Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and
Computational Mathematics (IMACM)

Preprint BUW-IMACM 11/11

Birgit Jacob, Kirsten Morris

Second-Order Systems with Acceleration Measurements

June 2011

http://www2.math.uni-wuppertal.de/ fa/jacob/
http://www.math.uwaterloo.ca/ kmorris/

Second-Order Systems with Acceleration Measurements

Birgit Jacob, Kirsten Morris*

Abstract

Accelerometers are often used to measure the output of second-order systems, such as structural vibrations. Conditions under which these systems are well-posed are obtained. We also establish conditions under which these systems have minimum-phase transfer functions.

Keywords: Accelerometers, second-order systems, well-posed systems, partial differential equations, control, systems theory, infinite-dimensional system

1 Introduction

Many physical systems are modelled by partial differential equations that include second-order derivatives with respect to time. Formally,

$$M\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t) \tag{1}$$

where $z(t)$ depends also on a spatial variable. Flexible structures, acoustic waves in cavities as well as coupled acoustic-structure systems are examples of systems modelled by equations of this type. A number of different types of measurement of these systems are possible.

*Birgit Jacob is with the Department of Mathematics, University of Wuppertal, Gaußstraße 20, 42119 Wuppertal, Germany, `jacob@math.uni-wuppertal.de`. Kirsten Morris is with the Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1, `kmorris@uwaterloo.ca`.

In [2] the well-posedness of a coupled structural-acoustic system with several different choices of measurement is established. In [32, 34] an abstract class of second-order systems similar to (1) but with $D = \frac{1}{2}B_oB_o^*$ is examined. In [34], this system with output $y(t) = -B_o^*\dot{z}(t) + u(t)$ was shown to be well-posed (actually, conservative). More general damping operators and measurements for the abstract class (1) were considered in [17]. The control system with velocity measurements $y(t) = B_o^*\dot{z}(t)$ and certain assumptions on the damping operator was shown to be well-posed. It was shown in [17] that the control system with position measurements $y(t) = B_o^*z(t)$ instead of velocity measurements is also well-posed.

The primary focus of this paper is acceleration measurements. Accelerometers are a very popular choice of sensor for second-order systems in many situations; see, for example [20, 25] and the references therein. In this paper we consider the well-posedness of second-order systems (1) with acceleration measurements

$$y(t) = C_o\ddot{z}(t). \tag{2}$$

Conditions under which these systems are well-posed are established. The damping operator is not restricted to $D = \frac{1}{2}B_oB_o^*$ and more general outputs than $C_o = B_o^*$ are considered. We obtain a representation for the input/output map and transfer function for the situation where the control system may not be well-posed. We provide several examples to show that in general (1) with acceleration measurements (2) is not well-posed.

We develop a model for acceleration measurements that, instead of (2), incorporates a model for the micro-electrical-mechanical systems (MEMS) devices used to measure acceleration. With this more complex model, the control system is in general well-posed with a natural choice of state space. We then provide conditions for the control system to be minimum-phase.

2 Framework

We will use the following notations throughout this article. We denote by $\mathcal{L}(X,Y)$ the set of linear, bounded operators from the Hilbert space X to the Hilbert space Y.

The notation $\mathcal{H}^2(X)$, and $\mathcal{H}^\infty(X)$, where X is a Hilbert space, indicates the usual Hardy spaces of X-valued functions on $\mathbb{C}_0$, the open half plane with $\operatorname{Re} s > 0$. If $X := \mathbb{C}$ we write for simplicity $\mathcal{H}^2$, and $\mathcal{H}^\infty$. The space of matrix-valued functions $\mathcal{H}^\infty(\mathbb{C}^{m\times m})$ will be indicated by $M(\mathcal{H}^\infty)$. The Lebesgue space $L^2(0,t_0;X)$ is the space of strongly measurable, square integrable X-valued functions on the interval $(0,t_0)$, $0<t_0\leq\infty$, and $H^2(0,t_0;X)$ is the Sobolev space of X-valued functions on the interval $(0,t_0)$ that have weak second derivatives.

We study second-order systems of the form (1), and we make the following assumptions throughout this paper, which are similar to those in [17].

(A0) The mass operator M is a self-adjoint bounded linear positive-definite operator on a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ with the property that there exists $m>0$ such that $\langle Mz,z\rangle\geq m\langle z,z\rangle$ for all $z\in H$.

(A1) The stiffness operator $A_o:\mathcal{D}(A_o)\subset H\to H$ is a self-adjoint, positive-definite linear operator on H such that zero is in the resolvent set of A_o. Here $\mathcal{D}(A_o)$ denotes the domain of A_o. Since A_o is self-adjoint and positive-definite, $A_o^{\frac{1}{2}}$ is well-defined. The Hilbert space $H_{\frac{1}{2}}$ is defined as follows: $H_{\frac{1}{2}}=\mathcal{D}(A_o^{\frac{1}{2}})$ with the norm induced by

$$\langle x,z\rangle_{H_{\frac{1}{2}}}=\langle A_o^{\frac{1}{2}}x,A_o^{\frac{1}{2}}z\rangle_H,\qquad x,z\in H_{\frac{1}{2}}$$

and $H_{-\frac{1}{2}}=H_{\frac{1}{2}}^*$. Here the duality is taken with respect to the pivot space H, that is, $H_{-\frac{1}{2}}$ is the completion of H with respect to the norm $\|z\|_{H_{-\frac{1}{2}}}=\|A_o^{-\frac{1}{2}}z\|_H$. Thus A_o extends to $A_o:H_{\frac{1}{2}}\to H_{-\frac{1}{2}}$. We use the same notation A_o to denote this extension.

We denote the duality pairing on $H_{-\frac{1}{2}}\times H_{\frac{1}{2}}$ by $\langle\cdot,\cdot\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$. Note that for $(z',z)\in H\times H_{\frac{1}{2}}$ we have $\langle z',z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}=\langle z',z\rangle_H$.

(A2 i) The control operator $B_o\in\mathcal{L}(U,H_{-\frac{1}{2}})$, where U is a finite-dimensional Hilbert space.

(A2 ii) The damping operator $D:H_{\frac{1}{2}}\to H_{-\frac{1}{2}}$ is linear, bounded and nonnegative, that is,

$$\langle A_o^{-1/2}DA_o^{-1/2}z,z\rangle\geq 0,\qquad z\in H.$$

Assumption (A0) implies that M has a positive definite square root $M^{\frac{1}{2}}$. Defining $\tilde{z}=M^{\frac{1}{2}}z$, we can rewrite the system (1) as

$$\ddot{\tilde{z}}(t)+M^{\frac{1}{2}}A_oM^{-\frac{1}{2}}\tilde{z}(t)+M^{\frac{1}{2}}DM^{-\frac{1}{2}}\dot{\tilde{z}}(t)=M^{\frac{1}{2}}B_ou(t).\tag{3}$$

The operators $M^{\frac{1}{2}}A_oM^{-\frac{1}{2}}$, $M^{\frac{1}{2}}B_o$ and $M^{\frac{1}{2}}DM^{-\frac{1}{2}}$ also satisfy (A1)-(A2) and so without loss of generality we will assume that the operator M has been absorbed into the definition of z and consider systems of the form

$$\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t). \tag{4}$$

These assumptions, although stated abstractly, are satisfied by the systems that typically arise in applications, as the following examples illustrate.

Example 2.1 *(Euler-Bernoulli Beam)* Consider a beam with a thin film of piezoelectric polymer applied to one side. A spatially uniform voltage $u(t)$ is applied to the film to control the vibrations. Let $z(r,t)$ denote the deflection of the beam from its rigid body motion at time t and position r. Use of the Euler-Bernoulli model for the beam deflection and the Kelvin-Voigt damping model leads to [4, 9]:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 z}{\partial r^2} + C_d\frac{\partial^3 z}{\partial r^2 \partial t}\right] = 0, \qquad r \in (0,1), t > 0 \tag{5}$$

where E and C_d are positive physical constants. Assuming the beam to be clamped at $r = 0$ and free at $r = 1$, we obtain the boundary conditions, for some constant c,

$$\begin{array}{lcllcl} z(0,t) & = & 0, & \frac{\partial z}{\partial r}|_{r=0} & = & 0, \\ \left[E\frac{\partial^2 z}{\partial r^2} + C_d\frac{\partial^3 z}{\partial r^2 \partial t}\right]_{r=1} & = & cu(t), & \left[E\frac{\partial^3 z}{\partial r^3} + C_d\frac{\partial^4 z}{\partial r^3 \partial t}\right]_{r=1} & = & 0. \end{array} \tag{6}$$

Assumptions (A1)-(A2) are satisfied with $H = L^2(0,1)$ and

$$H_{\frac{1}{2}} = \left\{z \in H^2(0,1) : z(0) = z'(0) = 0\right\}$$

with inner product $\langle z, v\rangle_{H_{\frac{1}{2}}} = E\langle z'', v''\rangle$.

