjugate* if σ contains exactly one eigenvalue from each pair, i.e., if we have the disjoint union

$$\sigma(T) = \sigma \uplus -\sigma^*, \quad \text{where} \quad -\sigma^* = \{-\overline{\lambda} \mid \lambda \in \sigma\}.$$

A subspace $W \subset H \times H$ is called J_1*-nonnegative, -neutral, -nonpositive* if $\langle v|v\rangle \geq 0, = 0, \leq 0$ for all $v \in W$, respectively. The J_1*-orthogonal complement* of W is defined by

$$W^{\langle\perp\rangle} = \{v \in H \times H \mid \langle v|w\rangle = 0 \text{ for all } w \in W\}.$$

Then W is J_1-neutral if and only if $W \subset W^{\langle\perp\rangle}$.

Proposition 5.3 *Suppose that $\sigma(T) \cap i\mathbb{R} = \varnothing$.*

(i) For every skew-conjugate $\sigma \subset \sigma(T)$ the T-invariant subspace W_σ from (7) *satisfies $W_\sigma = W_\sigma^{\langle\perp\rangle}$; in particular W_σ is J_1-neutral.*

(ii) The subspace $W_+ = W_{\sigma(T)\cap\mathbb{C}_-}$ is J_2-nonnegative, $W_- = W_{\sigma(T)\cap\mathbb{C}_+}$ is J_2-nonpositive.

Proof. Since T is a skew-symmetric operator in the Krein space associated with $\langle\cdot|\cdot\rangle$ and has a finitely spectral Riesz basis of subspaces, (i) is a direct consequence of Theorem 5.3 together with Remark 5.8 in [25]. Similarly, in view of the J_2-dissipativity of T, (ii) follows from [25, Proposition 5.7]. $\square$

Lemma 5.4 *Suppose that*

$$\ker(A^* - \lambda) \cap \ker B^* = \{0\} \qquad \forall \lambda \in \mathbb{C}. \tag{15}$$

If the subspace $W \subset H \times H$ is J_1-neutral and $(T - z)^{-1}$-invariant for some $z \in \varrho(T)$, then W is the graph of some linear operator X on H,

$$W = \Gamma(X) = \left\{ \begin{pmatrix} x \\ Xx \end{pmatrix} \,\middle|\, x \in \mathcal{D}(X) \right\}.$$

Proof. Initially, we note two consequences of the assumption that both A and T have a compact resolvent: First, the invariance of W under $(T-z)^{-1}$ for one $z \in \varrho(T)$ implies the invariance for all $z \in \varrho(T)$. Second, the spectra $\sigma(A)$ and $\sigma(T)$ are both discrete, and hence $\varrho(T) \cap \varrho(A)$ contains a non-empty interval on the imaginary axis.

To prove that W is a graph subspace, it suffices to show that $(0, w) \in W$ implies $w = 0$. So let $(0, w) \in W$. For $it \in \varrho(T) \cap \varrho(A) \cap i\mathbb{R}$ we set

$$\begin{pmatrix} x \\ y \end{pmatrix} = (T - it)^{-1} \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

Then the invariance and neutrality of W imply that $(x, y) \in W$ and

$$0 = \left\langle \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} 0 \\ w \end{pmatrix} \right\rangle = i(x|w).$$

Now since

$$(A - it)x - BB^*y = 0, \qquad -C^*Cx - (A^* + it)y = w,$$

we obtain

$$\begin{aligned} 0 &= (x| - C^*Cx - (A^* + it)y) = -(x|C^*Cx)_{s,-s} - (x|(A^* + it)y)_{s,-s} \\ &= -\|Cx\|_Y^2 - ((A - it)x|y)_{-s,s} = -\|Cx\|_Y^2 - (BB^*y|y)_{-s,s} \\ &= -\|Cx\|_Y^2 - \|B^*y\|_U^2. \end{aligned}$$

So $Cx = 0$, $B^*y = 0$, $-(A^* + it)y = w$, and therefore

$$B^*(A^* + it)^{-1}w = 0.$$

Let μ_j be the pairwise distinct eigenvalues of A and P_j the orthogonal projection onto $\ker(A - \mu_j)$. We have $w = \sum_{j=0}^{\infty} w_j$ with $w_j = P_j w$ and thus for any $u \in U$ and $it \in \varrho(T) \cap \varrho(A) \cap i\mathbb{R}$,

$$0 = (u|B^*(A^* + it)^{-1}w)_U = \sum_{j=0}^{\infty} \frac{1}{\mu_j - it}(Bu|w_j)_{-s,s}. \tag{16}$$

Now the series

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{\mu_j - z}(Bu|w_j)_{-s,s} \tag{17}$$

converges uniformly on compact subsets of $\varrho(A)$ since

$$\frac{1}{|\mu_j - z|}|(Bu|w_j)_{-s,s}| \le \frac{1}{|\mu_j - z|}\|P_jBu\|_{-s}\|w_j\|_s = \frac{(|\mu_j| + 1)^s}{|\mu_j - z|}\|P_jBu\|_{-s}\|w_j\|,$$

$\sup_j(|\mu_j|+1)^s|\mu_j - z|^{-1}$ is bounded on compact subsets of $\varrho(A)$ and

$$\sum_{j=0}^{\infty}\|P_jBu\|_{-s}\|w_j\| \leq \left(\sum_{j=0}^{\infty}\|P_jBu\|_{-s}^2\right)^{1/2}\left(\sum_{j=0}^{\infty}\|w_j\|^2\right)^{1/2} = \|Bu\|_{-s}\|w\| < \infty.$$

Hence f is analytic on $\varrho(A)$ and with the identity theorem we conclude from (16) that f vanishes on $\varrho(A)$. Integrating (17) along a circle enclosing exactly one μ_j, we thus obtain

$$0 = (Bu|w_j)_{-s,s} = (u|B^*w_j)_U$$

Since u was arbitrary, this implies $B^*w_j = 0$. Along with $w_j \in \ker(A - \mu_j)$, our assumption yields $w_j = 0$ for all j and hence $w = 0$. □

Remark 5.5 In the next section, we will relate conditions on the eigenspaces of A of the form (13) and (15) to controllability and observability concepts:

$$\ker(A^* - \lambda) \cap \ker B^* = \{0\} \qquad \forall \lambda \in \mathbb{C}$$

is equivalent to the approximate controllability of the pair (A, B).

$$\ker(A - it) \cap \ker C = \{0\} \qquad \forall t \in \mathbb{R}$$

means that there are no non-observable eigenvectors of A corresponding to eigenvalues on $i\mathbb{R}$.