Example 2.2 *(Vibrations in a bounded connected region with boundary damping)* Consider vibrations in a bounded connected region Ω with boundary Γ. The vibrations are controlled via a control u on part of the boundary, Γ_1 [15, 21, 27, 34]. The region $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary Γ, where $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ and Γ_0, Γ_1 are disjoint open subsets of Γ with both Γ_0 and Γ_1 not empty and Γ_1 is such that the interior sphere condition holds at least for one point in Γ_1. The partial differential equation describing the system is

$$\begin{array}{ll} \ddot{z} = \nabla^2 z, & \Omega \times (0,\infty), \\ z(x,0) = z_0,\ \dot{z}(x,0) = z_1, & \Omega, \\ z(x,t) = 0, & \Gamma_0 \times (0,\infty), \\ \frac{\partial z(x,t)}{\partial n} + d(x)^2 \dot{z}(x,t) = b(x)u(t), & \Gamma_1 \times (0,\infty), \end{array} \tag{7}$$

We also assume that $b, d \in C(\Gamma_1) \subset L^2(\Gamma_1)$ with $\inf_{x\in\Gamma_1} d(x) > 0$ and b not identically zero.

We have $H = L^2(\Omega)$. Defining

$$H^1_{\Gamma_0}(\Omega) = \{g \in H^1(\Omega) \mid g|_{\Gamma_0} = 0\},$$

$H_{\frac{1}{2}} = H^1_{\Gamma_0}(\Omega)$ and the inner product on $H_{\frac{1}{2}}$ is $(\nabla f, \nabla g)$. It is shown in [17] that this system satisfies assumptions (A1)-(A2).

The control system (4) is equivalent to the following standard first-order equation

$$\dot{x}(t) \;=\; Ax(t) + Bu(t) \tag{8}$$

where $A : \mathcal{D}(A) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$ and $B : U \to H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ are given by

$$A = \begin{bmatrix} 0 & I \\ -A_o & -D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_o \end{bmatrix}, \tag{9}$$

where

$$\mathcal{D}(A) = \left\{ \left[\begin{smallmatrix} z \\ w \end{smallmatrix} \right] \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_o z + Dw \in H \right\}.$$

The following theorem is well known, see e.g. [5], [6], [11], [16] , [23], or [34].

Theorem 2.3 *The operator A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of contractions on the state space $H_{\frac{1}{2}} \times H$.*

This guarantees that the spectrum of A is contained in the closed left half plane of $\mathbb{C}$. In [34] it is shown that $0 \in \rho(A)$. Otherwise, the spectrum of A can be quite arbitrary [19]. The generator associated with the system in Example 2.1 generates an exponentially stable semigroup while that in Example 2.2 is only strongly stable. Some conditions on the damping operator D under which the system is exponentially or strongly stable are listed in [17].

3 System theory

Distributed parameter systems theory is complicated by the fact that not only is the state-space infinite-dimensional, but also the control operator B is often not bounded into the state space. Similarly, in the case of velocity and acceleration measurements, the observation operator is not bounded from the state space. We will review some relevant systems theoretic concepts. Throughout this section A, B and C denote arbitrary operators between Hilbert spaces and not the specific operators introduced in Section 2. Denote by U, X and Y Hilbert spaces. Let $A : \mathcal{D}(A) \subset X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X, the state space. Let $W \subset X \subset V$ be Hilbert spaces with continuous dense injections such that $D(A) \subset W$ and the semigroup $(T(t))_{t\geq 0}$ can be extended or restricted to a strongly continuous semigroup on V or W, respectively. We will denote this extension (restriction) again by $(T(t))_{t\geq 0}$.

Consider for $B \in \mathcal{L}(U, V)$ the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_o, \tag{10}$$

where $x_o \in X$ and $u \in L^2_{\text{loc}}(0, \infty; U)$. The operator $T(t)$ defines the map from initial condition to state, that is, for zero input u we have

$$x(t) = T(t)x(0). \tag{11}$$

The boundedness of this map from X to X (or on W or V) is implied by the assumption that A generates a C_0-semigroup.

The mild solution of (10),

$$x(t) := T(t)x_o + \int_0^t T(t-s)Bu(s)\, ds, \qquad t \geq 0, \tag{12}$$

is well-defined on V. For $u \in H^2(0, \infty; U)$, the mild solution $x(t)$ is an X-valued function. Define

$$\mathcal{B}_t u = \int_0^t T(t-s)Bu(s)ds. \tag{13}$$

This operator is well-defined from $H^2(0, \infty; U)$ to X. An operator B is an *admissible control operator for the semigroup* $(T(t))_{t\geq 0}$, if for every $t > 0$ there is a constant $M_t > 0$ such that for all $u \in H^2(0, \infty; U)$,

$$\|\mathcal{B}_t u\|_X \leq M_t \|u\|^2_{L^2(0,\infty;U)}.$$

This allows us to extend $\mathcal{B}_t$ to a linear bounded operator from $L^2(0,\infty;U)$ to X. We call B an *infinite-time admissible control operator* if the above inequality holds with M_t independent of t.

We now add an output to our system (10). Let $C \in \mathcal{L}(W,Y)$. For initial conditions $x(0) = x_o$ in $D(A)$, $T(t)x_o \in D(A)$, and since $D(A) \subset W$, we can define the output operator $\mathcal{C}_t : W \to L^2(0,t;Y)$ by

$$(\mathcal{C}_t x_o)(s) = CT(s)x_o, \qquad 0 \le s \le t.$$

The operator $C \in \mathcal{L}(W,Y)$ is called an *admissible observation operator for the semigroup* $(T(t))_{t\ge 0}$, if for every $t > 0$ there is a constant $N_t > 0$ such that for $x_o \in W$,

$$\int_0^t \|\mathcal{C}_t x_o\|^2 ds \le N_t \|x_o\|_X^2.$$

This allows us to extend the operator $\mathcal{C}_t$ to a linear bounded operator from X to $L^2(0,t;Y)$. We call C an *infinite-time admissible observation operator* if the constant N_t is independent of t. Further information on admissible control and observation can be found in [18, 35, 36].

For an input $u \in H^2_{loc}(0,\infty;U)$ and $x(0) = 0$ the output y is given by

$$y(\tau) = (\mathcal{G}_t u)(\tau), \qquad \tau < t, \tag{14}$$

where $\mathcal{G}_t$ is a linear operator from $L^2(0,t;U)$ to $L^2(0,t;Y)$. Moreover, $\mathcal{G}_t u$ is the convolution of the input u with a distribution g. We define $\mathcal{G}$: $L^2_{\mathrm{loc}}(0,\infty;U) \to L^2_{\mathrm{loc}}(0,\infty;Y)$ by

$$(\mathcal{G}u)(\tau) := (\mathcal{G}_t u)(\tau), \qquad \tau \le t.$$

The transfer function G of system (10), (14), which is an analytic $\mathcal{L}(U,Y)$-valued function on some right-half-plane $\{s \in \mathbb{C} \mid \operatorname{Re} s > \mu\}$, can be defined as the Laplace transform of g. Boundedness of $\mathcal{G}_t$ is equivalent to the boundedness of the transfer function G on some right-half-plane.

Definition 3.1 *The system (10), (14) is* well-posed *on X if and only if the four maps from input and initial condition to state and output defined by $T(t), \mathcal{B}_t, \mathcal{C}_t$ and $\mathcal{G}_t$ are bounded for some $t > 0$ (and hence every $t > 0$).*

A control system is well-posed on some state-space if and only if the input/output map $\mathcal{G}_t$ is bounded, or equivalently, the transfer function G is bounded in some right-half-plane. For more information on well-posed linear systems and transfer functions we refer the reader to [12] and [31].

4 Well-posedness of Second-Order Systems

We now return to the specific class of second-order systems introduced in the Section II.

(A3) There exists a constant $\beta > 0$ such that

$$\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \geq \beta \|B_o^* z\|^2, \qquad z \in H_{\frac{1}{2}}.$$

Theorem 4.1 *([17, Prop. 4.1]) If (A3) holds, then the control operator B is infinite-time admissible.*

This assumption is satisfied by many control systems; for instance Examples 2.1 and 2.2 [17]. However, if B is a bounded operator, then it is always admissible, regardless of the extent of the damping D, so assumption (A3) is sufficient but not necessary for the admissibility of B.

A measurement of the position $C_o z(t)$ where $C_o \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ and Y is a finite-dimensional Hilbert space leads to the control system

$$\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t), \qquad y(t) = C_o z(t). \tag{15}$$

The control system (15) can be equivalently written as

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = C_p x(t) := \begin{bmatrix} C_o & 0 \end{bmatrix} x(t), \tag{16}$$

where the operators A and B are given by (9). If B defined by (9) is an admissible control operator for A, then the transfer function is the Laplace transform of the distribution $\mathcal{G}_p$ defined by [7, pg. 7]

$$\mathcal{G}_p(\phi) = C_p \int_0^\infty T(r)B\phi(r)dr, \qquad \phi \in \mathcal{C}_0^\infty. \tag{17}$$

The next result follows in a similar way to [17, Prop. 4.3] since $[C_o\ 0]$ is a linear bounded operator from the state space $H_{\frac{1}{2}} \times H$ to Y.

Theorem 4.2 *For $\operatorname{Re} s > 0$, we define $V(s) \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}})$ by $V(s) = (s^2 I + sD + A_0)^{-1}$. Let B defined by (9) be an admissible control operator for A_o. Then $G_p(s) = C_o V(s) B_o$, the position measurement system (15) is well-posed and the transfer function G_p is bounded on any right half plane $\mathbb{C}_\alpha$, $\alpha > 0$.*

Consider velocity measurements by studying the system

$$\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t), \qquad y(t) = C_o \dot{z}(t), \tag{18}$$

where $C_o \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ and Y is a finite-dimensional Hilbert space. The system (18) is equivalent to

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = C_v x(t) := \begin{bmatrix} 0 & C_o \end{bmatrix} x(t). \tag{19}$$

where A and B are given by (9) and $C_v : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to Y$.

Theorem 4.3 *If B defined by (9) is an admissible control operator for A, then the transfer function G_v of (18) is the Laplace transform of the distribution $\mathcal{G}_v$ defined by*

$$\mathcal{G}_v(\phi) = C_p \int_0^\infty T(r) B \frac{d}{dr}\phi(r) dr$$

for $\phi \in \mathcal{C}_0^\infty$ and its transfer function is

$$G_v(s) = sC_o V(s) B_o, \quad \operatorname{Re} s > 0.$$

Moreover, for every $\alpha > 0$ there exists a constant $M_\alpha > 0$ such that $\|G_v(s)\| \leq M_\alpha |s|$ for all s with $\operatorname{Re} s > \alpha$.

Proof: The formulae for the input/output map and the transfer function follows in a similar way to the position measurement system. Since $G_v(s) = sG_p(s)$ and $\|G_p(s)\|$ is bounded in any right-half plane $\operatorname{Re} s > \alpha$ for $\alpha > 0$, we obtain that $\|G_v(s)| \leq M_\alpha |s|$ in the same half plane. □

Proposition 4.4 *If assumption (A3) holds and $\|C_o z\| \leq \|B_o^* z\|$ for all $z \in H_{\frac{1}{2}}$, then*

1. *The observation operator C_v is infinite-time admissible for the semigroup generated by A.*
2. *The system (18) is well-posed.*
3. *The transfer function of (18) satisfies $G_v \in M(\mathcal{H}^\infty)$.*

Proof: This result is identical to that in [17, Prop. 4.1], except that the condition $C_o = B_o^*$ in [17, Prop. 4.1] is here generalized to $\|C_o z\| \leq \|B_o^* z\|$ for all $z \in H_{\frac{1}{2}}$. Since the proof in [17] only requires this inequality, the result is immediate. □

5 Acceleration measurement systems

Consider acceleration measurements

$$\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t), \qquad y(t) = C_o \ddot{z}(t). \tag{20}$$

where $C_o \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ and Y is a finite-dimensional Hilbert space. If B is admissible for the semigroup generated by A, then acceleration measurements based on the definition (20) lead to a well-defined input/output map.

Theorem 5.1 *[7, Thm. 2.1] Assume that the control operator B is admissible for the semigroup generated by A. Then there exists a Laplace transformable distribution*

$$\mathcal{G}_a(\phi) = \begin{bmatrix} C_o & 0 \end{bmatrix} \int_0^\infty T(\sigma) B \phi''(\sigma)\, d\sigma, \qquad \phi \in C_0^\infty, \tag{21}$$

such that $y = \mathcal{G}_a * u, u \in C_0^\infty$, where y is given by (20).

Proposition 5.2 *If B is an admissible control operator for A, then the transfer function of the acceleration measurement system (20) is given by*

$$G_a(s) = s^2 C_o V(s) B_o, \quad \operatorname{Re} s > 0,$$

where $V(s)$ is defined as in Theorem 4.2. For every $\alpha > 0$ there exists a constant $M_\alpha > 0$ such that $\|G_a(s)\| \leq M_\alpha |s|^2$ for all $\operatorname{Re} s > \alpha$.

Proof: Theorem 5.1 implies the representation (21). Since the semigroup generated by A is a contraction, for any $\operatorname{Re} s > 0$, the transfer function $G_a(s)$ is given by the Laplace transform of the distribution $\mathcal{G}_a$, yielding

$$\begin{aligned} G_a(s) &= \begin{bmatrix} C_o & 0 \end{bmatrix} \int_0^\infty T(r) B \frac{d^2}{dr^2} e^{-sr} dr \\ &= s^2 \begin{bmatrix} C_o & 0 \end{bmatrix} (sI - A)^{-1} B. \end{aligned}$$

Since $\begin{bmatrix} C_o & 0 \end{bmatrix} (sI - A)^{-1} B = C_o V(s) B_o$ [17, Prop. 3.7], the expression for the transfer function G_a follows. As A generates a contraction semigroup, the resolvent $(sI - A)^{-1}$ is bounded on every half plane $\operatorname{Re} s > \alpha$ with $\alpha > 0$, and this implies the norm estimate of the transfer function. $\square$

The following result now follows from Proposition 4.4.

Corollary 5.3 *If Property (A3) is satisfied and* $\|C_o z\| \le \|B_o^* z\|$ *for all* $z \in H_{\frac{1}{2}}$*, then there exists a constant* $M > 0$ *such that* $\|G_a(s)\| \le M|s|$ *for every* s *in the right half plane.*

The following examples illustrates that in general, G_a is not bounded on some right half plane.
Example 2.1 cont. Consider the clamped-free beam with control $u(t)$ applied via a voltage to a thin film of piezoelectric polymer. Suppose

$$y(t) = \frac{\partial^3 z}{\partial t^2 \partial r}(1, t).$$

Assumptions (A1)-(A3) are satisfied and the transfer function is well-defined (Proposition 5.2). Moreover, A generates an analytic semigroup. By direct calculation from the partial differential equation and boundary conditions,

$$G_a(s) = -c^2 m(s)^2 \frac{\cos(m(s))\sinh(m(s)) + \sin(m(s))\cosh(m(s))}{1 + \cos(m(s))\cosh(m(s))},$$

$$m(s)^4 = -\frac{s^2}{E + C_d s}.$$

Since

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}}} \frac{\cos(m(s))\sinh(m(s)) + \sin(m(s))\cosh(m(s))}{1 + \cos(m(s))\cosh(m(s))} = 1 + i,$$

it follows that the transfer function G_a is unbounded on every right half plane.

If the transfer function is unbounded in every right-half-plane, then the input/output map is not bounded from $L_2(0, t; U)$ to $L_2(0, t; Y)$ for any $t > 0$ and the system is not well-posed on any state-space. However, we have the following positive result, see also [7, Thm. 3.2] for a similar result that assumes B_o is bounded into a smaller space.

Theorem 5.4 *Let* $\overline{\mathbb{C}_0}$ *indicate the closed right-half-plane, that is, the set of all* $s \in \mathbb{C}$ *with* $\operatorname{Re} s \ge 0$*. If* $B_o \in \mathcal{L}(U, H)$*,* $C_o \in \mathcal{L}(H, Y)$ *and* A *generates a bounded analytic semigroup, then the transfer function* G_a *of (20) is analytic on a half-plane containing* $\overline{\mathbb{C}_0}$ *and bounded on* $\overline{\mathbb{C}_0}$*. Thus, the acceleration measurement system (20) is well-posed.*

Proof: We have

$$\begin{aligned} G_a(s) &= s^2 C_o V(s) B_o \\ &= s \begin{bmatrix} 0 & C_o \end{bmatrix} (sI - A)^{-1} B. \end{aligned}$$

Since A generates a bounded analytic semigroup and $(sI-A)^{-1}$ is analytic on a right-half-plane containing $\overline{\mathbb{C}_0}$ [26] and so the transfer function G_a is analytic on this same half-plane. Also, we have for every s in the closed right half plane $\|s(sI-A)^{-1}\|_{\mathcal{L}(H_{\frac{1}{2}}\times H)} \leq M$, for some constant $M > 0$ independent of s. Since $B_o \in \mathcal{L}(U,H)$, $B \in \mathcal{L}(U, H_{\frac{1}{2}} \times H)$ and $\begin{bmatrix} 0 & C_o \end{bmatrix} \in \mathcal{L}(H_{\frac{1}{2}} \times H, Y)$, the transfer function G_a is bounded on the closed right half plane and the result follows. □

The following example shows that, unfortunately, if the semigroup is not analytic, the transfer function $G_a(s)$ may be unbounded in every right-half-plane, even if $B_o \in \mathcal{L}(U,H)$.