Recall that for a closed operator X a subspace $D \subset \mathcal{D}(X)$ is called a *core* for X if $\overline{X|_D} = X$. Let us denote by pr_1 the projection onto the first component of $H \times H$.

Theorem 5.6 *Let $B \in L(U, H_{-s})$, $C \in L(H_s, Y)$ with $0 \leq s \leq 1$, let T have compact resolvent and a finitely spectral Riesz basis of subspaces. Suppose that*

$$\ker(A - it) \cap \ker C = \{0\} \qquad \forall t \in \mathbb{R}, \tag{18}$$

$$\ker(A^* - \lambda) \cap \ker B^* = \{0\} \qquad \forall \lambda \in \mathbb{C}. \tag{19}$$

Then we have:

(i) $\sigma(T) \cap i\mathbb{R} = \varnothing$.

(ii) If $\sigma \subset \sigma(T)$ is skew-conjugate and W_σ is the T-invariant subspace from (7), then $W_\sigma = \Gamma(X)$ where X is a selfadjoint solution of the Riccati equation

$$A^*Xx + X(Ax - BB^*Xx) + C^*Cx = 0 \quad \forall x \in D,$$

and $D = \mathrm{pr}_1(\Gamma(X) \cap \mathcal{D}(T))$ is a core for X.

(iii) The solution X_+ corresponding to $\sigma = \sigma(T) \cap \mathbb{C}_-$ is nonnegative, the solution X_- corresponding to $\sigma(T) \cap \mathbb{C}_+$ is nonpositive.

Proof. We have $\sigma(T) \cap i\mathbb{R} = \varnothing$ by Lemma 5.2. For $\sigma \subset \sigma(T)$ skew-conjugate, first recall that by Proposition 2.9 W_σ is T-invariant and $(T-z)^{-1}$-invariant for every $z \in \varrho(T)$. Proposition 5.3 implies that $W_\sigma = W_\sigma^{\langle * \rangle}$ and Lemma 5.4 then yields $W_\sigma = \Gamma(X)$ with some operator X. Now $\Gamma(X) = \Gamma(X)^{\langle * \rangle}$ implies that X is selfadjoint, see e.g. [25, Lemma 6.2].

For $x \in D$, i.e. $(x, Xx) \in \Gamma(X) \cap \mathcal{D}(T)$, by the T-invariance of $\Gamma(X) = W_\sigma$ there exists $y \in \mathcal{D}(X)$ such that

$$\begin{aligned} & T\begin{pmatrix} x \\ Xx \end{pmatrix} = \begin{pmatrix} y \\ Xy \end{pmatrix} \\ \Rightarrow \quad & Ax - BB^*Xx = y, \quad -C^*Cx - A^*Xx = Xy \\ \Rightarrow \quad & X(Ax - BB^*Xx) = -C^*Cx - A^*Xx. \end{aligned}$$

Since $\sum_{\lambda\in\sigma} \mathcal{L}(\lambda) \subset \Gamma(X)$ is dense and $\sum_{\lambda\in\sigma} \mathcal{L}(\lambda) \subset \mathcal{D}(T)$, we have that $\Gamma(X) \cap \mathcal{D}(T) \subset \Gamma(X)$ is dense too and hence D is a core for X.

Finally $\Gamma(X_+)$ is J_2-nonnegative by Proposition 5.3, and this is clearly equivalent to X_+ being nonnegative. Similarly for X_-. □

6 Controllability and observability concepts

Theorem 5.6 on the existence of solutions of the Riccati equation contains the conditions (18), (19), which are in terms of the eigenspaces of A and the kernels of B^* and C. In this section we relate these conditions to controllability and observability concepts. We consider again the general setting from Section 3: A is normal with compact resolvent and generates a C_0-semigroup $\mathbb{T}$, $B \in L(U, H_{-s})$, $C \in L(H_s, Y)$ with $0 \le s \le 1$. Let us first look at admissibility.

Recall that the control operator B is called *admissible* if for one (and hence for all) $t_0 > 0$,

$$\int_0^{t_0} \mathbb{T}(t)Bu(t)\,dt \in H \quad \forall u \in L^2([0,t_0],U).$$

The observation operator C is called admissible if for one (and hence for all) $t_0 > 0$ there exists $M > 0$ such that

$$\int_0^{t_0} \|C\mathbb{T}(t)x\|_U^2\,dt \le M\|x\|^2 \quad \forall x \in H_1.$$

Proposition 6.1 *If $\mathbb{T}$ is analytic and $s \leq 1/2$, then B and C are admissible control and observation operators, respectively.*

Note that these assumptions are satisfied in the context of Theorem 4.6.

Proof. Since B is admissible for $\mathbb{T}$ if and only if B^* is an admissible observation operator for the dual semigroup $\mathbb{T}^*$, it suffices to check admissibility for C. Let $x \in H_s$, $x = \sum_k \alpha_k e_k$. We have

$$\int_0^1 \|C\mathbb{T}(t)x\|_U^2 \, dt \leq \|C\|^2 \int_0^1 \|\mathbb{T}(t)x\|_s^2 \, dt$$

and

$$\|\mathbb{T}(t)x\|_s^2 = \sum_k (|\lambda_k| + 1)^{2s} e^{2 \operatorname{Re} \lambda_k t} |\alpha_k|^2.$$

Since $\mathbb{T}$ is analytic, there exists $c \geq 1$ such that almost all eigenvalues λ_k satisfy

$$\operatorname{Re} \lambda_k < 0, \qquad |\lambda_k| \geq 1, \qquad |\lambda_k| \leq c |\operatorname{Re} \lambda_k|.$$