Example 5.5 Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. We define the operators $A_o : \mathcal{D}(A_o) \subset H \to H$ and $D \in \mathcal{L}(H)$ by

$$\begin{aligned} A_o z &:= \sum_{n=1}^{\infty} n^4 \langle z, e_n\rangle e_n, \quad z \in \mathcal{D}(A_o), \\ \mathcal{D}(A_o) &:= \{z \in H \mid \sum_{n\in\mathbb{N}} n^8 |\langle z, e_n\rangle|^2 < \infty\}, \\ Dz &:= \sum_{n=1}^{\infty} \langle z, e_n\rangle e_n. \end{aligned}$$

Further, we choose $C_o^* = B_o \in \mathcal{L}(\mathbb{R}, H_{\frac{1}{2}})$ such that $b_n := \langle B_o, e_n\rangle = n^{-3/4}$ for $n \in \mathbb{N}$. An easy calculation shows that

$$\begin{aligned} |G_a(n^{1/4} + in^2)| &\geq n^4 \left| \sum_{k=1}^{\infty} \frac{|b_k|^2}{(n^{1/4}+in^2)^2 + (n^{1/4}+in^2) + k^4} \right| \\ &\geq n^4 \left| \operatorname{Im} \sum_{k=1}^{\infty} \frac{|b_k|^2}{(n^{1/4}+in^2)^2 + (n^{1/4}+in^2) + k^4} \right| \\ &\geq n^4 \frac{|b_n|^2 n^{2+1/4}}{(n^{1/2}+n^{1/4})^2 + (2n^{2+1/4}+n^2)^2}. \end{aligned}$$

Since this approaches $\frac{n^{1/4}}{4}$ as n tends to ∞, the transfer function G_a is unbounded on every right half plane.

The concept of *system nodes* is an even more general concept of control systems than well-posed systems [31]. It provides a framework for the study of systems that may not be well-posed. However, the highly unbounded nature of the observation means that in general, this control system is not even a system node.

More importantly, an ill-posed system cannot be stabilized in any practical sense. If the resolvent of A contains the closed right-half plane, as is common in applications, the corresponding transfer function will be analytic on a region containing the closed right-half-plane. Thus, $G_a(s) \notin M(\mathcal{H}^\infty)$ (or $G_v(s) \notin M(\mathcal{H}^\infty)$) indicates that

$$\limsup_{\substack{|s|\to\infty \\ s\in\mathbb{C}_0}} \|G(s)\| = \infty.$$

In [24, Thm. 8.5] it was shown that these systems are difficult to stabilize. If there is an infinite unbounded sequence in the right-half-plane, the system cannot be stabilized by a strictly proper regular controller (that is any realistic controller) in a manner that is robust with respect to time delays. Since all practical control system contain small time delays, such systems cannot be stabilized in any practical setting.

6 Accelerometers

In general, as illustrated by the examples in the preceding section, the acceleration measurement system (20) is ill-posed. However, the sensors commonly used to measure acceleration are micro-electro-mechanical system (MEMS). The effective mass is suspended between two capacitors and the measured voltage is proportional to the relative position of the mass - see, for example, [8, 25, 30]. The response of the accelerometer is modelled by a second-order system

$$m\ddot{a}(t) + ka(t) + d\dot{a}(t) = F(t)$$

where $F(t)$ is the effective force on the accelerometer due to the movement of the structure to which it is attached.

In [33] the connection of a finite-dimensional system to an infinite-dimensional system is considered, but there both input and output occur through the finite-dimensional system. Also, the finite-dimensional system is only connected to the infinite-dimensional system through external variables.

A common model is to regard the effective force on the accelerometer as $F(t) = C_o\ddot{z}(t)$ where C_o indicates the point of attachment of the accelerometer and to define the output as $y(t) = \alpha a(t)$ for some constant α. The corresponding transfer function is bounded on some right half plane, and thus there exists a state-space such that the system is well-posed. Formally, the corresponding equations are, with $z = \left[\begin{smallmatrix} w \\ a \end{smallmatrix}\right]$,

$$\begin{bmatrix} I & 0 \\ 0 & m \end{bmatrix} \ddot{z}(t) + \underbrace{\begin{bmatrix} A_o & 0 \\ C_oA_o & k \end{bmatrix}}_{\hat{A}_o} z(t) + \begin{bmatrix} D & 0 \\ C_oD & d \end{bmatrix} \dot{z}(t) = \begin{bmatrix} B_o \\ C_oB_o \end{bmatrix} u(t).$$

Appropriate domains for the new stiffness and damping operators are needed. The new "stiffness" operator, $\hat{A}_o$, is not symmetric, nor is $\operatorname{Re}\langle \hat{A}_o z, z\rangle \geq 0$. The new "damping" operator has similar problems. The input operator C_oB_o will only be well-defined if the assumptions on B_o and C_o are strengthened; for instance if B_o is bounded on H. Putting these concerns aside, choosing $m = 1$ and defining the state $\tilde{x} = \begin{bmatrix} w & a & \dot{w} & \dot{z} \end{bmatrix}^T$, we write the uncontrolled system in first-order form, obtaining

$$\dot{\tilde{x}}(t) = \left(\underbrace{\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \\ -A_o & 0 & -D & 0 \\ 0 & -k & 0 & -d \end{bmatrix}}_{\hat{A}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -C_oA_o & 0 & -C_oD & 0 \end{bmatrix}}_{P} \right) \tilde{x}(t).$$

Define $A_{aug} = \hat{A} + P$. Since the operator P is bounded on $D(\hat{A}^2)$, A_{aug} generates a strongly continuous semigroup on $D(A_{aug}) = D(\hat{A})$, see [14, Corollary III.1.5]. This semigroup can be extended to $[D(\hat{A})]_{-1}^{aug}$ which is the completion of $D(\hat{A})$ with respect to the norm $\|(sI - A_{aug})^{-1} \cdot \|$. This is not the natural state space $H_{\frac{1}{2}} \times \mathbb{R} \times H \times \mathbb{R}$. Furthermore, not only are C_o and B_o restricted so that C_oB_o is well-defined, but also the control operator needs to be admissible with respect to this state space.

These mathematical problems are reflecting the fact that this model is not physically correct.

Although the accelerometer is small compared to the structure, there is some coupling. The coupled system describing the interaction of such a device with a structure will be derived using Hamilton's Principle. In order to simplify the exposition, the case of a single accelerometer is analyzed, that is, U and Y are one-dimensional Hilbert spaces, and so $B_o \in \mathcal{L}(\mathbb{R}, H_{-\frac{1}{2}})$ and $C_o \in \mathcal{L}(H_{\frac{1}{2}}, \mathbb{R})$.

Theorem 6.1 *Consider a general second-order system (4) with potential energy defined through a stiffness operator A_o as*

$$V_{structure} = \frac{1}{2}\langle A_o z(t), z(t)\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}$$

where $z(t) \in H_{\frac{1}{2}}$ indicates the deflection of the structure and A_o satisfies (A1). Assume also the kinetic energy of the structure is

$$T = \frac{\rho}{2}\langle \dot{z}(t), \dot{z}(t)\rangle$$

where $\rho > 0$.

Let the real-valued function $a(t)$ indicate the deflection of the mass in the accelerometer mounted on the structure, with respect to the same frame as z, and assume that the uncoupled accelerometer has mass m and potential energy $\frac{k}{2}a(t)^2$ where $k > 0$ is the stiffness of the accelerometer.

Let the damping forces within the structure be described by $D\dot{z}(t)$ where D satisfies (A2ii). The damping force in the accelerometer is assumed proportional to the relative velocity of the accelerometer mass: $d(\dot{a}(t) - C_o\dot{z}(t))$. Also consider a control force on the structure $B_o u(t)$ where $B_o \in \mathcal{L}(\mathbb{R}, H_{-\frac{1}{2}})$.

If the second-order system is coupled to the accelerometer so that $C_o z(t)$ where $C_o \in \mathcal{L}(H_{\frac{1}{2}}, \mathbb{R})$ is the position of the structure where the accelerometer is attached, then the following equations describe the dynamics of the coupled system

$$\begin{aligned} m\ddot{a}(t) + k\big(a(t) - C_o z(t)\big) + d\big(\dot{a}(t) - C_o\dot{z}(t)\big) &= 0, \\ \rho\ddot{z}(t) + A_o z(t) + D\dot{z}(t) + kC_o^*\big(C_o z(t) - a(t)\big) + dC_o^*\big(C_o\dot{z}(t) - \dot{a}(t)\big) &= B_o u(t). \end{aligned} \tag{22}$$

Proof: The position of the accelerometer relative to the structure is $a(t) - C_o z(t)$ so the total potential energy of the system is

$$V = \frac{1}{2}\langle A_o z(t), z(t)\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \frac{k}{2}(a(t) - C_o z(t))^2.$$

The kinetic energy of the coupled system is

$$T = \frac{\rho}{2}\langle \dot{z}(t), \dot{z}(t)\rangle + \frac{m}{2}(\dot{a}(t))^2$$

where m is the accelerometer mass. The action integral $\mathcal{A} : C^2([t_o, t_f]; H_{\frac{1}{2}} \times \mathbb{R}) \to \mathbb{R}$ is then

$$\begin{aligned}
&\mathcal{A}(z,a) \\
&\quad = \int_{t_o}^{t_f} T - V dt \\
&\quad = \int_{t_o}^{t_f} \frac{\rho}{2}\langle \dot{z}(t), \dot{z}(t)\rangle - \frac{1}{2}\langle A_o z(t), z(t)\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \frac{m}{2}|\dot{a}(t)|^2 - \frac{k}{2}|a(t) - C_o z(t)|^2 dt.
\end{aligned}$$

The first variation $(\Delta\mathcal{A}(z,a))(h_1,h_2)$, or derivative, of this functional is defined for

$$h_1 \in C^2(t_o, t_f; H_{\frac{1}{2}}) \text{ with } h_1(t_o) = h_1(t_f) = 0,$$
$$h_2 \in C^2(t_o, t_f) \text{ with } h_2(t_o) = h_2(t_f) = 0,$$

and given by

$$\begin{aligned}
&(\Delta\mathcal{A}(z,a))(h_1,h_2) && (23)\\
&\quad = \int_{t_o}^{t_f} \rho\langle \dot{z}(t), \dot{h}_1(t)\rangle - \langle A_o z(t), h_1(t)\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + m\dot{a}(t)\dot{h}_2(t) \\
&\qquad\qquad -k(a(t) - C_o z(t))(h_2(t) - C_o h_1(t))dt \\
&\quad = -\int_{t_o}^{t_f} \langle \rho\ddot{z}(t), h_1(t)\rangle + \langle A_o z(t), h_1(t)\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + m\ddot{a}(t)\cdot h_2(t) && (24)\\
&\qquad\qquad +k(a(t) - C_o z(t))h_2(t) - k\langle C_o^*(a(t) - C_o z(t)), h_1(t)\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} dt.
\end{aligned}$$

For each $(z,a) \in H^2(t_o, t_f; H_{\frac{1}{2}} \times \mathbb{R})$, the operator $\Delta\mathcal{A}(z,a)$ in (24) is well-defined for $(h_1,h_2) \in L_2(t_o, t_f; H_{\frac{1}{2}} \times \mathbb{R})$. Since for each such (z,a) and (h_1,h_2),

$$|\Delta\mathcal{A}(z,a)(h_1,h_2)| \le M(\|h_1\|_{L_2(t_o,t_f;H_{\frac{1}{2}})} + \|h_2\|_{L_2(t_o,t_f;\mathbb{R})}),$$

where M is independent of (h_1,h_2), $\Delta\mathcal{A}(z,a)$ is a bounded linear functional on $L_2(t_o, t_f; H_{\frac{1}{2}} \times \mathbb{R})$; that is, $\Delta\mathcal{A}(z,a)$ can be regarded as an element of $L_2(t_o, t_f; H_{-\frac{1}{2}} \times \mathbb{R})$.

By Hamilton's Principle, z, a are such that $\Delta\mathcal{A}(z,a) = 0$ for all (h_1, h_2). Since h_1 and h_2 can be zero independently, this implies that if (z,a) is the system dynamics, $\Delta\mathcal{A}(z,a)$ is the zero functional. Using F_a to indicate other forces on the accelerometer and F_z to indicate other forces on the structure we obtain

$$m\ddot{a}(t) + k(a(t) - C_o z(t)) = F_a(t), \tag{25}$$

$$\rho\ddot{z}(t) + A_o z(t) + kC_o^*(C_o z - a(t)) = F_z(t) \tag{26}$$

where the first equation is understood in $L_2(t_o, t_f)$ and the second equation in $L_2(t_o, t_f; H_{-\frac{1}{2}})$. Including the control and damping forces $F_z(t) = B_o u(t) - D\dot{z}$ and $F_a = d(C_o\dot{z}(t) - \dot{a}(t))$ leads to (22). □

Notes:

1. The dual operator C_o^* can be calculated as follows. For any $v \in \mathbb{R}$ the Riesz representation theorem implies that there exists a unique $g \in H_{\frac{1}{2}}$ such that
$$\langle g, \phi \rangle_{\frac{1}{2}} = v \cdot (C_o\phi), \qquad \forall \phi \in H_{\frac{1}{2}}.$$
Let this define a map $N : \mathbb{R} \to H_{\frac{1}{2}}$ by $Nv = g$. Equivalently,
$$\langle A_o g, \phi \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = vC_o\phi, \qquad \forall \phi \in H_{\frac{1}{2}}.$$
Thus,
$$\langle A_o Nv, \phi \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = vC_o\phi, \qquad \forall \phi \in H_{\frac{1}{2}}$$
which implies that $C_o^* = A_o N$.

2. If the structure is the clamped-free beam studied in Example 2.1, $z(t)$ is the beam deflection, $H = L_2(0,1)$, with $\langle w, v \rangle = \int_0^1 w(x)v(x)dx$,
$$H_{\frac{1}{2}} = \{z \in H^2(0,1); z(0) = z'(0) = 0\},$$
$$\text{and} \qquad \langle A_o w, v \rangle = E \int_0^1 w''(x)v''(x)dx,$$
where E is the total beam elasticity. The potential energy of the beam is
$$V = \frac{E}{2}\int_0^1 w''(x)w''(x)dx = \frac{1}{2}\langle A_o w, w \rangle.$$
If the accelerometer is at $x = x_o$, $C_o w = w(x_o)$ and $C_o \in \mathcal{L}(H_{\frac{1}{2}}, \mathbb{R})$ is implied by Sobolev's Imbedding Theorem [1, Thm. 4.12].

3. The effect of damping can be incorporated into the model by utilizing an adjoint or mixed variational principle - see for example [3] and the references therein.

The equations (22) form a description for a structure coupled to an accelerometer. For accelerometer mass $m = 0$, we recover the original equation (4). Since the accelerometer is several orders of magnitude smaller than the structure, the deflection of the structure is not significantly changed by the accelerometer.

The measurement is proportional to the relative deflection of the accelerometer proof mass, that is, the output

$$y(t) = \alpha(C_o z(t) - a(t)), \tag{27}$$

where α is a parameter. We will derive an expression for the transfer function of this coupled system in terms of that of the original system, and show that, except for high frequencies, this leads to an output that is close to the true acceleration.

We first put the coupled system (22) into the standard second-order framework (4) and show that, along with the measurement (27), it defines a well-posed system. Defining $\tilde{H} = H \times \mathbb{R}$ and

$$\tilde{z}(t) = \begin{bmatrix} \sqrt{\rho} z(t) \\ \sqrt{m} a(t) \end{bmatrix},$$

we obtain

$$\ddot{\tilde{z}}(t) + \tilde{A}_o \tilde{z}(t) + \tilde{D}\dot{\tilde{z}}(t) = \begin{bmatrix} \frac{1}{\sqrt{\rho}} B_o \\ 0 \end{bmatrix} u(t) \tag{28}$$

where

$$\tilde{A}_o = \begin{bmatrix} \frac{1}{\rho}(A_o + kC_o^* C_o) & \frac{-k}{\sqrt{\rho m}} C_o^* \\ \frac{-k}{\sqrt{\rho m}} C_o & \frac{k}{m} \end{bmatrix}, \qquad \tilde{D} = \begin{bmatrix} \frac{1}{\rho}(D + dC_o^* C_o) & \frac{-d}{\sqrt{\rho m}} C_o^* \\ \frac{-d}{\sqrt{\rho m}} C_o & \frac{d}{m} \end{bmatrix}, \tag{29}$$

with

$$\begin{aligned} D(\tilde{A}_o) &= \left\{ (z, a) \in H_{\frac{1}{2}} \times \mathbb{R} \mid \frac{1}{\rho}(A_o + kC_o^* C_o) z - \frac{k}{\sqrt{\rho m}} C_o^* a \in H \right\}, \\ D(\tilde{D}) &= \left\{ (z, a) \in H_{\frac{1}{2}} \times \mathbb{R} \mid \frac{1}{\rho}(D + dC_o^* C_o) z - \frac{d}{\sqrt{\rho m}} C_o^* a \in H \right\}. \end{aligned}$$

Theorem 6.2 *Provided that*

$$\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t)$$

satisfies assumptions (A1)-(A2), the second-order system with accelerometer measurement system (28) satisfies the same assumptions with $\tilde{H}_{\frac{1}{2}} = H_{\frac{1}{2}} \times \mathbb{R}$ *and inner product* $\langle \tilde{A}_o^{\frac{1}{2}} \tilde{z}_1, \tilde{A}_o^{\frac{1}{2}} \tilde{z}_2 \rangle$*. This inner product is equivalent to that generated by* $A_o^{\frac{1}{2}}$ *plus the norm of* $\mathbb{R}$*. Thus,*

$$\tilde{A} = \begin{bmatrix} 0 & I \\ -\tilde{A}_o & -\tilde{D} \end{bmatrix} \tag{30}$$

generates a contraction semigroup on $H_{\frac{1}{2}} \times \mathbb{R} \times H \times \mathbb{R}$*.*

Proof: $\langle \tilde{A}_o \tilde{z}_1, \tilde{z}_2 \rangle = \langle \tilde{z}_1, \tilde{A}_o \tilde{z}_2 \rangle$ for every $\tilde{z}_1, \tilde{z}_2 \in D(\tilde{A}_o)$ implies that the operator $\tilde{A}_o$ is symmetric. Moreover, the operator $\tilde{A}_o$ is surjective. For any $\tilde{z} = (z, a) \in D(\tilde{A}_o)$,

$$\langle \tilde{A}_o \tilde{z}, \tilde{z} \rangle = \langle A_o z, z \rangle + \frac{k}{\rho}|C_o z|^2 - 2(\sqrt{\frac{k}{m}}a)(\sqrt{\frac{k}{\rho}}C_o z) + \frac{k}{m}|a|^2.$$