Then

$$\begin{aligned}(|\lambda_k| + 1)^{2s} \int_0^1 e^{2 \operatorname{Re} \lambda_k t} \, dt &= \frac{(|\lambda_k| + 1)^{2s}}{2|\operatorname{Re} \lambda_k|}(1 - e^{2 \operatorname{Re} \lambda_k}) \\ &\leq \frac{c(|\lambda_k| + 1)^{2s}}{2|\lambda_k|} \leq 2^{2s-1} c |\lambda_k|^{2s-1} \leq 2^{2s-1} c\end{aligned}$$

where we used $2s - 1 \leq 0$. Consequently

$$M = \sup_{k \in \mathbb{N}} (|\lambda_k| + 1)^{2s} \int_0^1 e^{2 \operatorname{Re} \lambda_k t} \, dt < \infty,$$

and we obtain

$$\int_0^1 \|C\mathbb{T}(t)x\|_U^2 \, dt \leq M \|C\|^2 \|x\|^2.$$

□

Remark 6.2 For the reverse implication, the following result holds, see [19, Theorem 1.4]: If B is admissible, then $B \in L(U, H_{-s})$ for all $s > 1/2$. Note that we do not get $s = 1/2$ here in general: Consider e.g. the case $\lambda_k = -k^2$, $U = \mathbb{C}$, $Bu = ub$, and $b = \sum_k k^{1/2} e_k$. Then B is admissible by the Carleson measure criterion, and we have $B \in L(U, H_{-s}) \Leftrightarrow b \in H_{-s} \Leftrightarrow s > 1/2$.

Since the only restriction on s in Theorem 5.6 is $0 \le s \le 1$, we see that this theorem allows for non-admissible operators B and C. Consequently, we will now look at controllability and observability without the assumption that B or C are admissible.

Let us consider the rescaled semigroup $\mathbb{T}_0(t) = e^{-\omega t}\mathbb{T}(t)$ with $\omega \ge 0$ such that $\mathbb{T}_0$ is exponentially stable. We then have the input and output maps

$$\begin{aligned} &\Phi \in L(L^2([0,\infty[,U), H_{-1}), && \Phi u = \int_0^\infty \mathbb{T}_0(t) B u(t)\, dt, \\ &\Psi \in L(H_1, L^2([0,\infty[,Y)), && (\Psi x)(t) = C\mathbb{T}_0(t)x. \end{aligned}$$

Definition 6.3 We say that

- (A, B) *approximately controllable (in infinite time)* if $\mathcal{R}(\Phi) \subset H_{-1}$ is dense,
- (A, C) is *approximately observable (in infinite time)* if $\ker \Psi = \{0\}$.

We call $\ker \Psi$ the *non-observable subspace.*

It is clear that approximate controllability and observability are dual concepts since the adjoint of Φ is

$$\Phi^* \in L(H_1, L^2([0,\infty[,U)), \qquad (\Phi^* x)(t) = B^*\mathbb{T}_0^*(t)x$$

and

$$\mathcal{R}(\Phi) \subset H_{-1} \text{ dense} \quad \Leftrightarrow \quad \ker \Phi^* = \{0\}.$$

In the literature, there are alternative definitions of approximate controllability and observability, both with and without the additional assumption of admissibility. We will see in Remark 6.6 and Proposition 6.7 that in our setting these alternative definitions coincide with Definition 6.3.

Proposition 6.4 *The non-observable subspace is of the form*

$$\ker \Psi = \bigoplus_{\lambda \in \sigma(A)} \ker(A - \lambda) \cap \ker C,$$

as an orthogonal direct sum in H_1.

Proof. Obviously, $\ker \Psi$ is a closed subspace of H_1 and invariant under the semigroup $\mathbb{T}_0$. Since A has a compact resolvent, $\sigma(A)$ is discrete and $\varrho(A)$ connected. Hence $\ker \Psi$ is also $(A - z)^{-1}$-invariant for all $z \in \varrho(A)$. This implies

$$\ker \Psi = \bigoplus_{\lambda \in \sigma(A)} N_\lambda \tag{20}$$

with $N_\lambda \subset \ker(A-\lambda)$: Indeed if we set $N_\lambda = P_\lambda(\ker\Psi)$ where P_λ is the orthogonal projection onto $\ker(A-\lambda)$, then "$\subset$" holds in (20). Moreover, since P_λ is the Riesz projection corresponding to the eigenvalue λ of A, the $(A-z)^{-1}$-invariance of $\ker\Psi$ implies that $P_\lambda(\ker\Psi) \subset \ker\Psi$; thus also "$\supset$" in (20).

Now for $x \in \ker(A-\lambda)$ we have $(\Psi x)(t) = e^{(\lambda-\omega)t}Cx$ and thus $x \in \ker\Psi$ if and only if $x \in \ker C$. Hence $N_\lambda = \ker(A-\lambda) \cap \ker C$. $\square$

Corollary 6.5 *(A,B) is approximately controllable if and only if*

$$\ker(A^*-\lambda) \cap \ker B^* = \{0\} \quad \forall\lambda \in \sigma(A^*).$$

(A,C) is approximately observable if and only if

$$\ker(A-\lambda) \cap \ker C = \{0\} \quad \forall\lambda \in \sigma(A).$$

Remark 6.6 If B and C are admissible, then $\mathcal{R}(\Phi) \subset H$ and Ψ can be extended to H, i.e.

$$\Phi \in L(L^2([0,\infty[,U),H), \qquad \overline{\Psi} \in L(H, L^2([0,\infty[,Y)).$$

In this case, a natural definition for approximate controllability and observability is that $\mathcal{R}(\Phi) \subset H$ is dense and that $\ker\overline{\Psi} = \{0\}$, respectively, see e.g. [20, Definition 6.5.1]. Now Proposition 6.4 and Corollary 6.5 also hold in this setting, with H_1 and Ψ replaced by H and $\overline{\Psi}$, respectively. Consequently, the controllability and observability concepts from Definition 6.3 coincide with the ones in the admissible case.

Instead of the condition that $\mathcal{R}(\Phi) \subset H_{-1}$ is dense, another possible condition for approximate controllability is that $\mathcal{R}(\Phi) \cap H \subset H$ is dense. This approach was used e.g. in [11]. We will show now that both conditions are equivalent.