Making use of Young's inequality, for any $0 < \varepsilon < 1$,

$$\begin{aligned} \langle \tilde{A}_o \tilde{z}, \tilde{z} \rangle &\geq \langle A_o z, z \rangle + \frac{k}{\rho}(1 - \frac{1}{\varepsilon})|C_o z|^2 + \frac{k}{m}(1-\varepsilon)|a|^2 \\ &\geq (1 - \frac{k}{\rho}(\frac{1}{\varepsilon} - 1)\|C_o\|^2)\|z\|^2_{H_{\frac{1}{2}}} + \frac{k}{m}(1-\varepsilon)|a|^2. \end{aligned}$$

Thus, by choosing ε sufficiently close to 1 we obtain $M_1 > 0$ such that

$$\langle \tilde{A}_o \tilde{z}, \tilde{z} \rangle \geq M_1(\langle A_o z, z \rangle + |a|^2).$$

In order to show that $\tilde{A}_o$ is selfadjoint on $H \times \mathbb{R}$ it is sufficient to show that $D(\tilde{A}_o)$ is dense in $H \times \mathbb{R}$. Let $(\tilde{z}, \tilde{a}) \in H \times \mathbb{R}$ with

$$\langle \left[\begin{smallmatrix} z \\ a \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{z} \\ \tilde{a} \end{smallmatrix}\right] \rangle = 0, \quad \left[\begin{smallmatrix} z \\ a \end{smallmatrix}\right] \in D(\tilde{A}_o).$$

There exists $\left[\begin{smallmatrix} z_1 \\ a_1 \end{smallmatrix}\right] \in D(\tilde{A}_o)$ with $\tilde{A}_o \left[\begin{smallmatrix} z_1 \\ a_1 \end{smallmatrix}\right] \left[\begin{smallmatrix} \tilde{z} \\ \tilde{a} \end{smallmatrix}\right]$ and thus

$$\langle \left[\begin{smallmatrix} z_1 \\ a_1 \end{smallmatrix}\right], \tilde{A}_o \left[\begin{smallmatrix} z_1 \\ a_1 \end{smallmatrix}\right] \rangle = 0,$$

which implies $(\tilde{z},\tilde{a})=(0,0)$. This shows $\overline{D(\tilde{A}_o)}=H\times\mathbb{R}$ and the selfadjointness of $\tilde{A}_o$. Also,

$$\begin{aligned}\langle \tilde{A}_o\tilde{z},\tilde{z}\rangle &= \langle A_o z,z\rangle + |\sqrt{\frac{k}{\rho}}C_o z-\sqrt{\frac{k}{m}}a|^2\\ &\le \langle A_o z,z\rangle + 2\frac{k}{\rho}|C_o z|^2+2\frac{k}{m}|a|^2\\ &\le M_2(|z|^2_{H_{\frac{1}{2}}}+|a|^2)\end{aligned}$$

for some $M_2>0$. It follows that the norm defined by $\tilde{A}_o^{\frac{1}{2}}$ is equivalent to $\|z\|_{H_{\frac{1}{2}}}+|a|$. In a similar manner it can be shown that $\tilde{D}$ satisfies (A2 ii). Finally, $\begin{bmatrix}\frac{1}{\sqrt{\rho}}B_o\\ 0\end{bmatrix}$ satisfied (A2 i). □

Theorem 6.3 *If A_o has a compact resolvent and $\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}>0$ for any eigenvector z of A_o, then $i\mathbb{R}$ is contained in the resolvent set of $\tilde{A}$ and $\tilde{A}$ generates a strongly stable C_0-semigroup.*

Proof: Let $\tilde{H}_{\frac{1}{2}}:=H_{\frac{1}{2}}\times\mathbb{R}$ and $\tilde{H}_{-\frac{1}{2}}:=H_{-\frac{1}{2}}\times\mathbb{R}$. We will first show that $\langle\tilde{D}\tilde{z},\tilde{z}\rangle_{\tilde{H}_{-\frac{1}{2}}\times\tilde{H}_{-\frac{1}{2}}}=0$ and $\lambda\tilde{z}=\tilde{A}_0\tilde{z}$ implies that $\tilde{z}=0$. Writing $\tilde{z}=(z,a)$, $\langle\tilde{D}\tilde{z},\tilde{z}\rangle_{\tilde{H}_{-\frac{1}{2}}\times\tilde{H}_{-\frac{1}{2}}}=0$ is equivalent to ([17, Rem 2.1])

$$\frac{1}{\rho}(D+dC_o^*C_o)z-\frac{d}{\sqrt{\rho m}}C_o^*a = 0 \tag{31}$$

$$\frac{-d}{\sqrt{\rho m}}C_o z+\frac{d}{m}a=0 \tag{32}$$

where the first equation is in $H_{-\frac{1}{2}}$. Applying C_o^* to (32) we obtain that

$$C_o^*a=\sqrt{\frac{m}{\rho}}C_o^*C_o z. \tag{33}$$

Substituting into (31), we obtain that

$$Dz=0. \tag{34}$$

If also $\tilde{z}$ is an eigenfunction of $\tilde{A}_o$, corresponding to an eigenvalue λ, then

$$\lambda z = \frac{1}{\rho}(A_o + kC_o^* C_o)z - \frac{k}{\sqrt{\rho m}} C_o^* a.$$

Substituting in (33), we obtain that

$$\lambda z = \frac{1}{\rho} A_o z.$$

This, together with (34) and the assumption that $\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} > 0$ for any eigenvector z of A_o implies that $z = 0$, and therefore $\tilde{z} = 0$ by (32). Summarizing, we have shown that $\langle \tilde{D}\tilde{z}, \tilde{z}\rangle_{\tilde{H}_{-\frac{1}{2}} \times \tilde{H}_{-\frac{1}{2}}} > 0$ for every eigenfunction $\tilde{z}$ of $\tilde{A}_o$.

We now show that for some $s \in \rho(\tilde{A}_o)$, the resolvent $(sI - \tilde{A}_o)^{-1}$ is a compact operator. Choose $s \in \rho(A_o) \cap \rho(\tilde{A}_o)$. Define

$$A_e = \begin{bmatrix} \frac{1}{\rho} A_o & 0 \\ 0 & \frac{k}{m} \end{bmatrix}$$

and write

$$(sI - \tilde{A}_o)^{-1} = (sI - A_e)^{-1} - (sI - A_e)^{-1} F (sI - \tilde{A}_o)^{-1},$$

where

$$F := \begin{bmatrix} \frac{k}{\rho} C_o^* C_o & \frac{-k}{\sqrt{\rho m}} C_o^* \\ \frac{-k}{\sqrt{\rho m}} C_o & 0 \end{bmatrix}.$$

By assumption, $(sI - A_e)^{-1}$ is a compact operator on $\mathcal{L}(H \times \mathbb{R}, H \times \mathbb{R})$ for any $s \in \rho(A_e)$. It is also a bounded operator from $H_{-\frac{1}{2}} \times \mathbb{R}$ to $H \times \mathbb{R}$. Similarly, $(sI - \tilde{A}_o)^{-1} \in \mathcal{L}(H \times \mathbb{R}, H_{\frac{1}{2}} \times \mathbb{R})$. The operator F in the above expression is in $\mathcal{L}(H_{\frac{1}{2}} \times \mathbb{R}, H_{-\frac{1}{2}} \times \mathbb{R})$ and has finite-rank. It follows that $(sI - \tilde{A}_o)^{-1}$ is a compact operator.

This implies that the resolvent set of $\tilde{A}$ includes the imaginary axis and that $\tilde{A}$ generates a strongly stable C_0-semigroup on $H \times \mathbb{R}$ [11, Lemma 4.1, Theorem 4.4]. □

Definition 6.4 *A control system is* regular *if it is well-posed and if for some* $E \in \mathcal{L}(U, Y)$*, its transfer function* G *satisfies*

$$\lim_{s \to +\infty, s \in \mathbb{R}} G(s)u = Eu, \qquad u \in U.$$

The operator E *is called the* feedthrough operator *of the system.*

It is known that position measurement systems are regular [17]. There is no general result for the regularity of velocity measurement systems, although results for particular systems have been obtained - see, for instance, [2]. However, the use of an accelerometer leads to a control system that is not only well-posed, but also regular. This is shown in the following theorem.