Proposition 6.7 *(A,B) is approximately controllable if and only if*

$$\mathcal{R}(\Phi) \cap H \subset H \text{ dense.}$$

Proof. The implication "$\Leftarrow$" is clear since $H \subset H_{-1}$ is dense. So let (A,B) be approximately controllable. Then $\Phi(C_c([0,\infty[,U)) \subset H_{-1}$ is dense, where $C_c([0,\infty[,U)$ denotes the set of continuous, compactly supported functions from $[0,\infty[$ to U. Let $A_0 = A-\omega$ be the generator of $\mathbb{T}_0$. So $A_0 - I : H \to H_{-1}$ is an isomorphism and thus

$$(A_0-I)^{-1}\Phi(C_c([0,\infty[,U)) \subset H \text{ is dense.}$$

Let $x = (A_0 - I)^{-1}\Phi u$ with $u \in C_c([0,\infty[,U)$ and set

$$v(t) = -e^{-t}\int_0^t e^{\tau}u(\tau)\,d\tau.$$

Then $v \in C^1([0,\infty[,U) \cap L^2([0,\infty[,U)$ and $\dot v = -v - u$. Consider the extrapolation space H_{-2} and the corresponding extension $A_0 : H_{-1} \to H_{-2}$. A straightforward computation shows that $t \mapsto \mathbb{T}_0(t)Bv(t)$ belongs to $C^1([0,\infty[,H_{-2})$ with

$$\frac{d}{dt}\mathbb{T}_0(t)Bv(t) = A_0\mathbb{T}_0(t)Bv(t) + \mathbb{T}_0(t)B\dot v(t).$$

Note here that $\mathbb{T}_0(t)Bv(t) \in H_{-1}$ and so $A_0\mathbb{T}_0(t)Bv(t) \in H_{-2}$ in general. Integrating the last equation, we obtain

$$\begin{aligned} 0 = \mathbb{T}_0(t)Bv(t)\big|_0^\infty &= \int_0^\infty A_0\mathbb{T}_0(t)Bv(t)\,dt + \int_0^\infty \mathbb{T}_0(t)B\dot v(t)\,dt \\ &= (A_0 - I)\int_0^\infty \mathbb{T}_0(t)Bv(t)\,dt - \int_0^\infty \mathbb{T}_0(t)Bu(t)\,dt \end{aligned}$$

and hence $\Phi v = (A_0 - I)^{-1}\Phi u = x$. Consequently

$$(A_0 - I)^{-1}\Phi(C_c([0,\infty[,U)) \subset \mathcal{R}(\Phi),$$

which completes the proof. □

Remark 6.8 In the proof of the previous proposition we have not used our assumption that the generator is normal and has a compact resolvent. The equivalence of the two controllability conditions is thus general.

7 Boundedness of solutions

In Theorem 5.6 we proved the existence of selfadjoint, but not necessarily bounded solutions of the Riccati equation. We will now show that under certain additional assumptions some of these solutions are bounded. The key observation is the following lemma, which characterises when a subspace is the graph of a linear operator, and when this operator is bounded.

Lemma 7.1 *Let $W \subset H \times H$ be a closed subspace.*

(i) We have $W = \Gamma(X)$ with a linear operator $X : \mathcal{D}(X) \subset H \to H$ if and only if

$$W \cap \{0\} \times H = \{0\}. \tag{21}$$

(ii) We have $W = \Gamma(X)$ with a bounded operator $X \in L(H)$ if and only if

$$W \oplus \{0\} \times H = H \times H. \tag{22}$$

Proof. (i) is clear since W is a graph subspace if and only if $(0, y) \in W$ implies $y = 0$. To prove (ii), let first $W = \Gamma(X)$ with $X \in L(H)$. For any $x, y \in H$ we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ Xx \end{pmatrix} + \begin{pmatrix} 0 \\ y - Xx \end{pmatrix}.$$

Hence $W + \{0\} \times H = H \times H$ and in view of (i) this sum is also direct. On the other hand, if (22) holds, then by (i) we have $W = \Gamma(X)$ where X is a closed operator since W is closed. Now for $x \in H$ we get from (22) that

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -y \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in W;$$

in particular $x \in \mathcal{D}(X)$ and hence $\mathcal{D}(X) = H$. The closed graph theorem thus yields $X \in L(H)$. $\square$

Remark 7.2 The previous lemma is closely related to the notions of angular subspaces and angular operators, see e.g. [2, §5.1]: If (22) holds, then W is said to be angular with respect to the projection onto the first component of $H \times H$. The operator X is called the angular operator for W. For the relation between angular subspaces and pairs of orthogonal projections, see e.g. [12, §3].

Corollary 7.3 *Let X be a closed, densely defined operator on H. Suppose there exists a Riesz basis $(\varphi_k)_{k\in\mathbb{N}}$ of $\Gamma(X)$, $k_0 \in \mathbb{N}$, and an orthonormal system $(f_k)_{k\geq k_0}$ of H such that*

$$\sum_{k=k_0}^{\infty} \left\| \varphi_k - \begin{pmatrix} f_k \\ 0 \end{pmatrix} \right\|^2 < \infty. \tag{23}$$

Then $X \in L(H)$.

Proof. Let $(\tilde{f}_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H and consider

$$\psi_{1k} = \begin{pmatrix} f_k \\ 0 \end{pmatrix}, \qquad \psi_{2k} = \begin{pmatrix} 0 \\ \tilde{f}_k \end{pmatrix}.$$

So $(\psi_{1k})_{k\geq k_0} \cup (\psi_{2k})_{k\in\mathbb{N}}$ is an orthonormal system. From Lemma 7.1 we know that $\Gamma(X) \cap \{0\} \times H = \{0\}$, which implies that $(\varphi_k)_{k\in\mathbb{N}} \cup (\psi_{2k})_{k\in\mathbb{N}}$ is ω-linearly independent. Since X is densely defined, $\Gamma(X) + \{0\} \times H$ is dense in $H \times H$. Hence $(\varphi_k)_{k\in\mathbb{N}} \cup (\psi_{2k})_{k\in\mathbb{N}}$ is also complete. In view of (23), Corollary 2.5 now shows that $(\varphi_k)_{k\in\mathbb{N}} \cup (\psi_{2k})_{k\in\mathbb{N}}$ is a Riesz basis of $H \times H$. This in turn implies that $\Gamma(X) \oplus \{0\} \times H = H \times H$ and so $X \in L(H)$ by Lemma 7.1. $\square$

Recall the main result of Theorem 5.6: For every skew-conjugate subset $\sigma \subset \sigma(T)$, the T-invariant subspace W_σ given by (7) is the graph $W_\sigma = \Gamma(X)$ of a selfadjoint operator X and

$$A^*Xx + X(Ax - BB^*Xx) + C^*Cx = 0 \quad \forall x \in D,$$

where $D = \mathrm{pr}_1(\Gamma(X) \cap \mathcal{D}(T))$ is a core for X.