Theorem 6.5 *If (A3) is satisfied by the structure (4) then (A3) holds for the coupled system with $\tilde{A}_o$ defined in (29) and $\tilde{B}_o = \left[\begin{smallmatrix} B_o \\ 0 \end{smallmatrix}\right]$. The observation operator defined by (27) is bounded from the state space $H_{\frac{1}{2}} \times \mathbb{R} \times H \times \mathbb{R}$ to $\mathbb{R}$ and the accelerometer control system (28) with measurement (27) is well-posed on the state-space $H_{\frac{1}{2}} \times \mathbb{R} \times H \times \mathbb{R}$.*

Defining

$$G_{acc}(s) = \frac{\alpha m}{ms^2 + ds + k},$$

and letting $\tilde{G}_a(s)$ indicate the transfer function of (28) with acceleration measurement

$$y(t) = \left[\begin{array}{cc} \frac{1}{\sqrt{\rho}} C_0 & 0 \end{array} \right] \ddot{\tilde{z}}(t) = C_0 \ddot{z}(t), \tag{35}$$

the transfer function of (27), (28) is

$$G_{am}(s) = G_{acc}(s)\tilde{G}_a(s). \tag{36}$$

Furthermore, the system is regular with zero feedthrough.
Proof: By assumption (A3) there exists a $\beta > 0$ such that

$$\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq \beta \|B_o^* z\|^2, \qquad z \in H_{\frac{1}{2}}.$$

Following the proof of Theorem 6.2 there exists a constant $M_1 > 0$ such that

$$\begin{aligned} \langle \tilde{D}\tilde{z}, \tilde{z}\rangle &\geq M_1 \langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq M_1 \beta \|B_o^* z\|^2 \\ &= M_1 \beta \|\tilde{B}_o^* \tilde{z}\|^2 \end{aligned}$$

for $\tilde{z} = (z, a) \in H_{\frac{1}{2}} \times \mathbb{R}$. Thus (A3) holds for $\tilde{A}_o$ and $\tilde{B}_o$. By Theorem 4.1 $\tilde{B}$ is an infinite-time admissible control operator for $\tilde{A}$. As the observation operator defined by (27) is bounded from the state space $H_{\frac{1}{2}} \times \mathbb{R} \times H \times \mathbb{R}$ to $\mathbb{R}$, the accelerometer control system (28) with measurement (27) is well-posed on the state-space $H_{\frac{1}{2}} \times \mathbb{R} \times H \times \mathbb{R}$.

Since B is an admissible control operator for $\tilde{A}$, Proposition 5.2 implies that the system (28) with acceleration measurement (35) has a well-defined transfer function $\tilde{G}_a(s)$ where $\tilde{G}_a(s) = s^2\tilde{G}_p(s)$ and $G_p(s)$ is the transfer function associated with the position measurement $\begin{bmatrix} \frac{1}{\sqrt{\rho}}C_0 & 0 \end{bmatrix} \tilde{z}(t) = C_0 z(t)$ of (28). An easy calculation then shows that the transfer function of (28), (27) is

$$\begin{aligned} G_{am}(s) &= \frac{\alpha m s^2}{ms^2 + ds + k}\tilde{G}_p(s) \\ &= G_{acc}(s)\tilde{G}_a(s). \end{aligned} \tag{37}$$

Since the position control system (28) with position measurement and transfer function $\tilde{G}_p$ is regular with zero feedthrough [17, Prop. 4.3] the transfer function G_{am} also satisfies

$$\lim_{x\to\infty} G_{am}(x) = 0$$

and the system is regular with zero feedthrough. □

It is clear from (36) that the difference between the transfer function of the accelerometer G_{acc} and 1 determines the deviation of the measurement from the actual acceleration. Defining $\omega_n = \sqrt{\frac{k}{m}}$, $\xi = \frac{d}{2m\omega_n}$, and setting $\alpha = \omega_n^2$,

$$G_{acc}(s) = \frac{1}{\frac{s^2}{\omega_n^2} + 2\xi\frac{s}{\omega_n} + 1}.$$

As illustrated in Figure 1, $G_{acc}(s) \approx 1$ for $|s| << \omega_n$. Thus, at frequencies sufficiently below the natural frequency ω_n of the accelerometer, the relative position of the accelerometer is proportional to the acceleration of the structure and the measurement $y(t)$ provides an accurate measurement of the structure's acceleration.

7 Minimum-Phase behaviour

The term minimum-phase was first coined by Bode during his work on design of feedback amplifiers [10]. The original concept is that a transfer function G is minimum-phase if over all other transfer functions having the same magnitude $|G(\jmath\omega)|$ as G, no other function has smaller phase. Every function in

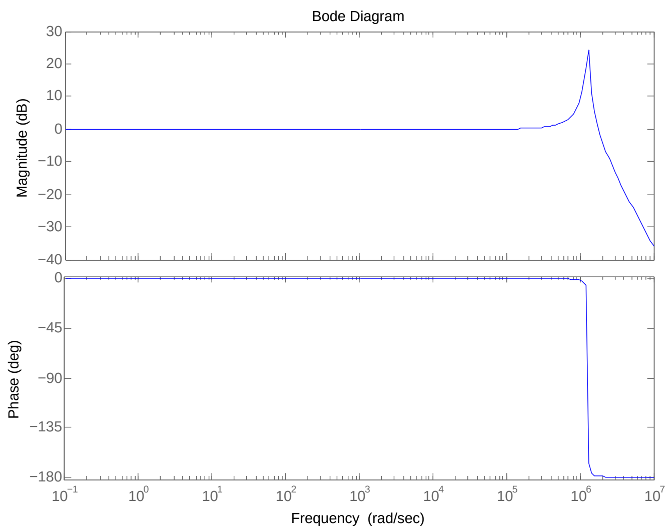

Figure 1: Frequency response $G_{acc}(\imath\omega)$ of a MEMS accelerometer with parameters $m = 3.561 \times 10^{-10} kg$, $k = 571.5 N/m$ and $d = 6.254 \times 10^{-6} Ns/m$ [30, Tables 3&4,D]. For this device, $\omega_n = \sqrt{\frac{k}{m}} = 1.267 \times 10^6$. For frequencies $\omega < \omega_n/2.5 rad/s$, $G_{acc} \approx 1$.

$\mathcal{H}^\infty$ has a factorization into a minimum-phase part (that contains the magnitude information) and an all-pass part (that contains the phase information and any singular part). A stable rational function is minimum-phase if and only if the function has no zeros in the open right-half-plane [13, sec. 6.2]. Minimum-phase functions are equivalent to the outer functions studied in analytic function theory [28]. These functions correspond to operators that have inverses defined on a dense subset of $\mathcal{H}^2(U)$.

Definition 7.1 *[28, page 94] For any function $G \in M(\mathcal{H}^\infty)$ define the operator $\Gamma_G : \mathcal{H}^2(U) \to \mathcal{H}^2(U)$ by $\Gamma_G f = Gf$ for any $f \in \mathcal{H}^2(\mathbb{C}_0; U)$. The function G is* minimum-phase *or* outer *if the range of Γ_G is dense in $\mathcal{H}^2(U)$.*

This explains their importance in controller design- such a system has an inverse defined on a dense subset of $\mathcal{H}^2(U)$. This often facilitates controller design and can lead to improved closed loop performance, as compared to control of non-minimum-phase systems.

Theorem 7.2 *[17, Thm. 5.9] Assume that B_o is injective, $C_o = B_o^*$, and the resolvent set of A contains the imaginary axis. If, in addition, (A3) is satisfied, then the transfer function G_p of the position measurement system (16) and the transfer function G_v of the velocity measurement system (19) are minimum-phase functions.*

We are now in a position to show that under some weak assumptions the transfer function of the accelerometer control system (28) is minimum-phase.

Theorem 7.3 *Assume that B_o is injective, $C_o = B_o^*$, and the resolvent set of $\tilde{A}$ contains the imaginary axis. If, in addition, B_o and D satisfy (A3), then the transfer function G_{am} of the accelerometer control system (27)-(28) is a minimum-phase function.*

Proof: Consider the coupled second-order system (28),

$$\ddot{z}(t) + \tilde{A}_o z(t) + \tilde{D}\dot{z}(t) = \begin{bmatrix} \frac{1}{\sqrt{\rho}} B_o \\ 0 \end{bmatrix} u(t)$$

where

$$\tilde{A}_o = \begin{bmatrix} \frac{1}{\rho}(A_o + kC_o^* C_o) & \frac{-k}{\sqrt{\rho m}} C_o^* \\ \frac{-k}{\sqrt{\rho m}} C_o & \frac{k}{m} \end{bmatrix}, \qquad \tilde{D} = \begin{bmatrix} \frac{1}{\rho}(D + dC_o^* C_o) & \frac{-d}{\sqrt{\rho m}} C_o^* \\ \frac{-d}{\sqrt{\rho m}} C_o & \frac{d}{m} \end{bmatrix}.$$

with output

$$y(t) = \left[\begin{array}{cc} \frac{1}{\sqrt{\rho}}C_o & -\frac{1}{\sqrt{m}} \end{array}\right]$$

and transfer function $\tilde{G}_p$. Since the operator $\tilde{A}$ has no spectrum on the imaginary axis, $\tilde{G}_p$ is minimum-phase by Theorem 6.5 and Theorem 7.2. Thus, by (37) G_{am} is minimum-phase. □

The assumption on $\tilde{A}$ in the above theorem is satisfied, for instance, if A_o has compact resolvent and $\langle Dw, w\rangle > 0$ for any eigenfunction w of A_o (Theorem 6.3).