Compared with Theorem 5.6, we now make the stronger assumption that the Hamiltonian T has a Riesz basis $(\varphi_k)_{k\in\mathbb{Z}}$ of generalised eigenvectors. Note here that in Theorem 5.6 as well as in the following theorem we have $\sigma(T) \cap i\mathbb{R} = \varnothing$.

Theorem 7.4 *Let T have a compact resolvent and a Riesz basis of generalised eigenvectors $(\varphi_k)_{k\in\mathbb{Z}}$, ordered such that all φ_k with $k \geq 0$ correspond to the spectrum in $\mathbb{C}_-$ and all φ_k with $k < 0$ to $\mathbb{C}_+$. Suppose that there exists an orthonormal system $(f_k)_{k\geq k_0}$ of H such that*

$$\sum_{k=k_0}^{\infty} \left\| \varphi_k - \begin{pmatrix} f_k \\ 0 \end{pmatrix} \right\|^2 < \infty, \tag{24}$$

and that

$$\begin{aligned} \ker(A - it) \cap \ker C &= \{0\} \qquad \forall t \in \mathbb{R}, \\ \ker(A^* - \lambda) \cap \ker B^* &= \{0\} \qquad \forall \lambda \in \mathbb{C}. \end{aligned}$$

If $\sigma \subset \sigma(T)$ is skew-conjugate and such that $\sigma \cap \mathbb{C}_+$ is finite, then $W_\sigma = \Gamma(X)$ with $X \in L(H)$ selfadjoint. Moreover the operator

$$A_X : \mathcal{D}(A_X) \subset H \to H, \quad A_X x = Ax - BB^*Xx, \quad \mathcal{D}(A_X) = \mathrm{pr}_1(\Gamma(X) \cap \mathcal{D}(T)),$$

has a compact resolvent and spectrum $\sigma(A_X) = \sigma$.

Proof. By Theorem 5.6 we have $W_\sigma = \Gamma(X)$ with X selfadjoint. Moreover since each $\mathcal{L}(\lambda)$ is spanned by some φ_k (see Lemma 2.7), there exists $J \subset \mathbb{Z}$ such that

$$W_\sigma = \overline{\mathrm{span}\{\varphi_k \,|\, k \in J\}}.$$

From the assumption that σ is skew-conjugate and $\sigma \cap \mathbb{C}_+$ is finite, it follows that there exists $k_1 \geq 0$ such that $J_1 = \{k \in \mathbb{Z} \,|\, k \geq k_1\} \subset J$ and $J \setminus J_1$ is finite. Using (24), we can thus apply Corollary 7.3 to obtain $X \in L(H)$.

Consider now the isomorphism

$$\Phi : H \to \Gamma(X), \qquad x \mapsto \begin{pmatrix} x \\ Xx \end{pmatrix}.$$

It is easy to see that $\Phi(\mathcal{D}(A_X)) = \Gamma(X) \cap \mathcal{D}(T) = \mathcal{D}(T|_{\Gamma(X)})$ and $\Phi^{-1}T|_{\Gamma(X)}\Phi = A_X$ on $\mathcal{D}(A_X)$. Consequently, A_X has a compact resolvent since the same is true for the restriction $T|_{\Gamma(X)}$. Moreover we have $\sigma(A_X) = \sigma(T|_{\Gamma(X)}) = \sigma$. $\square$

Remark 7.5 In the previous theorem, we have $\mathcal{D}(A_X) \neq \mathcal{D}(A)$ in general. For example, this will be the case for the heat equation with boundary control considered in the next section.

We conclude this section by deriving a sufficient condition for the existence of a Riesz basis that satisfies the assumptions in Theorem 7.4.

Lemma 7.6 *Let P, Q be two projections on a Hilbert space with $\|P - Q\| < 1$ and $\dim \mathcal{R}(P) = \dim \mathcal{R}(Q) = 1$. If $e \in \mathcal{R}(P)$, $f \in \mathcal{R}(Q)$ are such that $\|e\| = \|f\| = 1$ and $(e|f) \geq 0$, then*

$$\|e - f\|^2 \leq \frac{\|P - Q\|^2}{1 - \|P - Q\|}.$$

Proof. We have

$$\|Qe\| \geq \|Pe\| - \|P - Q\|\|e\| = 1 - \|P - Q\|.$$

Since $Qe = \alpha f$ with $\alpha \in \mathbb{C}$, $|\alpha| = \|Qe\|$, we obtain

$$\begin{aligned}
\|P - Q\|^2 &\geq \|Pe - Qe\|^2 = 1 - 2\,\mathrm{Re}(Pe|Qe) + \|Qe\|^2 \\
&= 1 - 2\,\mathrm{Re}(\alpha)(e|f) + \|Qe\|^2 \geq 2\|Qe\| - 2\|Qe\|(e|f) \\
&\geq \big(2 - 2(e|f)\big)\big(1 - \|P - Q\|\big).
\end{aligned}$$

Hence

$$\|e - f\|^2 = 2 - 2(e|f) \leq \frac{\|P - Q\|^2}{1 - \|P - Q\|}.$$

□

Lemma 7.7 *Let S be an operator with compact resolvent and a Riesz basis of Jordan chains. Then for every $a > 1$ there exists $c \in \mathbb{R}$ such that*

$$\|(S - z)^{-1}\| \leq \frac{c}{\operatorname{dist}(z, \sigma(S))} \qquad \text{for} \quad \operatorname{dist}(z, \sigma(S)) \geq a.$$

Proof. Suppose that $(v_{jk})_{j \in \mathbb{N}, k=1\dots r_j}$ is the Riesz basis where each $(v_{j1}, \dots, v_{jr_j})$ is a Jordan chain of S for the eigenvalue λ_j. For

$$x = \sum_{j=1}^{n} \sum_{k=1}^{r_j} \alpha_{jk} v_{jk}, \quad \beta_j = \begin{pmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jr_j} \end{pmatrix}, \quad N_j = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{C}^{r_j \times r_j},$$

we then have

$$\|(S-z)^{-1}x\|^2 \leq M\sum_{j=1}^{n}\|(\lambda_j - z + N_j)^{-1}\beta_j\|^2,$$

where M is the constant corresponding to the Riesz basis and we use the Euclidean norm on $\mathbb{C}^{r_j}$. Since $|\lambda_j - z| \geq \operatorname{dist}(z,\sigma(S)) \geq a > 1$, we have the estimate

$$\|(\lambda_j - z + N_j)^{-1}\| \leq \sum_{k=0}^{\infty}\frac{\|N_j\|^k}{|\lambda_j - z|^{k+1}} = \frac{1}{|\lambda_j - z| - 1} \leq \frac{a}{(a-1)|\lambda_j - z|}.$$

This yields

$$\|(S-z)^{-1}x\|^2 \leq \frac{Ma^2}{(a-1)^2\operatorname{dist}(z,\sigma(S))^2}\sum_{j=1}^{n}\|\beta_j\|^2 \leq \frac{Ma^2}{m(a-1)^2\operatorname{dist}(z,\sigma(S))^2}\|x\|^2.$$

□

In Theorem 4.6 we proved the existence of a Riesz basis of generalised eigenvectors of the Hamiltonian T for the case that C is bounded. Under stronger assumptions on the growth rate of the eigenvalues of A (formulated in terms of the r_{jl}), we will now show that the Riesz basis can be chosen such that it satisfies assumption (24) in Theorem 7.4. Recall that $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of A, $Ae_k = \lambda_k e_k$.

Theorem 7.8 *Suppose that almost all λ_k lie inside discs*

$$K_{jl} = \left\{\lambda \in \mathbb{C} \,\middle|\, |\lambda - e^{i\theta_j}r_{jl}| \leq \alpha\right\}, \quad j = 1,\dots,n,\ l \in \mathbb{N},$$

where $\pi/2 < \theta_j < 3\pi/2$, $\alpha \geq 0$, and the $r_{jl} > 0$ satisfy

$$r_{j,l+1}^{1-q} - r_{jl}^{1-q} \geq \beta, \qquad \sum_{l=0}^{\infty}\frac{1}{r_{jl}^{2q}} < \infty$$

for some $0 < q < 1$ and $\beta > 0$. Suppose also that almost all K_{jl} contain exactly one λ_k, that $B \in L(U, H_{-s})$ with $s \leq \frac{1-q}{2}$ and $C \in L(H,Y)$.

Then T has a compact resolvent and admits a Riesz basis $(\varphi_k)_{k\in\mathbb{Z}}$ of eigenvectors and finitely many generalised eigenvectors. This basis can be chosen such that all φ_k with $k \geq 0$ correspond to the spectrum in $\mathbb{C}_-$, all φ_k with $k < 0$ to $\overline{\mathbb{C}_+}$, and that there exists an orthonormal system $(f_k)_{k\geq k_0}$ of H with

$$\sum_{k=k_0}^{\infty}\left\|\varphi_k - \begin{pmatrix} f_k \\ 0 \end{pmatrix}\right\|^2 < \infty. \tag{25}$$

Proof. First observe that the conditions of Theorem 4.6 are satisfied: Since $(r_{jl})_l$ converges monotonically increasing to infinity we have

$$r_{j,l+1} - r_{jl} \geq (r_{j,l+1}^{1-q} - r_{jl}^{1-q}) r_{jl}^q \geq \beta r_{jl}^q \to \infty.$$

Moreover, if for large l, μ_{jl} is the eigenvalue of A contained in K_{jl}, then

$$|\mu_{jl}| \geq r_{jl} - \alpha \geq \frac{r_{jl}}{2} \quad \text{for} \quad l \geq l_0,$$

and l_0 large enough. Hence

$$\sum_{\substack{j=1\dots n \\ l \geq l_0}} \frac{1}{|\mu_{jl}|^{2(1-2s)}} \leq 2^{2(1-2s)} n \sum_{l \geq l_0} \frac{1}{r_{jl}^{2(1-2s)}} < \infty$$

since $1 - 2s \geq q$. Consequently, T has a compact resolvent and a Riesz basis $(\varphi_k)_{k \in \mathbb{Z}}$ of eigenvectors and finitely many generalised eigenvectors.

Consider now the decomposition $T = S + R$ from (8) and the discs

$$D_{jl} = \left\{ \lambda \in \mathbb{C} \,\middle|\, |\lambda - e^{i\theta_j} r_{jl}| \leq 2 d r_{jl}^q \right\} \quad \text{where} \quad d = \frac{\beta}{4}.$$

Let ∂D_{jl} be the positively oriented boundary of D_{jl}. Our next step is to show that

$$\operatorname{dist}(\partial D_{jl}, \sigma(S)) \geq d r_{jl}^q \quad \forall l \geq l_0, \tag{26}$$

where l_0 is sufficiently large. Recall from Lemma 4.2 that $\sigma(S) = \sigma(A) \cup \sigma(-A^*)$ and hence, with finitely many exceptions, the eigenvalues of S in $\mathbb{C}_-$ are the μ_{jl}. Let $z \in \partial D_{jl}$. For large l we have the estimates

$$\begin{aligned}
|z - \mu_{jl}| &\geq |z - e^{i\theta_j} r_{jl}| - |\mu_{jl} - e^{i\theta_j} r_{jl}| \geq 2 d r_{jl}^q - \alpha \geq d r_{jl}^q, \\
|z - \mu_{j,l+1}| &\geq |e^{i\theta_j} r_{jl} - e^{i\theta_j} r_{j,l+1}| - |z - e^{i\theta_j} r_{jl}| - |\mu_{j,l+1} - e^{i\theta_j} r_{j,l+1}| \\
&\geq (r_{j,l+1} - r_{jl}) - 2 d r_{jl}^q - \alpha \geq \big((r_{j,l+1}^{1-q} - r_{jl}^{1-q}) - 2d \big) r_{j,l+1}^q - \alpha \\
&\geq (\beta - 2d) r_{jl}^q - \alpha = 2 d r_{jl}^q - \alpha \geq d r_{jl}^q, \\
|z - \mu_{j,l-1}| &\geq (r_{jl} - r_{j,l-1}) - 2 d r_{jl}^q - \alpha \geq \big((r_{jl}^{1-q} - r_{j,l-1}^{1-q}) - 2d \big) r_l^q - \alpha \\
&\geq (\beta - 2d) r_{jl}^q - \alpha \geq d r_{jl}^q.
\end{aligned}$$