Example 2.2 (cont.)) Consider vibrations of a plate or membrane that is fixed on part of the boundary, as described in Example 2.2 . This system satisfies assumptions (A3), as well as (A1)-(A2), B_o is injective [17]. In [17] it is shown that if the position measurement is

$$y(t) = \int_{\Gamma_1} b(x)z(x,t)dx,$$

which leads to $y(t) = C_o z(t)$ where $C_o = B_o^*$, then the system is well-posed and minimum-phase. The corresponding velocity measurement $y(t) = C_o\dot{z}(t)$ also leads to a well-posed minimum-phase system. Suppose that an accelerometer is used to measure the vibrations. Then the system is described by

$$\begin{aligned} m\ddot{a}(t) + ka(t) + d(\dot{a}(t) - C_o\dot{w}(t)) - kC_o w(t) &= 0, \\ \rho\ddot{w}(t) + A_o w(t) + kC_o^*C_o w(t) - dC_o^*(\dot{a}(t) - C_o\dot{w}(t)) + D\dot{w}(t) - kC_o^* a(t) &= B_o u(t) \end{aligned}$$

where m, d and k are the relevant accelerometer parameters. The output is

$$y(t) = \alpha\big(\int_{\Gamma_1} b(x)z(x,t)dx - a(t)\big)$$

where α is a parameter. For frequencies within the bandwidth of the accelerometer, $y(t)$ is an accurate measurement of the acceleration. By Theorems 6.5 and 7.3, this control system is well-posed and strictly proper. Furthermore, the transfer function is a minimum-phase function.

8 Conclusions

Following a review and improvement on some existing results of the systems theory for position and velocity measurements of second-order systems, acceleration measurements are introduced. Unless $B_o \in \mathcal{L}(U, H)$ and the semigroup generated by A is analytic, the transfer function G_a of an accelerometer control system is in general not in $M(\mathcal{H}^\infty)$. These are quite restrictive conditions.

However, the simple measurement $y(t) = C_o\ddot{z}(t)$ is a simplification for the output of a second-order system with acceleration measurements. A model describing the action of MEMS accelerometers typically used in practice was developed using Hamilton's Principle. It is shown that provided that the position measurement system is well-posed, the control system with the position sensor replaced by an accelerometer is also well-posed. In fact, the system is regular, and strictly proper. In the final section it is shown that a second-order system with an accelerometer has a minimum-phase transfer function under the same conditions that lead to a minimum-phase transfer function for the corresponding position (and velocity) systems.

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces,* Academic Press, 2006.

[2] G. Avalos, I. Lasiecka and R. Rebarber, "Lack of time-delay robustness for stabilization of a structural acoustics model", *SIAM J. Control Optim.*, Vol. 37(5), pg. 1394-1418, 1999.

[3] L.Y. Bahar and H.G. Kwatny, "Mixed variational principles for some non-self-adjoint dynamical systems", *Mechanics Research Communications*, Vol. 11(4), pg. 253-263, 1984.

[4] T. Bailey and J.E. Hubbard, "Distributed piezoelectric-polymer active vibration control of a cantilever beam", *Jour. of Guidance, Control and Dynamics*, Vol. 8, pg. 605-611, 1985.

[5] H.T. Banks and K. Ito, "A unified framework for approximation in inverse problems for distributed parameter systems", *Control Theory and Adv. Tech.*, Vol. 4, pg. 73-90, 1988.

[6] H.T. Banks, K. Ito and Y. Wang, "Well posedness for damped second order systems with unbounded input operators", *Differential Integral Equations*, Vol. 8, pg. 587-606, 1995.

[7] H.T. Banks and K.A. Morris, "Input-output stability of accelerometer control systems", *Control: Theory and Advanced Technology,* Vol. 10(1), pg. 1-17, 1994.

[8] B. Bais and B.Y. Majlis, "Structure design and fabrication of an area-changed bulk micromachined capacitive accelerometer", *Proc. of the IEEE ICSE*, Kuala Lumpur, Malaysia, pg. 29-34, 2006.

[9] H.T. Banks, R.C. Smith and Y. Wang, "Modeling aspects for piezoelectric patch activation of shells, plates and beams", *Quart. Appl. Math.*, Vol. 53, pg. 353-381, 1995.

[10] H. Bode, *Network analysis and feedback amplifier design*, Van Nostrand Co. Inc, 1945.

[11] S. Chen, K. Liu and Z. Liu, "Spectrum and stability for elastic systems with global or local Kelvin-Voigt damping", *SIAM J. Appl. Math.*, Vol. 59(2), pg. 651-668, 1998.

[12] R.F. Curtain, "The Salamon-Weiss class of well-posed infinite-dimensional linear systems: a survey", *IMA J. Math. Control Inform.*, Vol. 14, pg. 207-223, 1997.

[13] J. C. Doyle, B. A. Francis and A. R. Tannenbaum, *Feedback Control Theory,* Macmillan, 1992.

[14] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer-Verlag, New York, 2000.

[15] E. Hendrickson and I. Lasiecka, "Numerical approximations and regularizations of Riccati equations arising in hyperbolic dynamics with unbounded control operators", *Comput. Optim. Appl.*, Vol. 2(4) , pg. 343-390, 1993.

[16] R.O. Hryniv and A.A. Shkalikov, "Exponential stability of semigroups related to operator models in mechanics", *Math. Notes*, Vol. 73(5), pg. 618-624, 2003.

[17] B. Jacob, K.A. Morris and C. Trunk, "Minimum-phase behaviour of damped second-order systems with position measurements", *IEEE Trans. on Auto. Control*, Vol. 52(9), pg. 1654-1665, 2007.

[18] B. Jacob and J.R. Partington. "Admissibility of control and observation operators for semigroups: a survey". In J.A. Ball, J.W. Helton, M. Klaus, and L. Rodman, editors, *Current Trends in Operator Theory and its Applications, Proceedings of IWOTA 2002*, Operator Theory: Advances and Applications, Vol. 149, Birkhäuser Verlag, 199-221, 2004.

[19] B. Jacob and C. Trunk, "Location of the spectrum of operator matrices which are associated to second order systems", Appears in: *Operators and Matrices*, Vol. 1, pg. 45-60, 2007.

[20] D.-H. Kim, " Real-time frequency compensatory filter for adjusting dynamic bandwidth of single-degree-of-freedom sensor system", *Sensors and Actuators A*, Vol. 140, pg. 251-256, 2007.

[21] J. Lagnese, "Decay of solutions of wave equations in a bounded region with boundary dissipation", *J. Differ. Equations* , Vol. 50, pg. 163-182, 1983.

[22] I. Lasiecka, *Mathematical Theory of Coupled PDE's*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 2002.

[23] I. Lasiecka, "Stabilization of wave and plate equations with nonlinear dissipation on the boundary", *J. Differential Equations*, Vol. 79(2), pg. 340-381, 1989.

[24] H. Logemann and R. Rebarber and G. Weiss, "Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop", *SIAM J. Control Optim.*, Vol. 34(2), pg. 572-600, 1996.

[25] A. Manut and M.I. Syono, "Effects of mechanical geometries on resonance sensitivity of MEMS out-of-plane accelerometer", *Proc. of the IEEE ICSE*, Kuala Lumpur, Malaysia, pg. 25-28, 2006.

[26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[27] A. Rodríguez-Berna and E. Zuazua, "Parabolic singular limit of a wave equation with localized boundary damping", *Discrete Contin. Dyn. Syst.*, Vol. 1(3) , pg. 303-346, 1995.

[28] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Dover Publications, Inc., Mineola, NY, 1985.

[29] D. Salamon, "Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach", *Trans. Amer. Math. Soc.*, Vol. 300(2), pg. 382-431, 1987.

[30] A. Soma and G. De Pasquale, "Numerical and experimental comparison of MEMS suspended plates dynamic behaviour under squeeze film damping effect", *Analog Integr. Circ. Sig. Process*, Vol. 57, pg. 213-224, 2008.

[31] O. Staffans, *Well-posed linear systems*, Encyclopedia of Mathematics and its Applications, 103, Cambridge University Press, Cambridge, 2005.

[32] M. Tucsnak and G. Weiss, "How to get a conservative well-posed system out of thin air, Part II: controllability and stability", *SIAM J. Control Optim.*, Vol. 42(3) , pg. 907-935, 2003.

[33] G. Weiss and X. Zhao, "Well-posedness & controllability of a class of coupled linear systems", *SIAM J. Control Optim.*, Vol. 48(4), pg. 2719-2750, 2009.

[34] G. Weiss and M. Tucsnak, "How to get a conservative well-posed system out of thin air', Part I: well-posedness and energy balance", *ESAIM Control Optim. Calc. Var.*, Vol. 9, pg. 247-274, 2003.

[35] G. Weiss, "Admissibility of unbounded control operators", *SIAM J. Control Optim.*, Vol. 27, pg. 527-545, 1989.

[36] G. Weiss, "Admissible observation operators for linear semigroups", *Israel Journal of Mathematics*, Vol. 65, pg. 17-43, 1989.