For $\mu_{j_1 l_1}$, $j_1 \neq j$, we get with $\omega = \min\{|\theta_j - \theta_{j_1}|, \pi/2\}$,

$$|z - \mu_{j_1 l_1}| \geq |e^{i\theta_j} r_{jl} - e^{i\theta_{j_1}} r_{j_1 l_1}| - 2 d r_{jl}^q - \alpha \geq \sin\omega \cdot r_{jl} - 2 d r_{jl}^q - \alpha \geq d r_{jl}^q,$$

again for large l. A similar estimate holds for $|z - \lambda|$ with $\operatorname{Re}\lambda \geq 0$. Since only finitely many $\lambda \in \sigma(S)$ are not covered by one of the above cases, we have indeed verified (26).

From Theorem 4.5 we know that S has a Riesz basis of Jordan chains. By Lemma 7.7 there exists $c \in \mathbb{R}$ such that

$$\|(S-z)^{-1}\| \le \frac{c}{\operatorname{dist}(z,\sigma(S))}, \qquad \operatorname{dist}(z,\sigma(S)) \ge 2.$$

Hence if $\operatorname{dist}(z,\sigma(S)) \ge \max\{2, 2c\|R\|\}$, then

$$\|R(S-z)^{-1}\| \le \frac{c\|R\|}{\operatorname{dist}(z,\sigma(S))} \le \frac{1}{2};$$

with $T - z = (I + R(S-z)^{-1})(S-z)$ this implies $z \in \varrho(T)$ and

$$\|(T-z)^{-1}\| \le \left\|\left(I + R(S-z)^{-1}\right)^{-1}\right\| \|(S-z)^{-1}\| \le \frac{2c}{\operatorname{dist}(z,\sigma(S))}.$$

In particular, almost all eigenvalues of T in $\mathbb{C}_-$ are contained in the discs D_{jl}.

By (26), there exists $l_1 \ge l_0$ such that $\operatorname{dist}(\partial D_{jl}, \sigma(S)) \ge \max\{2, 2c\|R\|\}$ for $l \ge l_1$. Hence $\partial D_{jl} \subset \varrho(T)$, and we can form the Riesz projections

$$P_{jl} = \frac{i}{2\pi}\int_{\partial D_{jl}} (T-z)^{-1}\,dz, \qquad Q_{jl} = \frac{i}{2\pi}\int_{\partial D_{jl}} (S-z)^{-1}\,dz.$$

Using the identity $(T-z)^{-1} - (S-z)^{-1} = -(T-z)^{-1}R(S-z)^{-1}$, we obtain

$$\begin{aligned}
\|P_{jl} - Q_{jl}\| &\le \frac{1}{2\pi}\int_{\partial D_{jl}} \|(T-z)^{-1} - (S-z)^{-1}\|\,|dz| \\
&\le \frac{1}{2\pi}\int_{\partial D_{jl}} \|(T-z)^{-1}\|\|R(S-z)^{-1}\|\,|dz| \\
&\le \frac{1}{2\pi}\int_{\partial D_{jl}} \frac{2c^2\|R\|}{\operatorname{dist}(z,\sigma(S))^2}\,|dz| \le \frac{4c^2\|R\|}{d r_{jl}^q}.
\end{aligned}$$

In particular $\|P_{jl} - Q_{jl}\| < 1$ for almost all pairs (j,l), which implies $\dim \mathcal{R}(P_{jl}) = \dim \mathcal{R}(Q_{jl})$, see e.g. [9, Lemma I.3.1]. Let us denote by e_{jl} the eigenvector from the basis $(e_k)_k$ that corresponds to μ_{jl}, and let $v_{jl} = (e_{jl}, 0)$ be the corresponding eigenvector of S, $Sv_{jl} = \mu_{jl}v_{jl}$. By assumption we have $\sigma(S) \cap D_{jl} = \{\mu_{jl}\}$ and $\mathcal{R}(Q_{jl}) = \operatorname{span}\{v_{jl}\}$ for almost all (j,l). Therefore there exist $l_2 \ge l_1$ and $c_0 \in \mathbb{R}$ such that for $l \ge l_2$

$$\|P_{jl} - Q_{jl}\| \le \frac{c_0}{r_{jl}^q} \le \frac{1}{2}, \quad \mathcal{R}(Q_{jl}) = \operatorname{span}\{v_{jl}\}, \quad \mathcal{R}(P_{jl}) = \operatorname{span}\{\varphi_{jl}\},$$

where we choose φ_{jl} such that $\|\varphi_{jl}\| = 1$ and $(\varphi_{jl}|v_{jl}) \ge 0$. Lemma 7.6 then yields

$$\sum_{\substack{j=1\dots n\\ l\ge l_2}} \|\varphi_{jl} - v_{jl}\|^2 \le \sum_{\substack{j=1\dots n\\ l\ge l_2}} \frac{\|P_{jl} - Q_{jl}\|^2}{1 - \|P_{jl} - Q_{jl}\|} \le 2c_0^2 \sum_{\substack{j=1\dots n\\ l\ge l_2}} \frac{1}{r_{jl}^{2q}} < \infty.$$

Consequently (25) holds if we choose $(\varphi_k)_{k\geq k_0}$ to comprise all φ_{jl} with $l \geq l_2$, $(f_k)_{k\geq k_0}$ to comprise the corresponding v_{jl}, and $\varphi_0, \dots, \varphi_{k_0-1}$ to be the finitely many remaining basis elements corresponding to the spectrum in $\mathbb{C}_-$. $\square$

8 Application to the heat equation

We apply our theory to the one-dimensional heat equation with Neumann boundary control. Consider the system

$$\begin{aligned}
\frac{\partial x}{\partial t}(\xi,t) &= \frac{\partial^2 x}{\partial \xi^2}(\xi,t), && \xi \in [0,1],\ t \geq 0,\\
\frac{\partial x}{\partial \xi}(0,t) &= u(t), \quad x(1,t) = 0, && t \geq 0,\\
x(\xi,0) &= x_0(t).
\end{aligned}$$

Following [20, §10.2.1], we can rewrite this in the abstract form

$$\dot{x} = Ax + Bu \qquad \text{on} \qquad H = L^2([0,1]),$$

where

$$\begin{aligned}
Ax &= \frac{d^2x}{d\xi^2}, && \mathcal{D}(A) = \left\{ x \in H^2([0,1]) \;\middle|\; \frac{dx}{d\xi}(0) = x(1) = 0 \right\},\\
B^*x &= -x(0), && B^* \in L(H_1, U),\ U = \mathbb{C}.
\end{aligned}$$

For the observation, we choose any bounded $C \in L(H, Y)$.

The operator A is selfadjoint with compact resolvent. The eigenvalues and an orthonormal basis of eigenvectors of A are given by

$$\begin{aligned}
\lambda_k &= -\pi^2\left(k + \frac{1}{2}\right)^2, \qquad k = 0, 1, 2, \dots,\\
e_k(\xi) &= \frac{1}{\sqrt{2}} \cos\left(\pi\left(k + \frac{1}{2}\right)\xi\right).
\end{aligned}$$

In particular, A generates an exponentially stable analytic semigroup.

The operator $B \in L(\mathbb{C}, H_{-1})$ is of the form $Bu = ub$ with some $b \in H_{-1}$. We expand b in the basis $(e_k)_{k\in\mathbb{N}}$, $b = \sum_k \alpha_k e_k$ with convergence in the norm of H_{-1}. Then

$$\overline{\alpha}_k = (e_k|b)_{1,-1} = B^*e_k = -e_k(0) = -\frac{1}{\sqrt{2}}$$

and thus $b = -\frac{1}{\sqrt{2}}\sum_{k=0}^{\infty} e_k$. Now

$$\sum_{k=0}^{\infty} e_k \in H_{-s} \;\Leftrightarrow\; \sum_{k=0}^{\infty} |\lambda_k|^{-2s} < \infty \;\Leftrightarrow\; \sum_{k=1}^{\infty} k^{-4s} < \infty \;\Leftrightarrow\; s > \frac{1}{4}.$$

Therefore $B \in L(\mathbb{C}, H_{-s})$ for all $s > 1/4$. In particular, B is admissible by Proposition 6.1.

To show that the Hamiltonian T associated with the system (A, B, C) has a compact resolvent and a finitely spectral Riesz basis of subspaces, we invoke Theorem 7.8: We choose $\theta_1 = \pi$, $\alpha = 0$ and

$$r_{1l} = r_l = \pi^2\Big(l + \frac{1}{2}\Big)^2, \qquad l \geq 0.$$

For $1/4 < q \leq 1/2$ we then have

$$r_{l+1}^{1-q} - r_l^{1-q} \geq (r_{l+1}^{1/2} - r_l^{1/2}) r_l^{1/2-q} \geq \pi\Big(\frac{\pi}{2}\Big)^{1-2q} = \beta,$$

and

$$\sum_{l=1}^{\infty} \frac{1}{r_l^{2q}} \leq \sum_{l=1}^{\infty} \frac{1}{l^{4q}} < \infty.$$

We can thus apply the theorem for any $s \leq \frac{1-q}{2} < 3/8$. In particular, the Riesz basis condition (24) for the existence of bounded solutions is also satisfied.

Now we check that the controllability and observability conditions (18) and (19) of Theorem 5.6 are fulfilled: Since A has no eigenvalues on the imaginary axis, condition (18) is satisfied. From $B^*e_k = -1/\sqrt{2}$ we obtain $e_k \notin \ker B^*$ and so (19) holds too. Therefore $\sigma(T) \cap i\mathbb{R} = \varnothing$ and for every skew-conjugate $\sigma \subset \sigma(T)$ we obtain a selfadjoint solution X of the Riccati equation

$$A^*Xx + X(Ax - BB^*Xx) + C^*Cx = 0, \quad x \in D,$$

where D is a core for X. Moreover, Theorem 7.4 implies that the solutions for the case where $\sigma \cap \mathbb{C}_+$ is finite are bounded and satisfy $\sigma(A_X) = \sigma$.

Finally, we show for the case $C = I$ that

$$\mathcal{D}(A_X) \neq \mathcal{D}(A),$$

where $\mathcal{D}(A_X) = D = \mathrm{pr}_1(\Gamma(X) \cap \mathcal{D}(T))$. By construction in Theorem 5.6, $\Gamma(X)$ contains an eigenvector $(x, y) \neq 0$ of T,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{i.e.} \qquad \begin{cases} (A - \lambda)x - BB^*y = 0, \\ \quad -x - (A + \lambda)y = 0. \end{cases}$$

We show first that $\lambda \in \varrho(A)$: Indeed, if $\lambda \in \sigma(A)$, then

$$BB^*y = (y|b)_{s,-s} b = (A - \lambda)x \perp \ker(A - \lambda)$$

implies $(y|b)_{s,-s} = 0$ since $b \not\perp e_k$ for all k. But then $(A-\lambda)x = 0$, i.e. $x = \alpha e_k$ and $\lambda = \lambda_k$ for some $k \in \mathbb{N}$, $\alpha \in \mathbb{C}$. Hence

$$y = -(A+\lambda)^{-1}x = \frac{-\alpha}{2\lambda_k} e_k$$

and thus $\alpha(e_k|b)_{s,-s} = 0$. This implies $\alpha = 0$ and $x = y = 0$, a contradiction. Now $\lambda \in \varrho(A)$ yields

$$x = (A-\lambda)^{-1}BB^*y = (y|b)_{s,-s}(A-\lambda)^{-1}b.$$

If $(y|b)_{s,-s} = 0$, then $x = 0$, $(A+\lambda)y = 0$ and $y \neq 0$, i.e. $y = \alpha e_k$ for some $k \in \mathbb{N}$ and $\alpha \neq 0$; we obtain $0 = \alpha(e_k|b)_{s,-s}$, a contradiction. Hence $(y|b)_{s,-s} \neq 0$. Consequently $x \notin H_1 = \mathcal{D}(A)$ since $b \notin H$. (In fact $b \in H_{-s}$ and $x \in H_{1-s}$) Since $x \in \mathcal{D}(A_X)$, we conclude $\mathcal{D}(A_X) \neq \mathcal{D}(A)$.

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